SYNCHRONIZATION SYSTEMS

in

Communication and Control

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To

Mother and Daddy
CONTENTS

PREFACE xv

PRELUDE xix

GLOSSARY OF NOTATION xxxv

PART I

1 RANDOM PROCESSES AND THEIR STATISTICAL CHARACTERIZATION 3

1-1 Introduction 3
1-2 Random Variables 4
1-3 Random Processes 10
1-4 Stationary Random Processes and Spectral Densities 13
1-5 The Gaussian Process 17
1-6 The First-Order Markov Process 18
1-7 Delta-Correlated Processes 19
1-8 The Narrowband Gaussian Process 20
1-9 Nonstationary Random Processes with Stationary Increments 23
1-10 Periodic Nonstationary Processes 25
## Contents

1-11 Nonlinear Transformations of Random Processes 26  
1-12 Characterization of Thermal Noise 29  
1-13 Characterization of Shot or Impulse Noise 30  
1-14 Passage of a Random Process Through a Time Invariant Linear Filter 32  
1-15 Further Studies 34  
Problems 34  
References 35

### 2 OPTIMUM LINEAR FILTERING THEORY 37

2-1 Introduction 37  
2-2 The Mathematical Model 38  
2-3 The Optimum Linear Filter 49  
2-4 Noncausal Filtering of Stationary Processes 41  
2-5 Causal Filtering of Stationary Processes and the Wiener-Hopf Equation 44  
2-6 Optimum Filtering of Polynomial Type Signals 47  
2-7 Application of the Filter Performance Formulas for White Noise 48  
2-8 Filtering to Maximize the Output Signal-to-Noise Ratio 53  
2-9 Further Studies 56  
Appendix I Development of the Yovits-Jackson Formulas 58  
Problems 60  
References 62

### 3 SYNCHRONOUS CONTROL SYSTEMS EMPLOYING THE PHASE-LOCK OR ENTRAINMENT PRINCIPLE 65

3-1 Introduction and Motivation 65  
3-2 The Phase-Locked Loop Mechanization 66  
3-3 Loop Equation in the Absence of Noise 73  
3-4 Loop Equation in the Presence of Noise 76  
3-5 Statistical Properties of the Phase-Noise Process $N(t, \varphi)$ 81  
3-6 Applications to Coherent Communications 82  
3-7 Suppressed Carrier (Subcarrier) and Modulation Tracking Loops 91  
3-8 Coherent Receivers Which Exploit Coherency of the Signal's Sidebands at the Carrier Frequency for Tracking 99  
3-9 Pseudo-Noise Tracking Receivers and Coherent Transponders 107  
3-10 The Digital Phase-Locked Loop (DPLL) 112  
3-11 Unified Loop Equation for Synchronous Control Systems 115  
3-12 Further Studies 117  
Problems 119  
References 126
4 LINEAR THEORY OF SINUSOIDAL PHASE-LOCKED LOOPS WITH APPLICATIONS

4-1 Introduction 130
4-2 The Linear PLL Model When the Loop is Designed to Track $\theta(t)$ 131
4-3 Linear PLL Theory When $\theta(t)$ is Tracked by the Loop 133
4-4 Carrier and Doppler Tracking in the Absence of Noise and Oscillation Instabilities 143
4-5 Optimum Design of Carrier and Doppler Tracking Loops in the Presence of Noise 146
4-6 Carrier and Doppler Tracking with a Second-Order PLL Preceded by a Bandpass Limiter (BPL) 153
4-7 Linear PLL Angle Demodulation Theory 158
4-8 Phase Demodulation Using the Linear PLL Model 161
4-9 Frequency Demodulation Using the Linear PLL Model 166
4-10 Bandpass PLL Design Using Multiple Filters in the Loop 175
4-11 Phase-Locked Loop Mechanization for Carrier Tracking 177
4-12 Sample Design of a Superheterodyne PLL Receiver Preceded by a BPL 180
4-13 Related Studies 184
Appendix I 186
Problems 199
References 207

PART II

5 FUNDAMENTALS OF NONLINEAR OSCILLATION 213

5-1 Introduction 213
5-2 Linear-Free Oscillations 215
5-3 Linear Oscillations in the Presence of a Deterministic External Force 218
5-4 Free Nonlinear Oscillations of Undamped Systems with Nonlinear Restoring Forces 221
5-5 Free Nonlinear Oscillations of Damped Systems with Nonlinear Restoring Forces 226
5-6 The Various Types of Singularities 227
5-7 The Pendulum with Damping Force Proportional to the Absolute Velocity $|\dot{\theta}|$ 230
5-8 Self-Sustained and Relaxation Oscillations 231
5-9 Coupled Nonlinear Oscillations, the Phase-Locked Regenerative Receiver, and the Principle of Synchronization 240
5-10 Electrical Problems Leading to Hill's Equation 250
6 STOCHASTIC FIELD THEORY FOR TRANSPORT PROCESSES

6-1 Introduction 257
6-2 Probability Flux and Gauss's Law 258
6-3 Divergence of the Probability Flux Density 259
6-4 Stoke's Theorem and the Probability Flux Density 260
6-5 The Changing Flux Theorem and the Probability Flux Density 260
6-6 Equation of Probability Flow 262
6-7 Maxwell's Curl Equations and the Potential Equations for Stochastic Fields 265
6-8 Stochastic Fields Produced by Brownian Motion 268
6-9 Poisson's and Laplace's Equations for Static Stochastic Fields 273
6-10 Transport Processes and Carriers 274
6-11 Further Studies 276
   Appendix I Vector Field Concepts 277
   Appendix II Proof of the Changing Flux Theorem 281
   Problems 283
   References 284

7 STOCHASTIC METHODS FOR DYNAMICAL SYSTEMS UNDERGOING DIFFUSION

7-1 Introduction 285
7-2 Random Walk Principles 286
7-3 The Law of Diffusive Flow 291
7-4 Random Walk with Absorbing Boundaries 292
7-5 Markov Processes and Applications 296
7-6 First-Order Markov Processes and the One-Dimensional Fokker-Planck Equation 298
7-7 Time Dependent Solutions to the One-Dimensional Fokker-Planck Equation 304
7-8 The First-Passage Time Problem in One Dimension 311
7-9 Vector Markov Process and Representations 320
7-10 The Multidimensional Fokker-Planck Equation 326
7-11 Physical Interpretation of the Probability Current Density 332
7-12 The Potential Case 333
7-13 The Relationship Between the Fokker-Planck Equation, Stochastic Field Theory, and Continuous Markov Processes 336
7-14 Dynamical Systems Described by a Set of Stochastic Differential Equations and the Diffusion Approximation 337
Contents

7-15 The Escape of Particles over a Potential Wall 339
7-16 Further Studies 341
   Problems 342
   References 342

8 STOCHASTIC DIFFERENTIAL EQUATIONS AND THE FOKKER-PLANCK EQUATION 345

   8-1 Introduction 345
   8-2 State-Space Representation of a Dynamical System 346
   8-3 Basic Rules of Itô Calculus 347
   8-4 Evaluation of the FP Intensity Coefficients Using Itô Calculus 355
   8-5 Further Studies
      Appendix I Matrix Notation 360
      Problems 361
      References 369

PART III

9 NONLINEAR THEORY OF FIRST-ORDER SYNCHRONOUS CONTROL SYSTEMS 373

   9-1 Introduction 373
   9-2 Acquisition Behavior and Synchronization Stability in the Absence of Noise 374
   9-3 Characterization of the Statistical Behavior in the Presence of Noise 379
   9-4 Moments of the Mean Time to First Slip in a First-Order SCS and the Probability of Sync Failure 407
   9-5 Signal Acquisition Probability and Moments of the Signal Acquisition Time 410
   9-6 PLL Operation in the Presence of Impulsive and Gaussian Noise 412
   9-7 Techniques for Approximating the Variance of the Phase-Error in the Nonlinear Region of a Sinusoidal PLL 417
   9-8 Related Studies
      Appendix I 421
      Appendix II 425
      Appendix III 429
      Appendix IV 434
      Problems 436
      References 439
10 NONLINEAR THEORY OF SECOND-ORDER SYNCHRONOUS CONTROL SYSTEMS 442

10-1 Introduction 442
10-2 Signal Acquisition with Imperfect Second-Order Loops in the Absence of Noise 449
10-3 Signal Acquisition Properties of Perfect Second-Order Loops in the Absence of Noise 474
10-4 Signal Acquisition Aids and Techniques 475
10-5 Nonlinear Theory of Second-Order SCSs in Noise 481
10-6 Frequency Acquisition Time and Signal Acquisition Probability in the Presence of Noise 497
10-7 Approximate Theories for Evaluating the Steady-State Variance of the Phase Error 500
10-8 Conclusions Regarding the Application of First- and Second-Order SCSs 501
10-9 Related Studies 502
Appendix I 504
Appendix II 507
Problems 511
References 517

11 NONLINEAR THEORY OF HIGHER-ORDER SYNCHRONOUS CONTROL SYSTEMS 521

11-1 Introduction 521
11-2 Synchronous Control System Representation and Equivalent Model 522
11-3 The $(N+1)$-Dimensional Fokker-Planck Equation 525
11-4 Initial Conditions and the Periodic Extension Method 527
11-5 Boundary Conditions 530
11-6 Differential Equations for the Transition Probability Density Functions and the Average Residual Frequency Detuning 532
11-7 Synchronous Control Systems with $F_0 = 0$ 535
11-8 Evaluating the Steady-State Conditional Expectations $E(y_0|\phi)$ and the Steady State Density $p(\phi)$ for an $(N+1)$-Order SCS 536
11-9 The First-Passage Time Model and Boundary Conditions on $P(y; t)$ 541
11-10 Moments of the First-Passage Time of the $k$-th Projection 542
11-11 Net Flow of Probability per Unit Time and the Average Number of Cycles Slipped per Unit of Time 546
11-12 Synthesis of Optimum Synchronous Control Systems 548
11-13 The Optimum Reference Signal for a Square-Wave Input 553
11-14 Extensions to an Arbitrary Loop Filter and Further Studies 554
11-15 Application of the Nonlinear Theory to Obtain Performance of a Third-Order Loop 556
12 NONLINEAR THEORY OF SYNCHRONOUS CONTROL SYSTEMS WITH RANDOM MODULATION INPUTS

12-1 Introduction 562
12-2 Transmitter-Receiver Characterization 563
12-3 The Periodic Extension of \( P(Y, t) = P(Y, t | Y_0, t_0) \) 564
12-4 Reduction of the Fokker-Planck Equation for the Case of Angle Modulation 566
12-5 Differential Equations for the Conditional Transition Probability Density Functions 567
12-6 The Steady-State Probability Density of the Phase-Error \( p(\phi) \) When the Conditional Expectations Are Approximated 569
12-7 Further Studies 570

Problems 571
References 572

PART IV

13 SOLUTIONS OF THE FOKKER-PLANCK EQUATION BY THE SEQUENCE METHOD 577

13-1 Introduction 577
13-2 The Sequence Method 578
13-3 The Gaussian Density Function as an Initial Estimate for the Sequence Method 585
13-4 The Steady-State Variance for a Sinusoidal Nonlinearity 588
13-5 Further Studies 593

Appendix I Evaluation of Gaussian Moments 595

References 598

14 SOLUTIONS OF THE FOKKER-PLANCK EQUATION BY THE CONDITIONAL EXPECTATION METHOD 600

14-1 Introduction 600
14-2 The Dynamical System 601
14-3 Approximating the Conditional Expectations 602
14-4 The Steady-State Solution to the Reduced Fokker-Planck Equation 605
### Contents

14-5 Solution to the Fokker-Planck Equation by a Combination of Methods 608
14-6 Steady-State Results for a Sinusoidal Nonlinearity 609
   References 613

### PART V

15 FREQUENCY DEMODULATION BY MEANS OF A PHASE-LOCKED LOOP 617

15-1 Introduction 617
15-2 System Model 618
15-3 Application of the Nonlinear Theory 621
15-4 Further Studies 632
   References 633

16 TIME-DEPENDENT SOLUTIONS TO THE FOKKER-PLANCK EQUATION 635

16-1 Introduction 635
16-2 Development of the Expansion for the Conditional Transition Probability Density Function 636
16-3 Spectral Analysis of First-Order Loops in the Absence of Detuning, $\beta = 0$ 644
16-4 Eigenfunction Expansion for First-Order Loops in the Absence of Detuning, $\beta = 0$ 652
16-5 Spectral Analysis of First-Order Sinusoidal PLLs in the Absence of Detuning, $\beta = 0$ 657
16-6 Perturbation Solutions 660
16-7 Iterative Techniques and the Calculation of Eigenvalues 665
16-8 Numerical Results for the Case $\beta = 0$ 667
16-9 Spectral Analysis of First-Order Loops in the Presence of Detuning, $\beta \neq 0$ 675
16-10 Eigenfunction Expansions for First-Order Loops in the Presence of Detuning, $\beta \neq 0$ 676
16-11 Perturbation Estimates of the Nonself-Adjoint Spectrum, $\beta \neq 0$ 679
16-12 Development of the Expansion for the Restricted Probability Density Function $Q(\phi; t)$ 682
16-13 Conclusions 685
   References 685

### INDEX 687
PREFACE

This book was written with the needs and interests of several classes of readers in mind. An attempt has been made to present the underlying principles and theory in such a way that they can be grasped readily by those engineers, physicists, and applied mathematicians who possess an elementary background in probability theory, random processes, modulation theory, and differential equations. My interest in and acquaintance with a greater part of the material in the book arose through work at the Jet Propulsion Laboratory and through graduate lecture courses conducted at the University of Southern California.

The material presented here cannot be covered completely in a one-semester course. An instructor using the book as a text, however, will find suggested course plans in Fig. 1, or he may wish to select other approaches that he feels are more interesting and important. Several problems are included. Many of these are applications of the techniques described; others work out special cases of the general theory, and some fill in details of tedious proofs or extend the theory.

It is also hoped that this book will be a means by which engineers and others may learn about synchronization system theory and applications without resorting to a formal course of study. Figure 2 offers various study flow paths via which the material can be studied in private. The book may also serve as a reference. Some new results and methods are presented. In view of the author's own general interest, not to say biases, it would be strange if the needs and interests of researchers in the field of telecommunica-

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Fig. 1. Various Course Outlines and Study Flow Paths (Level of Difficulty Proceeds from Left to Right).
Fig. 2. Various Study Flow Paths (--- Represents Alternate Path).
tions were neglected. For this reason I have emphasized various known types of physical problems that lead to questions for further research. Hopefully, the book will aid those who wish to be brought up to the threshold where new investigations are being made. Readers in this category are supplied with "hints" regarding new problems to be tackled and with a number of ideas and references that perhaps could be used to solve these problems.

It should be noted that the book is not a complete survey of the work accomplished in the field and does not resort to name calling contests of the many authors who have addressed the subject. Rather the intent is to point to the work of various researchers which I have found useful in preparing the manuscript. Although not intentional, I have relied heavily upon material generated by American authors while only recognizing the important, valuable, and extensive work of certain Soviet workers. This is primarily due to the unavailability of translations of the Russian literature. Based upon what is available, however, I have attempted to cite the early Russian work accomplished by R. L. Stratonovich and V. I. Tikhonov and the offspring of papers and books written thereafter.

In view of the length of the book, some advice to readers and instructors may be useful in selecting material to study and teach. The PRELUDE which follows was written for the purpose of motivating the reader and provides an overview of the material presented. It also illustrates the operational behavior of a broad class of synchronization systems from the simplest viewpoint and, in essence, introduces the reader to the glossary of technical terms. This approach avoids introducing any mathematical concepts prior to understanding their need.

There are five parts to the book (see Fig. 2), consisting of sixteen chapters and a few Appendices. The first four chapters (Part I) pertain to the development of various models for the various applications in coherent telecommunication and control systems. In particular, Chapter 1 presents a concise review of the necessary background material in probability and random process theory and sets the notation adopted in the text. Chapter 2 reviews linear filtering theory which is required in Chapter 4. In Chapter 3 the phase-lock principle is introduced and motivation is provided for various scientific fields and applications. Since any book on the subject would be quite incomplete without the linear phase-locked loop (PLL) theory, Chapter 4 is devoted to applying the theory given in Chapters 1, 2, and 3 to its development. Various practical loop mechanizations, e.g., coherent transponders and double-loop superheterodyne receivers, etc., are discussed.

Part II (Chapters 5, 6, 7 and 8) presents the basic theory required for the theoretical developments given in Parts III, IV, and V. Chapter 5 presents the fundamentals of nonlinear oscillators. Here, the phase-lock principle is traced back to Huygens (1665), and the important Van der Pol theory of nonlinear oscillations is reviewed. Throughout this chapter an attempt is made to
bring into focus the application of nonlinear oscillations to many scientific fields. In Chapter 6 the pedagogic aspects of stochastic field theory are developed in a language appealing to the electrical engineer. Then in Chapter 7 the Fokker-Planck equation is derived and dissected using the theory of continuous Markov processes. The subject of diffusion is discussed from a physics point of view, and the interconnections between stochastic field theory and Markov processes are also noted. Chapter 8 introduces the basic rules of the Itô stochastic calculus. The Fokker-Planck equation is developed from state-space equations which describe a rather general dynamical system.

Part III is a treatment of the nonlinear theory. (Chapters 9, 10, 11 and 12). Portions of the material in these four chapters could be condensed into one large chapter at the expense of missing the buildup of the theory from elementary concepts. However, the behavior and understanding of non-linearity certainly would be lost by the average reader. Chapter 9 characterizes the performance of first-order systems from the nonlinear theory, discusses the nonlinear concepts of loop threshold, acquisition, cycle slipping, the steady-state phase error probability density function, etc. Chapter 10 extends the results of Chapter 9 to the case of second-order systems, the case of greatest practical interest. Chapter 11 gives a generalized nonlinear theory of synchronization systems for an arbitrary-order system with an arbitrary periodic nonlinearity. This chapter illustrates application of the theory by studying the behavior of third-order systems. In Chapter 12 the problem of random angle modulation tracking is attacked and special cases of angle demodulation by means of sinusoidal PLL are treated in detail.

Part IV of the book consists of Chapters 13 and 14. Chapter 13 presents the sequence method for solving the Fokker-Planck equation while Chapter 14 generalizes the conditional expectation method used in Chapters 11 and 12 to solve the Fokker-Planck equation. This body of theory is then applied in Chapter 15 of Part V to the nonlinear problem of frequency demodulation by means of a PLL. Finally, Chapter 16 investigates the problem of obtaining time-dependent solutions to the Fokker-Planck equation. A large part of the signal acquisition problem in noise is embedded in such solutions. Details are also given pertaining to the power spectrum and autocorrelation function of any function of the phase error process. Further research efforts on this problem are placed clearly into evidence.

Finally, we point out a large defect; that is, the nonlinear theory of synchronization stability in the presence of noise is to a large extent not touched upon at all. Unfortunately, this represents a large part of nonlinear theory and deserves considerable attention. Liapunov's method is the only general method available for the study of stability in the absence of noise and has been widely used, particularly in the Soviet Union. Construction of Liapunov functions requires ingenuity and experience. There are, as yet, no general schemes to determine stability in the presence of noise.
Although I have tried conscientiously to eliminate errors, some will certainly remain in the published book. I would very much appreciate hearing about these and, in turn, I will be happy to provide an errata list upon request.

There are many colleagues who have provided me with assistance and support during the preparation of the manuscript. I am particularly indebted to Dr. Marvin K. Simon and Dr. Joseph H. Yuen of the Jet Propulsion Laboratory, Dr. Aubrey Bush of the Georgia Institute of Technology and Dr. Ian F. Blake of the University of Waterloo, for reading and commenting on certain portions of the manuscript.

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In preparing the typescript my greatest debt is to Miss Corinne J. Leslie who patiently waded through many drafts and revisions. Last, but by no means least, the efforts of Mrs. Kay Haines, Mrs. Bess Fairbanks, and other reference librarians at the Jet Propulsion Laboratory are acknowledged for securing a large portion of the reference material required in preparation of the manuscript.

An important factor in my writing this book was the sustained financial support of the National Aeronautics and Space Administration and the Joint Services Electronics Program at the University of Southern California. Finally, I wish to acknowledge the financial sponsorship provided by the Statistics and Probability Program of the Office of Naval Research under the direction of Dr. Bruce J. McDonald.

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1. Introduction

In recent years it has become evident that a broad class of carrier (or sub-carrier) tracking loops, suppressed carrier tracking loops, phase-coherent demodulators of the phase-locked loop (PLL) or automatic phase control (APC) type, and various closed-loop symbol (bit) synchronization systems can be studied en masse by postulating a single, but versatile, system model. Since the model includes such a broad category of systems of great practical interest, and since certain aspects of the mathematically equivalent model characterizes the behavior of a synchronous machine, I have decided to refer to this category of systems as the class Synchronous Control Systems (SCSs). This book will therefore be concerned with the formulation of a mathematical theory for use in system design and performance characterization.

Before postulating the general model, a number of applications which serve to motivate the reader as well as motivate the introduction of the model itself, are presented. Perhaps it is also worth noting here that integrated circuit technology has spawned a number of products that have and will continue to influence system design; however, few of these can rival the potential impact which monolithic circuits, recently introduced into the technology, may have on the future uses of SCSs.
2. Applications of Synchronous Control Systems

In this section a few applications of SCSs to problems arising in phase-coherent telecommunication and control systems are presented; in particular, problems pertaining to synchronization and carrier tracking, coherent demodulation of digital and analog signals, frequency synthesis, electric power generation, and biophysics are included. In addition, a few examples where synchronization effects of the phase-lock type occur in nature are pointed out.

**Carrier Tracking.** In all applications pertaining to coherent telecommunications, it is necessary to reconstruct a carrier reference from a noise-corrupted version of the received signal. At least three methods are available: (I) tracking the instantaneous frequency and phase of the carrier component (if present) in the received signal spectrum; (II) tracking the instantaneous frequency and phase (of a phantom carrier) of a suppressed carrier signal; or (III) use of a hybrid of the first two methods.

Method (I) can be accomplished by designing a narrowband carrier tracking loop of the PLL or APC type. The basic configuration of such a loop is shown in Fig. 1. Here the input signal component \( s(t, \Phi) \) and reference signal \( r(t, \hat{\Phi}) \) are characterized by \( s(t, \Phi) = \sqrt{2}A(t) \sin \Phi(t) \) and \( r(t, \hat{\Phi}) = \sqrt{2}K_1 \cos \hat{\Phi}(t) \), respectively. Here \( \sqrt{2} \) represents a convenient normalizing factor in what follows. The additive function \( n_s(t) \) represents background noise and \( e(t) \) represents a signal acquisition voltage. In general, the amplitude variations \( A(t) \) can be used to characterize any type of digital or analog amplitude modulation (AM), while the phase function \( \Phi(t) \) can be used to characterize any type of digital or analog angle modulation, namely, frequency (FM) or phase (PM) modulation. The phase function \( \hat{\Phi}(t) \) represents the loop estimate (or at least certain of its components) of \( \Phi(t) \).

The PLL of Fig. 1 is essentially a closed-loop, electronic servo-mechanism in which the reference signal will acquire and track the signal component in the received signal \( x(t) = s(t, \Phi) + n_s(t) \). Any phase error, say...
\( \varphi(t) = \hat{\Phi}(t) - \Phi(t) \), between these two signals, which is within the bandwidth of the loop, is converted by the loop into a correction voltage that changes the instantaneous frequency and phase of the reference signal in order to make the loop track the input signal. In fact, in the absence of noise the product \( s(t, \Phi)r(t, \hat{\Phi}) = AK_1 K_m \sin \varphi(t) \), appearing at the output of the multiplier with double frequency terms neglected, represents this voltage.

The implementation of this concept is quite simple (see Fig. 1). It can be accomplished with three basic parts: a voltage control oscillator (VCO), a multiplier or phase detector,* and a loop filter. The loop is so designed that when the average phase error is constant the loop is phase-locked. Should either of the signals change in phase, the loop filter output produces an average error voltage which is proportional in magnitude and direction to the original phase change. When applied to the VCO, this error voltage \( z(t) + e(t) \) changes the frequency and phase in such a way as to retain phase-lock with the input signal. In the simplest carrier tracking application \( A(t) = A \) and \( \Phi(t) = \omega t + \theta(t) \) where \( \omega \) is the carrier radian frequency and \( \theta(t) = \Omega t + \theta_0 \) represents the phase of the carrier to be tracked. Furthermore, \( \hat{\Phi}(t) \) is characterized by \( \hat{\Phi}(t) = \omega_0 t + \hat{\theta}(t) \); here \( \omega_0 \) is the resonant or quiescent radian frequency of the VCO and \( \hat{\theta}(t) \) is the loop estimate of \( \theta(t) \).

Phase-locked loops may someday replace present day envelope-type demodulators used in superheterodyne receivers for AM and FM reception by doing the job more efficiently and economically. PLLs may also replace banks of crystals in multichannel receivers and transmitters with a single crystal-controlled oscillator of high stability. They can provide precision control over motor speeds. In instrumentation they can be used in the implementation of variable time base electronic signal generators. In computers they can be used to synthesize and synchronize multiple clock frequencies from a single source. PLLs can be used as PM or FM modulators. They can be used to detect signal tones, demodulate angle modulation, measure Doppler and range, synchronize signals, track unstable signal sources, help provide automatic gain control (AGC), reconstruct signals, and generate or select precise signals for data transmission. Generally speaking, PLLs are used in the implementation of communication system modems, in telephone signaling and telemetry equipment, in tracking and navigation systems, in satellite and airborne systems in which Doppler effects must be avoided or measured as well as in electronic test equipment; in fact, PLLs appear in most electronic applications that cover the frequency spectrum from very low to ultra-high frequencies.

* The symbol \( \otimes \) will be used to represent phase detectors which perform the basic operation of multiplication.

**Suppressed Carrier Tracking.** Many system applications require for one reason or another that the carrier component present in \( s(t, \Phi) \) be
completely suppressed at the transmitter. Even though the spectrum of the received signal \( x(t) \) does not contain a carrier component it is possible to reconstruct one. For example, suppressed carrier tracking can be accomplished by a variety of techniques, namely, the Costas loop, \( N \)th power loops, data-aided loops, decision-directed loops, delay-locked loops, hybrid loops, etc. In any case the implemented loop is usually narrowband and employs the phase-lock principle.

A Costas loop, which provides for suppressed carrier tracking of the phase-shift keyed signal \( s(t, \Phi) = \sqrt{2}A(t) \sin \Phi(t) \), with \( A(t) = A \), \( \Phi(t) = \omega_0 t + \theta, \theta = 0 \) or \( \pi \), during the time interval \((n - 1)T \leq t \leq nT, n = 1, 2, \ldots \), is illustrated in Fig. 2. Since a careful description of the operational behavior of such a loop is given later, we omit it here.

![Costas Loop Diagram](image)

**Fig. 2.** The Costas Loop for Suppressed Carrier Tracking.

**Coherent Demodulation of Analog Signals.** With appropriate choice of the loop circuit parameters one can extract small angle modulation, phase modulation (PM), or frequency modulation (FM) of low index from the phase detector output of a narrowband PLL, or wideband FM or PM information from the loop filter output, see Fig. 3.

Amplitude modulation can be detected coherently by using a coherent amplitude detector driven from the voltage control oscillator (VCO) output as illustrated in Fig. 4. This circuit also provides amplitude detection for automatic gain control (AGC). Most PLLs incorporate phase detector circuits which are amplitude sensitive. Where close control of loop parameters is important, the signal level requires AGC processing. Examples of detection applications are frequency and phase-shift keying, tracking, IF receivers with coherent AGC, and FM receivers.
Coherent Demodulation of Digital Signals. In the coherent demodulation of digital data, the signal $s(t, \Phi)$ frequently contains a sinusoidal component for carrier tracking purposes. When the digital modulation appears as a subcarrier which has been phase modulated onto the transmitted carrier, then this modulation can be extracted at the phase detector output of Fig. 3. This output is applied to a digital data demodulator (output filter) for purposes of detecting the transmitted digital data stream. When the carrier is completely suppressed, for example, using phase-shift keyed modulation, then the noise-corrupted baseband modulation appears at the output of the lower phase detector (multiplier) of Fig. 2.
To coherently demodulate digital amplitude modulation which retains a carrier component in the transmitted signal, a ninety degree shift of the reference signal generated in Fig. 4 produces a coherent reference signal for coherent amplitude demodulation.

Since the reference signals in Figs. 3 and 4 are produced in the presence of noise, they are not perfect and give rise to the noisy reference problem arising in the theory of signal detection. Noisy reference signals produce deleterious effects on the data detection process when the PLL is operating synchronously and catastrophic effects when the synchronous reference signal fails.

**Symbol (Bit) Synchronization Systems.** In digital data transmission systems the usual first step in the demodulation/detection procedure is to establish synchronization (sync) between the transmitted and received carriers. If subcarriers are used, the second step is to establish subcarrier sync. Of no less importance in the synchronization procedure is the establishment of symbol (bit) sync. Symbol synchronization has to do with determining the instants in time when the modulation may change states. One approach to providing this timing information is to utilize a separate communication channel solely for establishing symbol sync. More efficiently, however, are the techniques whereby this level of timing information can be obtained directly from the received data bearing signal. For this case a broad class of symbol synchronization systems, suggested by maximum a posteriori estimation techniques, can be represented by the diagram of Fig. 5. Specific

![Diagram of Symbol Synchronization System](image)

**Fig. 5.** Model of a Class of Symbol Synchronization Systems.

phase detector characteristics included are the early-late gate type, those which incorporate an absolute value approach, a difference of squares approach or a hybrid of these approaches, and the decision-directed type.

**Frequency Synthesis.** The PLL is an important building block in indirect frequency synthesis of spectrally pure signals. Frequency multiplication and/or division may be performed using a PLL in conjunction with divider elements as shown in Fig. 6. Frequency addition and subtraction
may be effected by translation in Fig. 7. These operations may be combined in many ways to form programmable frequency synthesizers for spectrally pure signal generation and for special signal tracking and processing.

![PLL Frequency Mutiplier/Division](image)

**Fig. 6.** PLL Frequency Mutiplier/Division (Noise Free Operation Illustrated).

![PLL Frequency Translation](image)

**Fig. 7.** PLL Frequency Translation (Noise Free Operation Illustrated).

**Electric Power Generation.** Not only is the theory of SCSs useful in the synthesis and analysis of telecommunication systems, but it is also helpful in the branch of electrical engineering related to electric power generation. The *hunting* phenomenon and *skip-a-pole* phenomenon arising in alternating current machines (e.g., synchronous motors and generators, due to randomly time-varying loads), can be accounted for by a statistical description of the phase error process \( \{ \varphi(t) = \Phi(t) - \hat{\Phi}(t) \} \). If the theory is applied, it also provides certain answers regarding the behavior of electric power generating systems.

**Synchronization in Nature and Biophysics.** Outside the area of electrical engineering, the theory of SCSs is of interest to physiologists and biophysicists, particularly those concerned with the class of *circadian rhythms*. For some time neurophysiologists have been interested in the true significance
of electrical brain rhythms, in particular, the \textit{alpha} and \textit{beta} rhythms. In studying such waves by the electroencephalograph (EEG), only the statistical behavior of this activity appears as a waveform at the surface of the skull. It depends upon the synchronization of radially-oriented electrical dipoles. When this synchronization is disturbed, as during activation of the cerebral cortex, the rhythmic waves on the surface often disappear or become desynchronized. An analogous phenomena occurs in a SCS. During the so-called \textit{petit mal} epileptic seizure, normal electrical activity is suddenly interrupted over the entire head, and large regular impulsive and square-wave type waveforms appear rhythmically at regular intervals synchronized as though they were triggered by a common pacemaker deep in the brain. Experimental studies of the phase-locking mechanism of this form of epileptic discharge have led to some important discoveries as to how the brain mechanisms govern consciousness in normal individuals. The nonlinear theory of SCSs may well suggest a phenomenological or mathematical model upon which a sound theory for treatment may be advanced and developed.

The heart is yet another organ in biological organisms where synchronous activity is highly important. The most common disorder of the heart rhythm is the \textit{extrasystole} or \textit{premature beat}. This is a hesitation which seems to be a skipping or dropping out of a heart beat. An analogous phenomenon occurs in a SCS when it skips a cycle, in alternating current machinery when it skips a pole or in a pendulum when it rotates through $2\pi$ radians. Frequently, a series of premature beats occurs in succession at a rapid rate. This phenomenon is called \textit{paroxysmal tachycardia} and is analogous to the loss of phase synchronization in a SCS where several cycles are slipped in rapid succession before phase-lock is reached (see Fig. 12a). The \textit{electrocardiogram} (EKG) clearly indicates a complex electrochemical as well as biological synchronization phenomenon at work. It appears that such waveforms are due to the simultaneous phase-locking of many random nonlinear oscillators. In other words, the frequencies of these different oscillators are not independent of one another but are synchronized together. Such a phenomenon, i.e., pulling together of the frequencies produced by many nonlinear oscillators, occurs in the parallel operation of electric generators and in a communication network of mutually synchronized oscillators. In electric power generation this coupling effect is responsible for the excellent frequency regulation of commercial power sources, while in a communication network, it can be used to provide synchronization reliability. In molecular spectra a similar nonlinear binding of frequencies probably takes place. No doubt there are a great many other physical situations wherein such a phenomenon occur.

It has recently been observed that weightlessness causes the brain to sporadically disengage some of its normal controls over body activities. This fact is explained by the loss of synchronization of the so-called biological clock, the mechanism believed to control a variety of activities, such as sleep and wakefulness, which are based on a period of 24 hours.
3. Model of a Synchronous Control System (SCS)

In this section we postulate the model of a SCS, and as we shall see, any of the aforementioned applications (among others) are represented as special cases of this model. Next, the operational behavior of SCSs is discussed from a qualitative viewpoint. Such a discussion is given here in order to introduce the reader to a specific set of technical terms and to provide insight into the operational behavior of the nonlinear system while simultaneously avoiding the mathematics needed to characterize and explain the underlying phenomenon.

A SCS (Fig. 8) consists of five major subsystems: (1) the phase detectors, (2) the loop filter, (3) the acquisition aid, (4) the waveform generator to be synchronized, and (5) the output filters. For carrier tracking alone the output filters usually are not required. Frequently, the acquisition aid is not needed in the implementation of a SCS; however, when required, it assists in expediting the system to a rapid and satisfactory mode of operation. Once the signal is acquired, the aid usually is of no further assistance. For certain applications such as in phase-coherent communication receivers, a system is required which automatically adjusts the gain. To do this, an automatic gain control (AGC) system is employed which maintains a constant output signal level and provides linear operation throughout the system. Automatic gain control has many other uses; it keeps the output of automobile radio
receivers at a pleasant listening level even when the received signal level changes drastically as when driving among tall buildings. It also prevents sophisticated radar receivers from being saturated at high signal levels.

Choice of the phase detectors is a basic problem in signal design. Ultimately it is connected with the engineer’s choice (if he has one) of designing the synchronizing (input) signal $s(t, \Phi)$ and the reference signal $r(t, \hat{\Phi})$ produced by the waveform generator to be synchronized. Here $\Phi(t)$ represents a time-dependent phase function of the input signal and $\hat{\Phi}(t)$ represents the system estimate of $\Phi(t)$ in the presence of the input noise $n_i(t)$. The phase function will be used to model the frequency and phase information to be extracted and/or tracked by the SCS. Any periodic signal (and some classes of nonperiodic signals) can carry this information.

4. Operational Behavior and Performance Measures of SCSs

On the one hand, the operation of any SCS is conveniently discussed by considering its two modes of operation, namely, the acquisition mode (achieves the synchronous state) and the synchronous or tracking mode (retains the synchronous state). On the other hand, the performance of any SCS is characterized by certain performance measures which are statistical in nature when noise is present and deterministic when noise is absent. In order to introduce these performance measures, we shall discuss the operational behavior of a SCS from the viewpoint of rotating phasors. This point of view has the distinct advantage of being simple without losing generality in illustrating the operational behavior of a SCS for any application.

Let the real part of the phasors $\exp[i \Phi(t)]$ and $\exp[i \hat{\Phi}(t)]$ be associated with the signals $s(t, \Phi)$ and $r(t, \hat{\Phi})$, respectively. These phasors, which rotate with instantaneous angular velocities $\omega(t) \triangleq \dot{\Phi}(t)$ rad/sec and $\hat{\omega}(t) \triangleq \dot{\hat{\Phi}}(t)$, respectively, are shown in Fig. 9. The phase error $\varphi(t) \triangleq \Phi(t) - \hat{\Phi}(t)$ has

![Phasor Diagram for Describing Operation of a SCS.](image)

**Fig. 9.** Phasor Diagram for Describing Operation of a SCS.
temporal characteristics which are strongly dependent on the application and mode of operation; however, the performance measures associated with the random process $\{\varphi(t)\}$ are essentially independent of the application. We note that when the phase function $\Phi(t)$ is characterized by a random process, the signal phasor $\exp[i\Phi(t)]$ and reference phasor $\exp[i\hat{\Phi}(t)]$ rotate randomly at rates proportional to $\hat{\Phi}(t)$ and $\hat{\Phi}(t)$.

**Acquisition Mode.** Suppose the system enters the acquisition mode at $t = t_0$. Initially the phasors $\exp[i\Phi(t)]$ and $\exp[i\hat{\Phi}(t)]$ rotate at angular velocities $\omega \triangleq \omega(t_0)$ and $\omega_0 = \hat{\omega}(t_0)$ rad/sec, respectively. Thus initially $\exp[i\Phi(t)]$ rotates at rate $\Omega_0 \triangleq \dot{\varphi}(t_0) = \dot{\varphi}_0 = \omega - \omega_0$ rad/sec relative to $\exp[i\hat{\Phi}(t)]$. The parameter $\Omega_0/2\pi$ is called the *initial frequency detuning* in the system.

Figure 10 shows a typical behavior of $\varphi(t)$ and $\varphi(t)$ for a second-order SCS operating in the acquisition mode without an acquisition aid in the absence of noise. From this figure it is clear that the behavior of $\varphi(t)$ and $\varphi(t)$ during acquisition is highly dependent upon the initial state $y_0 \triangleq [\varphi(t_0), \varphi(t_0)] = [\varphi_0, \varphi_0]$. Let the synchronous mode be defined by the conditions $|\varphi(t)| \leq \varepsilon_0$, $|\varphi(t) - 2\pi n| \leq \varepsilon_0$, $n$ any integer, for $t = t_a$. Suppose these conditions are satisfied at $t_a = t_{acq} - t_0$ for the first time. The parameter $t_{acq}$ is called the *signal acquisition time* (often called *pull-in time* or *lock-in*

![Fig. 10. Typical Behavior of $\varphi(t)$ and $\varphi(t)$ for an SCS in the Acquisition Mode (Noise Absent).](image-url)
time) since it is the time required to reach the synchronous mode. When noise is present, the situation is much more complicated and we do not discuss all the details here; however, for this case $t_{eq}$ becomes a random variable. Thus an important performance measure for the acquisition mode is the statistical moments $T_{eq}^n = E(t_{eq}^n)$; $E(\cdot)$ denotes expectation in the statistical sense. Obviously, $T_{eq}^n$ depends upon the initial state of the system. When the initial state is random and characterized by a probability density function (p.d.f.), then $T_{eq}^n$ must be averaged over this density.

Shortly after acquisition, say $t > t_a$ seconds, the "steady-state" mean $\bar{\varphi}/2\pi = (\omega - \bar{\varphi}_V)/2\pi$ represents the mean residual frequency detuning. Here $\bar{\varphi}_V$ is the average radian frequency of the synchronized generator and $(\omega_0 - \bar{\varphi}_V)/2\pi$ is the mean frequency shift of the waveform generator to be synchronized. When $\dot{\varphi}(t) = \omega$ is constant, the largest value of $\Omega_0$, say $|\Omega_0|_{m}$, from which the system can reach the synchronous mode is called the signal acquisition (often called pull-in or lock-in) range; that is, $|\Omega_0|_{m}$ represents the maximum relative angular velocity between phasors $\exp[i\Phi(t)]$ and $\exp[i\dot{\varphi}(t)]$ for which acquisition is possible. Since noise is present, characterization of the probability of acquisition in the interval $[t_a,t]$ is also of interest.

In the acquisition mode, a SCS is highly nonlinear. In fact, the so-called linear theory cannot be used to carry out a particular design nor can it be used to account for the system's operational behavior; that is, the acquisition problem is characteristically nonlinear.

**Synchronous or Tracking Mode.** Once the system is in the synchronous mode, a more or less steady-state condition exists and the system tends to automatically make adjustments so as to maintain $|\varphi(t) - 2\pi n| \leq \epsilon_\varphi$ and $|\dot{\varphi}(t)| \leq \epsilon_\varphi$ with high probability; that is, it tends to maintain the synchronous state. Hence the name Synchronous Control System. Due to the effects of noise $n(t)$ and instabilities in the waveform generator, random fluctuations in $\varphi(t)$ take place (see Fig. 11).

![Fig. 11. Fluctuation of $\varphi(t)$ in the Synchronous Mode.](image)

In the synchronous mode a number of performance measures are required to characterize and explain system behavior. The reason is that the performance measures that characterize this mode of operation are complicated to account for when both the system nonlinearity and noise are taken into consideration. If the system is linearized to simplify the problem, the performance measures which serve to characterize the fundamental behavior usually have no meaning; that is, operation in the synchronous mode is also characteristically nonlinear.
In this mode the time-dependent, conditional transition p.d.f. $P(\varphi, t | y_0, t_0)$ of the absolute phase error $\varphi(t)$ is of interest. Furthermore for some applications, for example, phase-coherent data transmission systems, the reduced phase error $\phi(t) \triangleq [\varphi(t) + \pi] \mod 2\pi + (2n - 1)\pi$ (such as displayed by a phase meter) is of interest. Here $n$ is considered to be any fixed integer. [See Section 12 for a typical sample function of $\varphi(t)$ and $\phi(t)$.] In loose terms, the phase error process $\{\phi(t)\}$ is called the modulo-2$\pi$ process.

![Diagram](image.png)

**Fig. 12.** Cycle Slipping in SCSs: (a) Phase Error $\varphi(t)$; (b) Reduced Phase Error $\phi(t)$.

Thus the time-dependent, conditional transition p.d.f. $p(\phi, t | y_0, t_0, n)$ of the phase error $\phi(t)$, which a phase meter would read, and the steady-state conditional transition p.d.f. $p(\phi | n)$ are of importance. In contrast in two-way phase-coherent radar, sonar, and navigation systems the phase of the reference waveform $r(t, \hat{\varphi})$ can help in providing for the measurement of range; therefore a statistical characterization of the relative phase error $\varphi(t)$ is required in order to specify system performance. The reference $r(t, \hat{\varphi})$ is frequently used to measure Doppler shift. Where $\phi$ is the quantity of interest, some applications may require the mean and variance of $\phi$ to be less than a few degrees while other applications allow the mean and standard deviation of $\phi$ to be as large as 30 or 45 degrees.
With the passage of time, noise causes synchronization failures to occur. Here we define a synchronization (sync) failure as the change of $\varphi$ from $\varphi_0$ to $\varphi_0 + 2\pi$ or $\varphi_0 - 2\pi$; i.e., $\Delta\varphi/2\pi = \pm 1$ cycle (see Fig. 12a). This corresponds to the situation in Fig. 9 where, for any initial state $y_0 = [\varphi_0, \dot{\varphi}_0]$, the phasor $\exp[i\Phi(t)]$ rotates $\pm 2\pi$ radians relative to the phasor $\exp[i\hat{\Phi}(t)]$. In physical terms the generator to be synchronized drops or adds one cycle of oscillation in $r(t, \hat{\Phi})$ relative to $s(t, \hat{\Phi})$; or stated another way, the signal $s(t, \Phi)$ is shifted $\pm 2\pi/\omega$ seconds relative to the reference signal $r(t, \hat{\Phi})$. This phenomenon is called cycle-slipping.

In the presence of noise and oscillator instabilities, cycle slipping leads to frequency and phase diffusion in the generator to be synchronized. Therefore the average cycle slipping rate $\bar{S}$, i.e., the average number of cycles-slipped per second independent of the direction of rotation of $\exp[i\Phi(t)]$ relative to $\exp[i\hat{\Phi}(t)]$, is of interest. Since the time $T_{fp}$ for a sync failure to occur for the first time is a random variable (see Fig. 12a), we are interested in its statistical description by means of its p.d.f. or its moments, i.e., $E(T_{fp})$. The first moment of this time is called the average time to first loss of phase sync or the mean time to sync failure and denoted by $\tau(2\pi|\varphi_0) = E(T_{fp})$. Evaluation of this time is related to the so-called first-passage time problem studied in the theory of Markov processes.

The probability of sync failure $P(t)$ in the time interval $[t_0, t]$ as well as the probability of slipping $n$ cycles $P(N = n)$ in the time interval $[t_0, t]$, are of interest. Since biases are frequently present in a SCS, the average number of clockwise (counterclockwise) rotations per second of the phasor $\exp[i\Phi(t)]$ through $2\pi$ relative to the phasor $\exp[i\hat{\Phi}(t)]$ and the respective probabilities are of interest. We shall refer to the dropping and adding of oscillations in $r(t, \hat{\Phi})$ relative to $s(t, \Phi)$ as cycle-slips to the right and left, respectively. The average number of slips to the right (left) per second is characterized by $N_+(N_-)$. Other parameters which require a statistical characterization by means of the nonlinear theory include the average time, $\Delta T$ between cycle slipping events (sync failures) or the average time the system is in lock and the average time the system remains out of the synchronous state (see Fig. 12a). Finally, a statistical description of the frequency error $\delta$ (together with $\bar{S} \triangleq N_+ + N_-$) serves to characterize the frequency stability of the synchronized generator.
## GLOSSARY OF NOTATION

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Typical Page Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A(t)$</td>
<td>rms signal amplitude at time $t$, volts (V)</td>
<td>73</td>
</tr>
<tr>
<td>$B_i$</td>
<td>equivalent (one-sided) noise bandwidth of input noise process ${n_i(t)}$, Hertz (Hz)</td>
<td>82</td>
</tr>
<tr>
<td>$B_L$</td>
<td>One-sided noise loop bandwidth, Hz</td>
<td>136</td>
</tr>
<tr>
<td>$d(t)$</td>
<td>Doppler signal function on input signal, rad.</td>
<td>79</td>
</tr>
<tr>
<td>$d(s)$</td>
<td>Laplace transform of $d(t)$</td>
<td>135</td>
</tr>
<tr>
<td>$D_e$</td>
<td>Phase error coefficient of diffusion</td>
<td>404</td>
</tr>
<tr>
<td>$e(t)$</td>
<td>Signal acquisition voltage, volts</td>
<td>74</td>
</tr>
<tr>
<td>$E(\ )$</td>
<td>Statistical expectation operator</td>
<td>7</td>
</tr>
<tr>
<td>$F(s)$</td>
<td>Loop filter transfer function</td>
<td>74</td>
</tr>
<tr>
<td>$F_0$</td>
<td>Ratio $\tau_2/\tau_1$ of loop filter time constants</td>
<td>137</td>
</tr>
<tr>
<td>$g(\phi)$</td>
<td>Loop phase detector characteristic or system nonlinearity</td>
<td>116</td>
</tr>
<tr>
<td>$\mathcal{H}$</td>
<td>Filter operator</td>
<td>33</td>
</tr>
<tr>
<td>$i$</td>
<td>$\sqrt{-1}$ or a summing index</td>
<td>41</td>
</tr>
<tr>
<td>$j(x,t)$</td>
<td>Probability current</td>
<td>262</td>
</tr>
<tr>
<td>$J(t)$</td>
<td>Net number of phase jumps $[t_0,t]$</td>
<td>384</td>
</tr>
<tr>
<td>$K$</td>
<td>$K_i K_m K_{VCO}$, Loop gain constant, volts$^{-1}$ — sec$^{-1}$</td>
<td>74</td>
</tr>
<tr>
<td>$K_1$</td>
<td>rms VCO signal output, volts</td>
<td>73</td>
</tr>
</tbody>
</table>
Glossary of Notation

\[ K_v \]
VCO gain constant, rad/sec-volt 74

\[ K_n \]
Intensity coefficient of phase noise process \( N(t, \varphi) \) 82

\[ K_m \]
Phase detector gain, volts/rad 74

\[ L \]
Linear operator, e.g. Laplace transform, etc. 38

\[ L \]
Differential spatial operator 637

\[ M(t) \]
Transmitter phase modulation, volts 79

\[ n_i(t) \]
Input noise waveform, volts 77

\[ N(t, \varphi) \]
Equivalent phase noise waveform, volts 79

\[ N_0 \]
Single-sided noise spectral density, volts^2/Hz or w/Hz 30

\[ N_f \]
Average number of cycles slipped per second to the right 398

\[ N_- \]
Average number of cycles slipped per second to the left 398

\[ p \]
d/dt, Heaviside operator 73

\[ p(x) \]
Probability density function (p.d.f.) or r.v. \( X \) 6

\[ P_c \]
Power at carrier frequency 87

\[ Q(x; t) \]
Restricted p.d.f. 313

\[ r \]
Second-order loop parameter, \( AKF_0 \tau_2 = 4 \xi^2 \) 137

\[ R_x(r) \]
Correlation function of the process \( \{ x(t) \} \) 14

\[ s \]
Complex frequency variable, rad/sec 133

\[ \bar{S} \]
Cycle slipping rate 402

\[ S_x(s) \]
Spectral density of the process \( \{ x(t) \} \) 14

\[ t \]
Time in seconds, sec 10

\[ T_{acq} \]
Signal acquisition time, sec 454

\[ T_p \]
Phase acquisition time, sec 465

\[ T_{acq} \]
Phase acquisition time, sec 378

\[ W_H \]
Two-sided equivalent noise bandwidth of filter \( H(s) \) 33

\[ W_i \]
Two-sided noise bandwidth of input noise \( \{ n_i(t) \}, \text{Hz} \) 154

\[ W_L \]
Two-sided noise bandwidth of loop, Hz 136

\[ W_{L0} \]
Two-sided design point noise bandwidth of loop, Hz 155

\[ \cdot \]
Overdot over variable denotes time derivative 216

\[ ' \]
Prime over variable denotes spatial derivative 638

\[ \dot{x} \]
Denotes operation on \( x(t) \) by \( p = d/dt \), that is, \( px = dx/dt = \dot{x} \) 216

\[ \alpha \]
Parameter used to characterize \( p(\phi) \) 386

\[ \alpha_t \]
Signal amplitude suppression factor 154

\[ \alpha_{t0} \]
Signal amplitude suppression factor at the design point 155

\[ \beta \]
Parameter used to characterize \( p(\phi) \) 386

\[ \gamma \]
\( \Lambda_0/AK \) normalized initial detuning 375

\[ \gamma_m \]
Normalized (to loop gain, \( AK \)) signal acquisition range 455

\[ \gamma_{mh} \]
Normalized (to loop gain, \( AK \)) loop hold-in range 455

\[ \Gamma_p \]
Limiter performance factor 154
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>Linear loop damping factor</td>
<td>139</td>
</tr>
<tr>
<td>$\theta_0$</td>
<td>Initial value of signal phase, rad</td>
<td>144</td>
</tr>
<tr>
<td>$\theta(t)$</td>
<td>Input signal phase modulation, rad</td>
<td>75</td>
</tr>
<tr>
<td>$\theta(t)$</td>
<td>System estimate of $\theta(t)$, rad</td>
<td>74</td>
</tr>
<tr>
<td>$\Theta(t)$</td>
<td>System estimate of $\theta(t)$ when instabilities are present, sec</td>
<td>78</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Lagrange multiplier</td>
<td>47</td>
</tr>
<tr>
<td>$\lambda_0$</td>
<td>$\omega_0 - K_\tau e$, initial frequency detuning, rad</td>
<td>375</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$\alpha_{10}/\alpha_{1}$, limiter suppression factor</td>
<td>156</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>Input signal-to-noise ratio</td>
<td>154</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Signal-to-noise ratio</td>
<td>136</td>
</tr>
<tr>
<td>$\sigma_x^2$</td>
<td>Variance of random variable $x$</td>
<td>14</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>Variance of total loop phase error, rad²</td>
<td>155</td>
</tr>
<tr>
<td>$\sigma_{\phi, c}^2$</td>
<td>Variance of carrier tracking loop phase error, rad²</td>
<td>156</td>
</tr>
<tr>
<td>$\sigma_d^2$</td>
<td>Mean square Doppler tracking error, rad²</td>
<td>135</td>
</tr>
<tr>
<td>$\sigma_M^2$</td>
<td>Variance due to modulation, $M(t)$, rad²</td>
<td>135</td>
</tr>
<tr>
<td>$\sigma_{\Delta \phi}^2$</td>
<td>Variance due to trans receiver instabilities, $\Delta \phi(t)$</td>
<td>135</td>
</tr>
<tr>
<td>$\tau_2$</td>
<td>Second-order loop filter time constant, sec</td>
<td>137</td>
</tr>
<tr>
<td>$\tau_1$</td>
<td>Second-order loop filter time constant, sec</td>
<td>137</td>
</tr>
<tr>
<td>$\phi(t) = \varphi$</td>
<td>Absolute value of loop phase error at time $t$</td>
<td>74</td>
</tr>
<tr>
<td>$\phi(t) = \phi$</td>
<td>Phase error read by phase meter at time $t$</td>
<td>74</td>
</tr>
<tr>
<td>$\varphi_0$</td>
<td>Initial value of phase error</td>
<td>381</td>
</tr>
<tr>
<td>$\varphi_{ss}$</td>
<td>Linear, steady-state phase error in no noise, rad</td>
<td>143</td>
</tr>
<tr>
<td>$\tilde{\varphi}$</td>
<td>Residual frequency detuning</td>
<td>398</td>
</tr>
<tr>
<td>$\Phi(t) = \Phi$</td>
<td>Input signal phase function, rad</td>
<td>73</td>
</tr>
<tr>
<td>$\Phi(t) = \Psi$</td>
<td>Estimate of input signal phase function, rad</td>
<td>73</td>
</tr>
<tr>
<td>$\varphi_{\ell}$</td>
<td>Potential lock points</td>
<td>39</td>
</tr>
<tr>
<td>$\omega = 2\pi f$</td>
<td>Radian frequency variable, rad/sec</td>
<td>42</td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>Average rest frequency of the VCO, rad/sec</td>
<td>73</td>
</tr>
<tr>
<td>$\omega_n$</td>
<td>Natural frequency of linear loop, rad</td>
<td>139</td>
</tr>
<tr>
<td>$\Omega_0$</td>
<td>$\omega - \omega_0$, initial frequency detuning, rad/sec</td>
<td>143</td>
</tr>
<tr>
<td>$\Omega_1$</td>
<td>Doppler phase-rate, rad/sec</td>
<td>143</td>
</tr>
<tr>
<td>$\psi_1(t) = \psi_1$</td>
<td>Transmitter oscillator instabilities, rad</td>
<td>79</td>
</tr>
<tr>
<td>$\psi_2(t) = \psi_2$</td>
<td>Instabilities of the generator to be synchronized, rad</td>
<td>78</td>
</tr>
<tr>
<td>$\Delta \psi$</td>
<td>Transreceiver instabilities $\psi_1 - \psi_2$, rad</td>
<td>79</td>
</tr>
<tr>
<td>$\sigma_{\psi, n}^2$</td>
<td>Variance (modulo-$2\pi$) of the phase error due to $N(t, \varphi)$</td>
<td>393</td>
</tr>
<tr>
<td>$\omega_{m}$</td>
<td>Maximum sweep rate rad/sec</td>
<td>477</td>
</tr>
<tr>
<td>$</td>
<td>\Delta \omega_{m}</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>\lambda_0</td>
<td>_m$</td>
</tr>
<tr>
<td>$P(t)$</td>
<td>Probability of sync failure in time interval $[t_0, t]$</td>
<td>408</td>
</tr>
<tr>
<td>$\tau^n(\varphi, \ell</td>
<td>\varphi_0)$</td>
<td>$n$th moment of the time to first slip a cycle, sec</td>
</tr>
</tbody>
</table>
### Glossary of Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{aoq}^{n}(t</td>
<td>\varphi_0)$</td>
<td>nth moment of the signal acquisition time</td>
</tr>
<tr>
<td>$P_{aoq}(t</td>
<td>\varphi_0)$</td>
<td>Probability of signal acquisition in time interval $[t_0,t]$</td>
</tr>
<tr>
<td>c.f.</td>
<td>characteristic function</td>
<td>6</td>
</tr>
<tr>
<td>p.d.f.</td>
<td>probability density function</td>
<td>5</td>
</tr>
<tr>
<td>r.v.</td>
<td>random variable</td>
<td>5</td>
</tr>
<tr>
<td>BPL</td>
<td>Bandpass Limiter</td>
<td>153</td>
</tr>
<tr>
<td>dB</td>
<td>decibel</td>
<td>50</td>
</tr>
<tr>
<td>dBm</td>
<td>decibels relative to one milliwatt</td>
<td>182</td>
</tr>
<tr>
<td>DPLL</td>
<td>Digital Phase-Locked Loop</td>
<td>112</td>
</tr>
<tr>
<td>FDD</td>
<td>Frequency Difference Detector</td>
<td>497</td>
</tr>
<tr>
<td>FM</td>
<td>Frequency Modulation</td>
<td>67</td>
</tr>
<tr>
<td>FP</td>
<td>Fokker-Planck</td>
<td>386</td>
</tr>
<tr>
<td>IF</td>
<td>Intermediate Frequency</td>
<td>177</td>
</tr>
<tr>
<td>PD</td>
<td>Phase Detector</td>
<td>66</td>
</tr>
<tr>
<td>PLL(s)</td>
<td>Phase-Locked Loop(s)</td>
<td>66</td>
</tr>
<tr>
<td>PM</td>
<td>Phase Modulation</td>
<td>67</td>
</tr>
<tr>
<td>SCS(s)</td>
<td>Synchronous Control System(s)</td>
<td>117</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal-to-Noise Ratio</td>
<td>53</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>Equal by definition</td>
<td>74</td>
</tr>
<tr>
<td>$\approx$</td>
<td>approximately</td>
<td>82</td>
</tr>
<tr>
<td>$(&lt;)$</td>
<td>less (greater) than</td>
<td>82</td>
</tr>
<tr>
<td>$\leq (&lt;)$</td>
<td>less (greater) than or equal to</td>
<td>379</td>
</tr>
<tr>
<td>$\leq (&lt;)$</td>
<td>less (greater) than; however, an approximation was used to obtain the result</td>
<td>455</td>
</tr>
<tr>
<td>-</td>
<td>Overbar denotes statistical expectation operator</td>
<td>5</td>
</tr>
</tbody>
</table>
SYNCHRONIZATION SYSTEMS
in
Communication and Control
PART ONE
1

RANDOM PROCESSES AND THEIR STATISTICAL CHARACTERIZATION

1-1 Introduction

The concepts and fundamentals pertaining to the characterization, filtering, and processing of random processes are of considerable interest to scientists and engineers working in various branches of telecommunication engineering. This interest is quite understandable, for at the present stage of technological development random noise constitutes the chief obstacle to further improvement of engineering devices. This chapter is devoted to the systematic outlining and reviewing of certain analytical tools and mathematical models needed for analysis and synthesis of the communication techniques considered in subsequent chapters. From a mathematical point of view, the topics studied in this chapter involve the concepts of random variables, random processes, and linear filtering of random processes. Zero-memory nonlinear transformations of random processes are also considered.

In the general discussion the term signal will be used in a generic sense with reference to either true signal or noise. Any function of time, \( x(t) \), may be considered to represent a signal and, in general, communication and data signals can be modeled mathematically in terms of one or a combination of two signal types—deterministic (nonrandom) and nondeterministic (random).
A deterministic signal is one that can be specified by an explicit mathematical expression which states how the signal behaves at any instant of time. One way to specify the signal completely is to state its exact functional dependence on the independent variable time.

A nondeterministic signal is one that cannot be specified over future time intervals by an explicit mathematical expression enabling one to determine all future values in advance. A priori statements about future values of such signals can be made only in probabilistic terms. Consequently, the characterization of nondeterministic signals rests on the concepts of random variables and random processes.

1.2 Random Variables

We are given an experiment $\mathcal{E}$ whose outcomes are specified by the universe $\mathcal{I}$, the field $\mathcal{F}$ of subsets of $\mathcal{I}$ called events, and the set of probabilities $\{P\}$ assigned to the events. The experiment $\mathcal{E} \triangleq \{\mathcal{I}, \mathcal{F}, P\}$ deserves some elaboration. First, the set $\mathcal{I}$ of elements of all possible outcomes of the experiment comprises the universe and is usually called the certain event. Second, the field $\mathcal{F}$ consists of certain subsets of $\mathcal{I}$ called events, the sets of practical interest. Third, the number $P(A)$ assigned to every event $A$ in $\mathcal{F}$ is called the probability of $A$ and it must satisfy the following conditions:

1. $P(A) \geq 0$
2. $P(\mathcal{F}) = 1$
3. $P(A \cup B) = P(A) + P(B)$ if the intersection $AB = \emptyset$, and $A, B \in \mathcal{F}$.
4. $P(\ )$ should be countably additive.

To every outcome $\zeta$ of $\mathcal{E}$—that is, every element of $\mathcal{I}$—we assign a number $x(\zeta)$. We have thus defined a function $x$ whose domain is the set $\mathcal{I}$ of all possible outcomes; its range is a set of numbers. This function is called a random variable provided it satisfies the conditions

1. The set of $\zeta$’s $\{x \leq \lambda\}$ is an event in $\mathcal{F}$ for any real number $\lambda$.
2. The probability of the event $\{x = +\infty\}$ and $\{x = -\infty\}$ equals zero—that is, $P\{x = \infty\} = P\{x = -\infty\} = 0$.

For the moment the experiment $\mathcal{E}$ will remain somewhat abstract. Examples of experiments of considerable practical interest might be the measurement of the velocity of electromagnetic energy in space, measurement of Faraday rotation of an electromagnetic signal as it propagates past the Sun, mapping of the features of a planetary surface from the Earth, traffic flow at the interchange in a transportation network, and so on.
For our purposes, a *random variable* \( x \) is characterized by the fact that instead of knowing its precise value, we only know how to obtain various values of \( x \) under certain experimental conditions that are physically well defined. As a result of unknown factors, in a given experiment \( x \) takes a particular value \( x_i \), that we call a realization (or sample value) of the random variable \( x \). Therefore a random variable can also be characterized by a certain statistical ensemble of realizations. By subjecting the realizations of a given random variable to statistical data processing, we can find certain of its statistical characteristics. Thus when we say that a random variable is given, we mean that we know the statistical data that completely characterize it. In most cases of interest here, the random variable will be of the continuous type; that is, \( x \) may assume all possible values in an interval of the real line, say \([a, b]\). In what follows, we set \( a = -\infty \) and \( b = \infty \) without any loss in generality.

The simplest concept used to characterize a random variable is to define its distribution function (occasionally abbreviated d.f.). Specifically, a *probability distribution function* \( F_x(\lambda) \) is defined as the probability that the random variable \( x \) (occasionally abbreviated r.v.) is less than or equal to some quantity, say \( \lambda \). For example, the distribution function (d.f.) of \( x \) is

\[
F_x(\lambda) \triangleq \text{probability of the set } \{\xi | x(\xi) \leq \lambda\} = P[x \leq \lambda] \tag{1-1}
\]

where we write \( P[\cdot] \) for "probability of \([\cdot]\)." The derivative (assumed to exist) of the d.f. is called the *probability density function* (p.d.f.) of \( x \). That is,

\[
p_x(\lambda) = \frac{dF_x(\lambda)}{d\lambda} \tag{1-2}
\]

For brevity, it is sometimes convenient to denote the value of \( \lambda \) by the same letter as the random variable \( x \) itself, so that \( p_x(\lambda) \triangleq p(x) \). We then drop the redundant subscript \( x \) and write \( p(x) \) for the probability density of the r.v. \( x \). This is a one-dimensional probability density function. The probability density \( p(x) \) satisfies the *normalization condition*

\[
\int_{-\infty}^{\infty} p(x) \, dx = 1 \tag{1-3}
\]

and we note that \( p(x) \geq 0 \) for all \( x \in (\infty, \infty) \). A r.v. \( x \) is completely characterized by its p.d.f., through which we can calculate the *average value* of any well-behaved function \( g(x) \). This average value is variously called the average, mean, expectation, or expected value of the function \( g(x) \) and is written \( g(x) \), \( E[g(x)] \), or \( \langle g(x) \rangle \). It is given by
\[ E[g(x)] = \int_{-\infty}^{\infty} g(x)p(x) \, dx \]  

(1-4)

Sometimes the symbol \( \langle \cdot \rangle \) is reserved for time average. We will not make that distinction here. A special case of great practical interest occurs when \( g(x) = x \) and yields the first moment or mean of r.v. \( x \), \( m_1 = \bar{x} \); when \( g(x) = x^n \), one obtains the \( nth \) noncentral moment of the r.v. \( x \).

Next we show how the p.d.f. of the r.v. \( x \) behaves under transformations of the original r.v. \( x \). Let a new r.v. \( y \) be defined by

\[ y \triangleq g(x) \]  

(1-5)

with the inverse of the transformation defined by \( x = h(y) = g^{-1}(y) \). Then the p.d.f. of the r.v. \( y \) is

\[ p(y) = p_x[x = h(y)] \]  

(1-6)

where \( |J| \) is the absolute value of the Jacobian \( J \triangleq dy/dx \). We assume that \( g(x) \) is a one-to-one onto function.

Besides the probability density \( p(x) \), the characteristic function (sometimes abbreviated c.f.)

\[ C_x(u) \triangleq E[\exp(iux)] = \int_{-\infty}^{\infty} p(x) \exp(iux) \, dx \]  

(1-7)

also completely characterizes the r.v. \( x \). In fact, \( p(x) \) is just the Fourier transform of \( C_x(u) \). That is

\[ p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-iux)C_x(u) \, du \]  

(1-8)

The moments

\[ m_n \triangleq E(x^n), \quad n = 1, 2, 3, \ldots \]  

(1-9)

of the r.v. \( x \) can be obtained from the characteristic function by differentiation.

\[ m_n = \frac{1}{i^n} \left. \frac{d^n C_x(u)}{du^n} \right|_{u=0}, \quad n = 1, 2, 3, \ldots \]  

(1-10)

From a knowledge of the moments—the values of the derivatives of the characteristic function at the origin—we can write \( C_x(u) \) as a Maclaurin series
Random Variables

\[ C_x(u) = 1 + \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} m_n \quad (1-11) \]

For several reasons it is sometimes convenient to characterize a r.v. not by its moments but by its semi-invariants or cumulants \( k_n \), defined by

\[ C_x(u) = \exp \left[ \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} k_n \right] \quad \text{or} \quad k_n \triangleq \frac{1}{i^n} \left. \frac{d^n \ln C_x(u)}{du^n} \right|_{u=0} \quad (1-12) \]

From the structure of these formulas it is clear that if we know the first \( n \) moments, we can calculate the first \( n \) semi-invariants, and conversely. The first semi-invariant is the same as the first moment \( m_1 \). The second semi-invariant

\[ k_2 = E[(x - \bar{x})^2] \quad (1-13) \]

is called the variance or dispersion of \( x \), and it will be denoted by one of the several forms

\[ \sigma_x^2 = \text{Var} (x) = E(x^2) - (\bar{x})^2 \quad (1-14) \]

The variance, as we shall see, is a very important characteristic of a r.v., for the smaller the variance, the more exact our knowledge of the random variable. The square root \( \sigma_x \) is called the standard deviation of the r.v. \( x \).

The analogous probability distribution function (sometimes called the cumulative distribution) for \( n \) random variables \( x = (x_1, \ldots, x_n) \) is a function of \( n \) variables \( \lambda = (\lambda_1, \ldots, \lambda_n) \), namely,

\[ F_x(\lambda) = P[x_1 \leq \lambda_1, \ldots, x_n \leq \lambda_n] = P[x \leq \lambda] \quad (1-15) \]

This is called an \( n \)-dimensional d.f. or a joint d.f. of the random vector \( x \). The corresponding joint probability density function for \( F_x(\lambda) \) is denoted by

\[ p_x(\lambda) = \frac{\partial^n F_x(\lambda)}{\partial x_1 \partial x_2 \cdots \partial x_n} \quad (1-16) \]

Dropping subscripts as before, the probability density of \( x \) is \( p(x) \). It follows by definition that the mean value of any well-behaved function \( g(x) = g(x_1, \ldots, x_n) \) can be calculated by \( n \)-fold integration over the probability space, that is,

\[ E[g(x)] = \int \cdots \int g(x) p(x) \, dx \quad (1-17) \]
If we let \( g(x) \equiv 1 \), we obtain the normalization condition

\[
\int_{\text{n-fold}} p(x) \, dx = 1 \tag{1-18}
\]

for \( n \) r.v.'s.

For example, suppose that we have two r.v.'s \( x_1 \) and \( x_2 \) described by the joint probability density \( p(x_1, x_2) \). Then the marginal probability density of the r.v. \( x_1 \) is given by

\[
p(x_1) = \int_{-\infty}^{\infty} p(x_1, x_2) \, dx_2 \tag{1-19}
\]

In some applications, knowing the observed value of the random variable \( x_2 \) helps us to obtain some information about the other r.v. \( x_1 \). In general, our knowledge of \( x_1 \) is still imperfect; hence \( x_1 \) remains undetermined and is now described by another p.d.f.

\[
p(x_1|x_2) = \frac{p(x_1, x_2)}{\int_{-\infty}^{\infty} p(x_1, x_2) \, dx_1} = \frac{p(x_1, x_2)}{p(x_2)} \tag{1-20}
\]

Sometimes the expression \( p(x_1|x_2) \) is referred to as the *a posteriori probability* of \( x_1 \) given \( x_2 \), and at other times the expression \( p(x_1|x_2) \) is called the *conditional probability density* of \( x_1 \) given the value of \( x_2 \). In some cases no information is gained about \( x_1 \) by observing the value of \( x_2 \), which means that

\[
p(x_1|x_2) = p(x_1) \tag{1-21}
\]

and

\[
p(x_1, x_2) = p(x_1)p(x_2) \tag{1-22}
\]

This factoring of the joint density \( p(x_1, x_2) \) implies that the r.v.’s are *statistically independent*. As we shall see, the opposite extreme also arises in practice—that is, the case where knowledge of \( x_2 \) completely determines the other r.v. \( x_1 = f(x_2) \). Then \( x_1 \) and \( x_2 \) are said to be completely correlated and their joint p.d.f. contains a *delta function* \( \delta(x_1 - x_2) \)—that is,

\[
p(x_1, x_2) = \delta(x_1 - x_2)p(x_2) \tag{1-23}
\]

where \( \delta(x) \) is the *delta* function. The delta function is defined by the conditions
Random Variables

\[ \int_{-\infty}^{\infty} \delta(x - x_0) \, dx = 1 \]  
(1-24)

and

\[ \int_{-\infty}^{\infty} \delta(x - x_0)g(x) \, dx = g(x_0) \]  
(1-25)

where \( g(x) \) is any continuous function of \( x \).

The calculation of \( p(x_1|x_2) \) from (1-20) is essentially an exercise in inverse probability and is used in specifying optimum communication systems. The reason for the term inverse is that we are trying to discover the cause \( x_1 \) that has produced the effect \( x_2 \). The following problem is typical of those lending themselves to exact treatment by the methods of inverse probability.

A binary-digital communication system transmits one of two basic symbols \( x_0 \) and \( x_1 \) during the \( n \)th transmission interval \( nT \leq t \leq (n+1)T \). The symbols \( x_0 \) and \( x_1 \) chosen for transmission are assumed to occur with probabilities \( \Pr(x_0) \) and \( \Pr(x_1) \) respectively. If \( x_i \) is transmitted and the observed effect at the receiver is \( y \), then from the product law (sometimes referred to as Bayes rule) for probabilities we have

\[ p(x_1, y) = \Pr(x_1)p(y|x_1) = p(y)p(x_1|y). \]  
(1-26)

or

\[ p(x_1|y) = \frac{\Pr(x_1)p(y|x_1)}{\sum_{i=0}^{1} \Pr(x_i)p(y|x_i)} \]

A similar expression holds if the symbol \( x_0 \) is transmitted. The optimum or ideal receiver may then be defined as something that, when supplied with \( y \) at its input, evaluates the probabilities \( p(x_0|y) \) and \( p(x_1|y) \) at its output. The probability densities \( p(x_i|y) \) and \( p(x_0|y) \) tell us as much as it is possible to know about \( x_1 \) or \( x_0 \) from a knowledge of \( y \). Hence choosing the larger of the two quantities tells us the message that was “most likely” to have been sent.

Equation (1-26) is the equation of inverse probability, \( \Pr(x_0) \) and \( \Pr(x_1) \) are the \textit{a priori} probabilities of \( x_0 \) and \( x_1 \), \( p(x_0|y) \) and \( p(x_1|y) \) are the \textit{a posteriori} probabilities of \( x_0 \) and \( x_1 \), and \( p(y|x_0) \) and \( p(y|x_1) \) are known as the \textit{likelihood functions} of \( x_0 \) and \( x_1 \) respectively. The first two explain themselves, but the third term is purely conventional. It is in this sense, however, that the preceding optimum receiver is known as a \textit{a posteriori} probability computing receiver. The example may be restated so as to consider the transmission of \( N \) symbols.

All that has been said about two r.v.'s can immediately be generalized
to the case of several r.v.'s. The conditional p.d.f. of the r.v.'s \(x_1, \ldots, x_k\) can be written in terms of \(p(x_1, \ldots, x_n)\).

\[
p(x_1, \ldots, x_k | x_{k+1}, \ldots, x_n) = \frac{p(x_1, \ldots, x_n)}{p(x_{k+1}, \ldots, x_n)}
\]

(1-27)

If knowledge of \(x_{k+1}, \ldots, x_n\) does not increase our information about \(x_1, \ldots, x_k\), then the r.v.'s \(x_1, \ldots, x_k\) and \(x_{k+1}, \ldots, x_n\) are said to be statistically independent.

As in the case of two r.v.'s, we introduce characteristics or properties of the variables that describe the degree of statistical dependence between them. The covariance of two r.v.'s \(x_1\) and \(x_2\), denoted by \(\text{Cov} [x_1, x_2]\), is defined by the relation

\[
\text{Cov} [x_1, x_2] \triangleq E[(x_1 - \bar{x})(x_2 - \bar{x})]
\]

\[
= E(x_1 x_2) - E(x_1)E(x_2)
\]

(1-28)

By definition, the covariance of a r.v. with itself is just its variance \(\sigma^2\).

The most concise expression for the mean values of the product of several random variables is written in terms of the \(n\)-dimensional characteristic function

\[
C_x(u_1, \ldots, u_n) = E \left[ \exp \left( i \sum_{k=1}^{n} u_k x_k \right) \right]
\]

(1-29)

where \(\mathbf{x} = (x_1, x_2, \ldots, x_n)\). Using the multidimensional Taylor series expansion, it is easy to show

\[
E(x_1 x_2 \cdots x_n) = \frac{1}{i^n} \left. \frac{\partial^n C_x(u_1, \ldots, u_n)}{\partial u_1 \cdots \partial u_n} \right|_{u_1=\cdots=u_n=0}
\]

(1-30)

To obtain the formula for the semi-invariant of order \(n\), we replace the c.f. by its logarithm. That is,

\[
K[x_1, \ldots, x_n] \triangleq \frac{1}{i^n} \left. \frac{\partial^n \ln C_x(u_1, \ldots, u_n)}{\partial u_1 \cdots \partial u_n} \right|_{u_1=\cdots=u_n=0}
\]

(1-31)

1-3 Random Processes

Next we consider a random process \(X \triangleq \{x(t)\}\), with member (sample) functions \(x(t)\), of a real argument \(t\) (denoting time) that varies over the observation interval, say \([0, T]\). Frequently, \(X\) is called a stochastic process. The random function \(x(t)\) is regarded as given if we are allowed to observe \(x(t)\) in
the time interval $0 \leq t \leq T$. For example, $x(t)$ might be an electromagnetic signal radiated from a deep-space vehicle with modulation existing for $0 \leq t \leq T$.

If we take $n$ fixed times $t_1, \ldots, t_n$ from the interval $0 \leq t \leq T$, where $n$ and $t_1, \ldots, t_n$ are arbitrary, then the values $x(t_1) = x_1, \ldots, x(t_n) = x_n$ constitute a set of r.v.'s. We then consider the random process $\{x(t)\}$ to be specified if the joint p.d.f.'s for each set of the r.v.'s generated above are specified for all $n$, regardless of how large. We denote these p.d.f.'s by the sequence $W_n = [p_1(x_1), p_2(x_1, x_2), \ldots, p_n(x_1, \ldots, x_n)]$, which satisfies all the necessary requirements—that is, non-negativity, the normalization condition, and consistency or $\int_{-\infty}^{\infty} p(x_1, \ldots, x_n) \, dx_n = p(x_1, \ldots, x_{n-1})$. In this respect it should be noted that the ideal state of affairs would be to describe the random process by a single p.d.f. of the largest possible order, if such existed. Since the argument $t$ is continuous, no such p.d.f. exists. However, it is sometimes possible to introduce a probability density functional that defines a "continuous probability density." Such an approach defines a p.d.f. in an appropriate waveform space.

Instead of the sequence of p.d.f.'s, one can characterize $X$ by the sequence of characteristic functions

$$C_x(u_1, \ldots, u_n) \triangleq E \left[ \exp \left( i \sum_{k=1}^{n} u_k x_k \right) \right], \quad n = 1, 2, \ldots \tag{1-32}$$

which uniquely determines the sequence of p.d.f.'s $W_n$. By letting $n$ approach infinity, we have the characteristic functional

$$C_x[u(t)] = E \left\{ \exp \left[ i \int u(t)x(t) \, dt \right] \right\} \tag{1-33}$$

which completely describes the process $X$. The probability density functional $p[x(t)]$ of the process $X$ is uniquely related to the characteristic functional under certain general conditions (see Gel'fand-Vilenkin, Chapters 4 and 5, Ref. 1).

In its general form, a random process cannot be described by a finite number of variables; hence a random process is either characterized by an infinite sequence of functions or by a functional. Analogous to the first moment of a r.v., the generalized mean values

$$m_{r_1, \ldots, r_n}(t_1, \ldots, t_n) \triangleq \frac{1}{i^q} \frac{\partial^q C_x(u_1, \ldots, u_n)}{\partial u_1^{r_1} \cdots \partial u_n^{r_n}} \bigg|_{u_1 = \ldots = u_n = 0}; \quad n = 1, 2, \ldots \tag{1-34}$$

such that $\sum_{j=1}^{n} r_j = q$, are called generalized moment functions and regarded as functions of $t_1, \ldots, t_n$. The most concise expression for moment functions
can be written in terms of the characteristic functions. In fact, using the Taylor series expansion for \( \exp(x) \) in (1-32), we have

\[
C_x(u_1, \ldots, u_n) = \sum_{q=0}^{\infty} \frac{i^q}{q!} E[(u_1 x_1 + \cdots + u_n x_n)^q]
\]  

(1-35)

Expanding the terms in the bracket of the expectation operator by using the multinomial theorem, we can write

\[
C_x(u_1, \ldots, u_n) = \sum_{q=0}^{\infty} \frac{i^q}{q!} \sum_{r_1} \cdots \sum_{r_n} \left( \prod_{i=1}^{n} u_i^r \right) \frac{m_{r_1, \ldots, r_n}(t_1, \ldots, t_n)}{r_1! \cdots r_n!}
\]

such that \( \sum_{j=1}^{n} r_j = q \)

(1-36)

One may also characterize the process \( X \) by generalized multiple semi-invariant functions, which are defined by

\[
k_q(t_1, \ldots, t_n) \triangleq K[x_1^t, x_2^t, \ldots, x_n^t] = \frac{1}{i^q} \frac{\partial^q \ln C_x(u_1, u_2, \ldots, u_n)}{\partial u_1^{r_1} \cdots \partial u_n^{r_n}} \bigg|_{u_1 = \cdots = u_n = 0}
\]

(1-37)

and regarded as functions of the times \( t_1, t_2, \ldots, t_n \). Note that for \( t_1 = t_2 = \cdots = t_n = t \) and \( r_1 = 1, r_2 = 1, \ldots, r_n = 1, k_q(t_1, \ldots, t_n) = k_n \) are the semi-invariants previously defined. If we know the moment functions or the multiple semi-invariants, we can specify the characteristic functions or p.d.f.'s. Using the multidimensional Taylor series expansion, we can write

\[
\ln C_x(u_1, \ldots, u_n) = \ln C_x(0, \ldots, 0) + \\
\sum_{q=1}^{\infty} \left[ \sum_{r_1} \cdots \sum_{r_n} \frac{1}{r_1! \cdots r_n!} \frac{\partial^q \ln C_x}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} \right] \left( \prod_{i=1}^{n} u_i^{r_i} \right) \bigg|_{x_1 = \cdots = x_n = 0}
\]

(1-38)

where \( C_x = C_x(x_1, \ldots, x_n) \). Using the fact that \( \ln C_x(0, \ldots, 0) = 0 \) and the definition of the multiple semi-invariant functions of order \( n \), we have, from (1-37), that

\[
\ln C_x(u_1, \ldots, u_n) = \sum_{q=1}^{\infty} i^q \left[ \sum_{r_1} \cdots \sum_{r_n} k_q(t_1, \ldots, t_n) \prod_{i=1}^{n} u_i^{r_i} \right]
\]

such that \( \sum_{j=1}^{n} r_j = q \)

(1-39)
Since \( x = \exp (\ln x) \), (1-39) becomes

\[
C_x(u_1, \ldots, u_n) = \exp \left[ \sum_{q=1}^{\infty} i^q \left( \sum_{r_1} \cdots \sum_{r_n} \frac{k_{\tau}(t_1, \ldots, t_n)}{r_1! \cdots r_n!} \prod_{i=1}^{n} u_i^{r_i} \right) \right] \quad (1-40)
\]

such that \( \sum_{j=1}^{n} r_j = q \)

In the limit as \( n \) becomes arbitrarily large, we have the characteristic functional (1-33) of \( X \).

The first- and second-moment functions and their interconnecting relationships with the corresponding correlation functions play a special role in later sections; we give the following formulas relating them.

\[
k_1(t_1) = m_1(t_1) \quad (1-41)
\]

\[
k_2(t_1, t_2) = m_2(t_1, t_2) - k_1(t_1)k_1(t_2) \quad (1-42)
\]

Frequently the second-moment function with \( r_1 = r_2 = 1 \), is written as

\[
m_2(t_1, t_2) = E(x_1x_2) = E[x(t_1)x(t_2)] \quad (1-43)
\]

and deserves a special notation

\[
R_x(t_1, t_2) \triangleq m_2(t_1, t_2) = E(x_1x_2) \quad (1-44)
\]

This function is called the correlation function of \( X \).

1-4 Stationary Random Processes and Spectral Densities

A random process \( X \triangleq \{x(t)\} \) is termed stationary if its statistical characteristics are invariant under shifts in the time origin. In terms of this definition of stationarity, the semi-invariant function can be written as

\[
k_n(t_1, \ldots, t_n) = k_n(t_1 + \tau, \ldots, t_n + \tau) \quad (1-45)
\]

and we have set \( r_1 = \cdots = r_n = 1 \), so that \( q = n \). Letting \( t_1 = -\tau \), we find that

\[
k_n(t_1, \ldots, t_n) = k_n'(0, t_2 - t_1, \ldots, t_n - t_1) \quad (1-46)
\]

Thus the semi-invariant functions of a stationary process depend only on time.
differences. In particular, the mean value of a stationary process equal to \( k_i(t) \) must be a constant for all \( t \). That is,

\[
k_i(t) = m_i(t) = m_x
\]  

(1-47)

Similarly, the semi-invariant \( k_x(t_1, t_2) \) (sometimes called the covariance function) of a stationary process is a function of \( \tau = t_2 - t_1 \) for which we introduce the special notation

\[
k'_x(0, t_2 - t_1) \triangleq k_x(\tau) = R_x(\tau) - m_x^2
\]  

(1-48)

for which \( R_x(\tau) \) is referred to as the correlation function of \( X \). It also follows that \( R_x(\tau) = R_x(-\tau) \); that is, the correlation function of a real stationary process is an even function of \( \tau \). If (1-47) and (1-48) hold for arbitrary times, then the process is said to be wide-sense stationary.

Next we introduce the concept of power spectral density \( S_x(\omega) \) of the wide-sense stationary process \( X \). This is defined by

\[
S_x(\omega) \triangleq \int_{-\infty}^{\infty} R_x(\tau) \exp(-i\omega\tau) \, d\tau = \int_{-\infty}^{\infty} E(xx_\tau) \exp(-i\omega\tau) \, d\tau
\]  

(1-49)

Here and subsequently the subscript \( \tau \) on a function like \( x_\tau \) in the presence of a similar function like \( x \) without a subscript is used to denote a shift of the argument of \( x \) by an amount \( \tau \)—that is, \( x(t + \tau) = x_\tau \), while \( x(t) = x \). We also note that \( S_x(\omega) \geq 0 \). For a stationary process, \( S_x(\omega) \) is the inverse Fourier transform of the correlation function.

\[
R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega\tau) S_x(\omega) \, d\omega
\]  

(1-50)

When \( X \) has zero mean we note that

\[
R_x(0) = \sigma_x^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) \, d\omega = k_x(0)
\]  

(1-51)

and that \( S_x(\omega) = S_x(-\omega) \) since \( R_x(\tau) \) is an even function.

In the treatment to follow, a special role is played by a zero-mean process \( X \) for which \( S_x(\omega) = K_x \) for all \( |\omega| < \infty \). In this case

\[
R_x(\tau) = K_x \delta(\tau)
\]  

(1-52)

where \( \delta(\tau) \) is the delta function defined by (1-24) and (1-25). Such a process is usually referred to as a white process, by analogy with white light, which has a constant spectral density for all frequencies.
In the following material we shall make use of the concept of the spectral bandwidth of a zero-mean process $X$ defined by

$$\mathcal{B}_X \triangleq \frac{1}{S_x(0)} \int_0^\infty S_x(\omega) \, d\omega$$ (1-53)

That is, we defined $\mathcal{B}_X$ in such a way that if we approximate $S_x(\omega)$ by a rectangle of height $S_x(0)$ and width $2\mathcal{B}_X$, the area of the rectangle will be the same as the area under the spectrum of $S_x(\omega)$. The correlation time of a zero-mean process is defined similarly by

$$\tau_x \triangleq \frac{1}{R_x(0)} \int_0^\infty |R_x(\tau)| \, d\tau$$ (1-54)

Intuitively speaking, the time $\tau_x$ gives some measure of the time interval over which correlation exists between different samples of the process $X$. When $R_x(\tau) \geq 0$ for all $\tau$, then using the formulas

$$S_x(0) = \int_{-\infty}^{\infty} R_x(\tau) \, d\tau \quad R_x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) \, d\omega$$ (1-55)

we easily arrive at a precise expression for the product $\mathcal{B}_x \tau_x$—namely,

$$\mathcal{B}_x \tau_x = \frac{\pi}{2}$$ (1-56)

The intensity coefficient of $X$, defined by

$$K_x \triangleq \int_{-\infty}^{\infty} R_x(\tau) \, d\tau = \int_{-\infty}^{\infty} k_x(\tau) \, d\tau = S_x(0)$$ (1-57)

plays an important role in the development of the theory of tracking and synchronization that follows.

As an example of the preceding concepts, let $X$ be an exponentially correlated, stationary random process with correlation function

$$R_x(\tau) = \sigma_x^2 \exp(-\beta|\tau|)$$ (1-58)

Then according to (1-49) and (1-58), $X$ has the spectral density

$$S_x(\omega) = \frac{2\sigma_x^2 \beta}{\omega^2 + \beta^2}$$ (1-59)

and the correlation time is $\tau_x = 1/\beta$.

If the correlation function has the form
\[ R_x(\tau) = \sigma_x^2 \exp(-\beta|\tau|) \cos \omega_0 \tau \] (1-60)

then the spectral density equals
\[ S_x(\omega) = 2\beta \sigma_x^2 \left[ \frac{\omega^2 + \omega_0^2 + \beta^2}{(\omega^2 - \omega_0^2 - \beta^2)^2 + 4\beta^2\omega^2} \right] \] (1-61)

When \( \beta \ll \omega_0 \), the spectral density is concentrated in a relatively narrow frequency band \( |\omega - \omega_0| \approx \beta \), and we have the approximations
\[ S_x(\omega) \approx \beta \sigma_x^2 \left( \frac{1}{(\omega - \omega_0)^2 + \beta^2} \right) \]
\[ \tau_x \approx \frac{1}{\beta} \]

Sometimes such a process is said to be narrowband or slowly varying, since for time intervals equal to a large number of periods \( 2\pi/\omega_0 \), the random function \( x(t) \) is close to being sinusoidal oscillations. Table 1-1 exhibits other commonly encountered correlation functions together with their corresponding spectral densities and correlation times.

<table>
<thead>
<tr>
<th>( R_x(\tau) )</th>
<th>( S_x(\omega) )</th>
<th>( \tau_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_x^2 \exp \left( -\frac{\beta^2 \tau^2}{2} \right) )</td>
<td>( \frac{\sqrt{2\pi} \sigma_x^2}{\beta} \exp \left( -\frac{\omega^2}{2\beta^2} \right) )</td>
<td>( \sqrt{\frac{\pi}{2\beta}} )</td>
</tr>
<tr>
<td>( \sigma_x^2 \frac{\beta \exp(-\alpha</td>
<td>\tau</td>
<td>) - \alpha \exp(-\beta</td>
</tr>
<tr>
<td>( \frac{\sigma_x^2}{1 + \beta^2 \tau^2} )</td>
<td>( \frac{\pi \sigma_x^2}{\beta} \exp \left( -\frac{</td>
<td>\omega</td>
</tr>
<tr>
<td>( \sigma_x^2 \sin \beta \tau )</td>
<td>( \frac{\pi \sigma_x^2}{\beta} ) if (</td>
<td>\omega</td>
</tr>
<tr>
<td>( \frac{\sigma_x^2}{\beta} ) if (</td>
<td>\omega</td>
<td>&gt; \beta )</td>
</tr>
</tbody>
</table>

Next we introduce the concept of cross-correlation between two random processes. We define the cross-correlation function between two real random processes \{\( x(t) \)\} and \{\( y(t) \)\} by
\[ R_{xy}(t_1, t_2) \triangleq E[x(t_1)y(t_2)] \] (1-63)

If the processes are stationary and correlated, then \( R_{xy}(t_1, t_2) \) depends only on the time difference \( t_2 - t_1 = \tau \). Thus
\[ R_{xy}(\tau) = R_{xy}(0, t_2 - t_1) \]  

(1-64)

Obviously a cross-correlation function need not be an even function of its argument. By analogy, the Fourier transform of the cross-correlation function gives the cross-spectral density

\[ S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) \exp(-i\omega \tau) d\tau \]  

(1-65)

which will be useful later. Moreover, for real processes, we have the relation

\[ S_{xy}(-\omega) = S_{xy}^*(\omega) \]  

(1-66)

where the asterisk (*) denotes complex conjugate.

1.5 The Gaussian Process

The study of a Gaussian process is essentially the study of a certain class of functions of \( n \) variables. These functions involve a quadratic form of \( n \) variables and the determinant of a matrix \( \Lambda \). It is therefore appropriate that we begin our discussion with the notation to be used. If \( \Lambda \) is a matrix, we shall denote its transpose by a prime—\( \Lambda' \)—its inverse by \( \Lambda^{-1} \). If \( x \) is a row vector, with components, \( x_1, \ldots, x_n \), a primed vector \( x' \) denotes a column vector. We further assume that all entries in our matrices belong to the field of real numbers.

Suppose that all the higher-order semi-invariant functions of the random process \( X \) are zero except \( k_1(t_i) \) and \( k_2(t_1, t_2) \). Then the process is said to be Gaussian or normal. In this case the joint p.d.f. of \( n \) samples of \( X \), say \( x = (x_1, \ldots, x) \), is given in matrix notation by

\[ p(x) = \frac{1}{(2\pi)^{n/2}} \frac{1}{|\Lambda_x|^{1/2}} \exp \left[ -\frac{1}{2} (x - m_x)[\Lambda_x]^{-1}(x - m_x)' \right] \]  

(1-67)

where the function \( m_x \) is composed of the first-order mean functions

\[ m_x \triangleq [m_1(t_1), \ldots, m_1(t_n)] \]  

(1-68)

and \( |\Lambda_x| \) is the determinant of the \( n \times n \) covariance matrix

\[ \Lambda_x = [\lambda_{ij}] \]  

(1-69)

with elements

\[ \lambda_{ij} \triangleq k_2(t_i, t_j) = \text{Cov} [x_i, x_j] \]

\[ = E[(x(t_i) - m(t_i))[x(t_j) - m(t_j)]] \]  

(1-70)
If the process is stationary, the covariance elements \( \lambda_{ij} \) depend only on the time differences \( t_i - t_j \) and the means become constants and independent of time.

If \( X \) has zero mean, then the characteristic functions \( C_x(u) \) and \( C_x(u_1, u_2) \) are given by

\[
C_x(u) = \exp \left( -\frac{\sigma_x^2 u^2}{2} \right)
\]

\[
C_x(u_1, u_2) = \exp \left[ -\frac{\sigma_x^2}{2} (u_1^2 + 2\rho(\tau)u_1 u_2 + u_2^2) \right]
\]

where \( \rho(\tau) = R_x(\tau)/\sigma_x^2 \). In general,

\[
C_x(u) = \exp \left( -\frac{i}{2} u \Lambda_x u' + i u m_x' \right)
\]

where \( u = (u_1, \ldots, u_n) \) and \( x = (x_1, \ldots, x_n) \). This follows from taking the multidimensional Fourier transform of the p.d.f., \( p(x) \).

Some particularly important moment functions are of considerable interest in practice if the Gaussian process is zero mean. For example, the fourth-order moment function \( E(x_1 x_2 x_3 x_4) \) is given by

\[
E(x_1 x_2 x_3 x_4) = E(x_1 x_2)E(x_3 x_4) + E(x_1 x_3)E(x_2 x_4) + E(x_1 x_4)E(x_2 x_3)
\]

This result can be obtained using (1-30). Finally, if a random process \( \{y(t)\} \) is related to the zero-mean, stationary random process \( \{x(t)\} \) through \( \{y(t)\} = \{x^2(t)\} \), then the correlation function of the process \( \{y(t)\} \) is easily shown to be

\[
R_y(\tau) = \sigma_x^4 + 2R_x^2(\tau)
\]

by using (1-73). These results will be useful in several subsequent applications.

### 1-6 The First-Order Markov Process

Consider the random process \( X \) whose sample values \( x(t_1) = x_1, \ldots, x(t_n) = x_n \), where \( t_1 > t_1 > \cdots > t_n \) and the conditional p.d.f. of the value of \( x(t) \) at \( t = t_1 \) [i.e., \( p(x_i | x_2, \ldots, x_n) \)] given by

\[
p(x_i | x_2, \ldots, x_n) = \frac{p(x_1, \ldots, x_n)}{p(x_2, \ldots, x_n)}
\]

The process \( X \) is termed a first-order Markov process if the conditional probabi-
lity (1-75) depends only on the last value $x_2$ and not on the preceding values $x_3 \cdots x_n$; that is, if

$$p(x_1|x_2) = p(x_1|x_2, \ldots, x_n)$$

(1-76)

In Markov process theory, $p(x_1|x_2)$ is frequently referred to as the transition p.d.f. We shall explore the subject of Markov processes and their relationship with telecommunication engineering more deeply in later chapters.

Markov processes are convenient mathematical abstractions of physically realizable random processes. Random processes encountered in communication engineering are not exactly Markovian; however, it is sometimes a good approximation to regard them as such. For our purposes, it allows us to develop many results needed in our study of tracking, synchronization, and coherent demodulation theory.

1-7 Delta-Correlated Processes

If the semi-invariant functions of order $n$ for the random process $X$ are given by

$$k_n(t_1, \ldots, t_n) = K_n(t_1)[\delta(t_1 - t_2) \cdots \delta(t_1 - t_n)], \quad n = 2, 3, \ldots$$

(1-77)

with intensity coefficient $K_n(t)$, the process is said to be delta-correlated. If the process is stationary, the intensity coefficients are constant. By setting $t_1 = t_2 = \cdots = t_n$ in (1-77), we see that a delta-correlated process has infinite semi-invariants. This indicates that delta-correlated processes are not physically realizable. Nevertheless, such processes are frequently useful as approximations to the actual process at work.

If $X$ is zero mean, stationary, and Gaussian, then $k_1(t) = 0$ and

$$k_2(t_2, t_1) = R_x(t_2 - t_1) = K_2 \delta(t_2 - t_1)$$

(1-78)

Since the process is assumed to be Gaussian, all the higher-order intensity coefficients are zero. According to (1-49), the process $\{x(t)\}$ has the constant spectral density

$$S_x(\omega) = K_2$$

(1-79)

Therefore $X$ is sometimes called stationary white Gaussian noise.

The spectral density of many processes encountered in practice can frequently be considered constant over a certain frequency range. For the physi-
cally realizable process $Y \triangleq \{y(t)\}$, there always exists some cutoff frequency, say $\beta$, beyond which for all practical purposes $S_y(\omega)$ is approximately zero. On the other hand, if it is exactly zero, the process is deterministic. This means that the variance of $\{y(t)\}$ is finite. In fact, if

$$S_y(\omega) = \begin{cases} K_2 & |\omega| \leq \beta \\ 0 & |\omega| > \beta \end{cases}$$  \hspace{1cm} (1-80)$$

then from (1-51) we have

$$\sigma_y^2 = R_y(0) = \frac{1}{2\pi} \int_{-\beta}^{\beta} K_2 d\omega = \frac{K_2 \beta}{\pi} \approx \frac{K_2}{\tau_y}$$ \hspace{1cm} (1-81)$$

(see also the fourth entry in Table 1-1). Replacing an actual or physically realizable process by a delta-correlated process is frequently done in communication system analysis. This means that the cutoff frequency is not explicitly taken into account, which is frequently permissible if $\beta$ is much larger than other frequencies seen by the system. On the other hand, the correlation time $\tau_y \approx \pi/\beta$ is often much smaller than all other relevant time constants of the system. If such conditions hold, as is the case in most practical tracking systems, we can make the substitution

$$R_y(t_2 - t_1) \approx \left[ \int_{-\infty}^{\infty} R_y(\tau) d\tau \right] \delta(t_2 - t_1)$$

$$\approx K_2 \delta(t_2 - t_1)$$ \hspace{1cm} (1-82)$$

where the intensity coefficient of the delta function is so chosen that both sides of (1-82) give the same result when integrated with respect to $(t_2 - t_1)$. This is tantamount to the assumption that $Y$ is a white Gaussian process. As we shall see later, it is precisely this fact that allows one to analyze and synthesize tracking systems on the basis of Markov process theory. We shall have occasion to make use of (1-82) several times in later chapters.

---

1-8 The Narrowband Gaussian Process

The concept of a narrowband process is important in a variety of communication engineering problems. In spectral language, a narrowband process is described by the fact that its spectral density is negligibly small everywhere except in the radian frequency band, say $|\omega - \omega_0| \ll \beta/2$. This is always the case in practice, for bandpass filters are always present in any communication receiver.
Suppose that we are given two zero-mean, Gaussian random processes \( \{n_1(t)\} \) and \( \{n_2(t)\} \) from which we form the sample function

\[
n(t) = n_1(t) \cos \omega_0 t - n_2(t) \sin \omega_0 t
\]  

(1-83)

where \( \omega_0 \) is the center radian frequency. To obtain the correlation function of \( \{n(t)\} \), we multiply (1-83) by the similar expression corresponding to time \( t + \tau \) and average the result, obtaining

\[
E(nn_\tau) = E(n_1 n_{1\tau}) \cos \omega_0 t \cos \omega_0 (t + \tau) - E(n_1 n_{2\tau}) \cos \omega_0 t \sin \omega_0 (t + \tau)
\]

\[
- E(n_2 n_{1\tau}) \sin \omega_0 t \cos \omega_0 (t + \tau) + E(n_2 n_{2\tau}) \sin \omega_0 t \sin \omega_0 (t + \tau)
\]

(1-84)

which can be rewritten as

\[
E(nn_\tau) = \frac{1}{2} \left[ E(n_1 n_{1\tau}) \cos \omega_0 \tau - E(n_1 n_{2\tau}) \sin \omega_0 \tau + E(n_2 n_{1\tau}) \sin \omega_0 \tau + E(n_2 n_{2\tau}) \cos \omega_0 \tau \right]
\]

\[
+ \frac{1}{2} \left[ E(n_1 n_{1\tau}) \cos \omega_0 (2t + \tau) - E(n_1 n_{2\tau}) \sin \omega_0 (2t + \tau)
\]

\[
- E(n_2 n_{1\tau}) \sin \omega_0 (2t + \tau) - E(n_2 n_{2\tau}) \cos \omega_0 (2t + \tau) \right]
\]

(1-85)

From (1-85) it is clear that \( \{n(t)\} \) is nonstationary, for \( E(nn_\tau) \) depends on \( t \). In fact, \( \{n(t)\} \) is stationary if and only if

\[
E(n_1 n_{1\tau}) = E(n_2 n_{2\tau}) \quad E(n_1 n_{2\tau}) = -E(n_2 n_{1\tau})
\]

(1-86)

Assuming that these relations hold, then the correlation function of the noise becomes

\[
R_s(\tau) = r(\tau) \cos \omega_0 \tau - q(\tau) \sin \omega_0 \tau
\]

(1-87)

where we have introduced the special notation

\[
r(\tau) = E(n_1 n_{1\tau}) = E(n_2 n_{2\tau})
\]

\[
q(\tau) = E(n_1 n_{2\tau})
\]

(1-88)

with \( g(-\tau) = -g(\tau) \) and \( r(\tau) = r(-\tau) \). Now (1-87) can be written as

\[
R_s(\tau) = \frac{1}{2} \left[ (r + iq) \exp (i\omega_0 \tau) + (r - iq) \exp (-i\omega_0 \tau) \right]
\]

(1-89)

Taking Fourier transforms to obtain the spectral density, we have
\[ S_s(\omega) = \frac{1}{2} [S_s(\omega - \omega_0) + S_s(\omega + \omega_0)] + \frac{i}{2} [S_q(\omega - \omega_0) - S_q(\omega + \omega_0)] \]  
(1-90)

where

\[ S_s(\Omega) \triangleq \int_{-\infty}^{\infty} r(\tau) \exp(-i\Omega \tau) \, d\tau \]

\[ S_q(\Omega) \triangleq \int_{-\infty}^{\infty} q(\tau) \exp(-i\Omega \tau) \, d\tau \]  
(1-91)

It is easy to see that \( S_s(\Omega) \) is even, whereas \( S_q(\Omega) \) is odd. Thus the functions \( S_s(\Omega) \) and \( S_q(\Omega) \) uniquely determine \( S_s(\omega) \), and, similarly, the correlation functions \( r(\tau) \) and \( q(\tau) \) uniquely determine the correlation function \( R_s(\tau) \).

The slowly varying processes \( \{n_1(t)\} \) and \( \{n_2(t)\} \) are related to the original narrowband process \( \{n(t)\} \) through (1-83). However, instead of \( \{n_1(t)\} \) and \( \{n_2(t)\} \), it is sometimes interesting to consider the envelope process \( \{N(t)\} \) and the phase process \( \{\theta(t)\} \) of \( \{n(t)\} \). The envelope and phase processes are defined by

\[ N(t) \triangleq \sqrt{n_1^2(t) + n_2^2(t)} \quad \theta(t) \triangleq \tan^{-1} \left[ \frac{n_2(t)}{n_1(t)} \right] \]  
(1-92)

since the sample function can be written as

\[ n(t) = N(t) \cos \left[ \omega_0 t + \theta(t) \right] \]  
(1-93)

If we assume that \( q(\tau) = 0 \)—that is, \( S_q(\omega) \) is symmetrical about \( \omega_0 \), \( \{n_1(t)\} \) and \( \{n_2(t)\} \) are uncorrelated—then the correlation function (1-87) becomes

\[ R_s(\tau) = r(\tau) \cos \omega_0 \tau \]  
(1-94)

where \( r(\tau) \) is the correlation function of the envelope. That is,

\[ r(\tau) = E(n_1 n_1) = E(n_2 n_2) \]  
(1-95)

When \( q(\tau) = 0 \), the cross-correlation between the values of \( \{n_1(t)\} \) and \( \{n_2(t)\} \) at the same time are zero so that the joint p.d.f. \( p(n_1, n_2) \) is given by

\[ p(n_1, n_2) = \frac{1}{2\pi\sigma^2} \exp \left( -\frac{n_1^2 + n_2^2}{2\sigma^2} \right) \]  
(1-96)

where \( \sigma^2 = r(0) \). Introducing the polar coordinates \( n_1 = N \cos \theta \) and \( n_2 = \)
$N \sin \theta$ and integrating over all possible values of $\theta \in [0, 2\pi]$, we obtain the Rayleigh p.d.f.

$$p(N) = \frac{N}{\sigma^2} \exp \left( -\frac{N^2}{2\sigma^2} \right), \quad N \geq 0$$

$$= 0 \text{ elsewhere}$$

This p.d.f. is of interest in the characterization of the amplitude of the noise voltage present in a communication receiver front end. Similarly, integrating on $N$ gives a uniform p.d.f. for the phase, $\theta$.

1.9 Nonstationary Random Processes with Stationary Increments

Here we consider processes with stationary increments, in particular, that class of processes described by the simplest of stochastic differential equations

$$\frac{dy}{dt} = x(t)$$

where $X = \{x(t)\}$ is stationary. Due to the stationarity of $X$, the increment

$$\Delta y = y(t + t_0) - y(t_0) = \int_{t_0}^{t+t_0} x(\lambda) \, d\lambda$$

(1-99)

during the interval of length $t$ has the same statistical properties regardless of the initial time $t_0$. Consequently, the increment $\Delta y$ is stationary and $Y$ is called a process with stationary increments.

Without loss in generality, we now assume that $t_0$ and $y(t_0)$ both equal zero so that (1-99) becomes

$$y(t) = \int_0^t x(\lambda) \, d\lambda$$

(1-100)

Subtracting the mean $\bar{y}$ from (1-100), squaring, and averaging the square produces

$$\sigma_y^2(t) = \int_0^t \int_0^t [x(\lambda_1)x(\lambda_2) - (\bar{x})^2] \, d\lambda_1 \, d\lambda_2$$

$$= 2 \int_0^t (t - \tau)R_x(\tau) \, d\tau - (\bar{x}t)^2$$

(1-101)
The covariance function of the process \( \{ y(t) \} \) is easily calculated using the identity
\[
2uv = [u^2 + v^2 - (u - v)^2]
\]
with \( u = y(t_1) - \overline{y(t_1)} \) and \( v = y(t_2) - \overline{y(t_2)} \). In fact,
\[
k_2(t_1, t_2) = \frac{1}{2} [\sigma^2_\delta(t_1) + \sigma^2_\delta(t_2) - \text{Var} \ [y(t_1) - y(t_2)]]
\]  
(1-102)

where
\[
\text{Var} \ [y(t_1) - y(t_2)] \triangleq E[(y(t_1) - y(t_2) - \overline{y(t_1)} + \overline{y(t_2)})^2]
\]  
(1-103)

Since the increments of \( \{ y(t) \} \) are stationary and the variance of the increment depends only on the time elapsed, we have
\[
\text{Var} \ [y(t_1) - y(t_2)] = \sigma^2_\delta(t_1 - t_2)
\]  
(1-104)

so that
\[
k_2(t_1, t_2) = \frac{1}{2} [\sigma^2_\delta(t_1) + \sigma^2_\delta(t_2) - \sigma^2_\delta(t_1 - t_2)]
\]  
(1-105)

Let us now consider a concrete example of our result.

Assume that \( X \) is a delta-correlated, zero-mean Gaussian process with correlation function \( R_x(\tau) = K_2 \delta(\tau) \). Then from (1-101) we find
\[
\sigma^2_\delta(t) = 2K_2 t
\]  
(1-106)

Since \( t_0 = 0, y(t_0) = 0 \), the p.d.f. of the increment \( y \) during time \( t \) evolves to
\[
p(y) = \frac{1}{\sqrt{4\pi K_2 t}} \exp \left( -\frac{y^2}{4K_2 t} \right)
\]  
(1-107)

This kind of normal process (frequently called \textit{Brownian motion}) is said to be of the \textit{diffusion} type, for at \( t = 0 \) all the probability is concentrated about the origin and as \( t \) approaches infinity the p.d.f. is everywhere zero but possesses unit area.

Because of the delta-function form of the correlation function \( R_x(\tau) \) and (1-98), increments of the process \( y(t) \) during nonoverlapping time intervals are statistically independent. Therefore \( \{ y(t) \} \) is a first-order Markov process with \textit{transition} p.d.f.
\[
p(y, \tau | y) = \frac{1}{\sqrt{4\pi K_2 \tau}} \exp \left[ -\frac{(y - y)^2}{4K_2 \tau} \right]
\]  
(1-108)
Such a normal process with independent increments is sometimes called a \textit{Wiener process}. As seen from (1-102) and (1-106), the process \{y(t)\} has the covariance function

\[
k_2(t_1, t_2) = K_2 [t_1 + t_2 - |t_1 - t_2|] = 2K_2 \min(t_1, t_2)
\]  
(1-109)

\textbf{1.10 Periodic Nonstationary Processes}

The statistical characteristics of a stationary process are invariant under arbitrary time shifts. A \textit{periodic nonstationary process} is a process whose statistical characteristics are only invariant under shifts of an integer multiple of the period $T$—namely, $nT$. Consequently, the moment functions and semi-invariant functions of a periodic nonstationary process \{x(t)\} depend on the absolute time, as well as the time differences. However, the dependence on the absolute time is periodic.

Assuming that the random function $x(t)$ can be expanded in any time interval of length $T$ in the Fourier series

\[
x(t) = \sum_{n=-\infty}^{\infty} x_n \exp \left( \frac{2\pi i nt}{T} \right)
\]  
(1-110)

where $x_n = (1/T) \int_0^T x(t) \exp (-i2\pi nt/T)$ are the random Fourier coefficients at time $t$. Its mean value is easily written as

\[
m_1(t) = E[x(t)] = \sum_{n=-\infty}^{\infty} M_n \exp \left( \frac{2\pi i nt}{T} \right)
\]  
(1-111)

where $M_n = \bar{x}_n$ and the correlation function is represented by

\[
R_x(t - \frac{\tau}{2}, t + \frac{\tau}{2}) = \sum_{n=-\infty}^{\infty} \alpha_n(\tau) \exp \left( \frac{2\pi i \tau n}{T} \right)
\]  
(1-112)

where, in general, the coefficients $M_n$ and $\alpha_n(\tau)$ are complex.

If we shift \{x(t)\} by an amount $\Delta t \neq T$ to produce \{x(t + \Delta t)\}, then

\[
m_1(t + \Delta t) \neq m_1(t)
\]

\[
m_2(t_1 + \Delta t, t_2 + \Delta t) \neq m_2(t_1, t_2)
\]  
(1-113)

and we obtain, in general, a random process whose statistical characteristics are different from those of the original process at time $t$. The communications
engineer would say that the processes \( \{x(t)\} \) and \( \{x(t + \Delta t)\} \) have different phases. In phase-coherent communication systems, the phase of a periodic process is very important. If the phase is unimportant, the system is said to be noncoherent. Later on in our study of tracking and synchronization theory we will be dealing with periodic processes.

1-11 Nonlinear Transformations on Random Processes

Frequently, in telecommunication engineering, random processes are passed through nonlinear devices. The simplest nonlinear systems are those for which the output function \( y(t) \) at any instant of time is determined by the value of the input function \( x(t) \) at the same instant of time. That is,

\[
y(t) = G[x(t)]
\]  

(1-114)

where \( G(x) \) is a nonlinear function called the transfer characteristic. Such a nonlinear transformation is said to have zero memory. In most cases the random process \( Y \cong \{y(t)\} \) is subjected to an additional transformation by a linear system. The engineer is then faced with the following problem. Given the transfer characteristic and the statistical characteristics of the input process \( X \), find the statistical characteristics of the output process \( Y \). In what follows, we consider a more general problem.

Suppose that we know the \( n \)-dimensional p.d.f. \( p(x_1, \ldots, x_n) \) of the random r.v.'s \( x_1 = x(t_1), \ldots, x_n = x(t_n) \) and suppose that we want to find the p.d.f. \( p(y_1, \ldots, y_n) \) of the r.v.'s \( y_1 = y(t_1), \ldots, y_n = y(t_n) \), where

\[
\begin{align*}
y_1 &= G_1(x_1, \ldots, x_n) \\
y_2 &= G_2(x_1, \ldots, x_n) \\
    & \vdots \\
y_n &= G_m(x_1, \ldots, x_n)
\end{align*}
\]  

(1-115)

and the functions \( G_1, \ldots, G_n \) are piecewise continuous. Assuming that the inverse of \( G_i(x_1, \ldots, x_n), i = 1, 2, \ldots, n \) exists and can be represented by

\[
\begin{align*}
x_1 &= H_1(y_1, \ldots, y_n) \\
    & \vdots \\
x_n &= H_n(y_1, \ldots, y_n)
\end{align*}
\]  

(1-116)

where the \( H_n \)'s are single valued, then the p.d.f. \( p(y_1, \ldots, y_n) \) is given by

\[
p_y(y_1, \ldots, y_n) = |J_n| p_x[H_1(y_1, \ldots, y_n), \ldots, H_n(y_1, \ldots, y_n)]
\]  

(1-117)

where \( J_n \) is the Jacobian of the transformation from the variables \( x_1, \ldots, x_n \) to the random variables \( y_1, \ldots, y_n \). That is,
\[ J_n \triangleq \begin{vmatrix} \frac{\partial H_1}{\partial y_1} & \cdots & \frac{\partial H_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial H_n}{\partial y_1} & \cdots & \frac{\partial H_n}{\partial y_n} \end{vmatrix} \]  

(1-118)

and \(|J_n|\) denotes the absolute value of \(J_n\). If the inverse functions \(H_1, \ldots, H_n\) are not single valued, we have to sum the right-hand side of (1-117) over each of the subregions involved.

We now consider a few special cases of practical interest. Let

\[ y_1 = x_1 \quad y_2 = G_2(x_1, x_2) \]  

(1-119)

where the inverse functions

\[ x_1 = y_1 \quad x_2 = H_2(y_1, y_2) \]  

(1-120)

are single valued. The Jacobian of this transformation is

\[ |J_2| = \left| \frac{\partial H_2}{\partial y_2} \right| \]  

(1-121)

and

\[ p_T(y_1, y_2) = \left| \frac{\partial H_2}{\partial y_2} \right| p_x[y_1, H_2(y_1, y_2)] \]  

(1-122)

The marginal p.d.f. of \(y_2\) becomes

\[ p(y_2) = \int_{-\infty}^{\infty} \left| \frac{\partial H_2}{\partial y_2} \right| p_x[y_1, H_2(y_1, y_2)] dy_1 \]  

(1-123)

so that the p.d.f. for the sum, difference, product, and quotient of two r.v.'s \(x_1\) and \(x_2\) are given respectively by

\[ p(x_1 + x_2) = p(y) = \int_{-\infty}^{\infty} p_x(x_1, y - x_1) \, dx_1 \]

\[ p(x_1 - x_2) = p(y) = \int_{-\infty}^{\infty} p_x(x_1, x_1 - y) \, dx_1 \]

\[ p(x_1 x_2) = p(y) = \int_{-\infty}^{\infty} p_x(x_1, \frac{y}{x_1}) \, dx_1 \left| \frac{1}{x_1} \right| \]  

(1-124)

\[ p\left( \frac{x_2}{x_1} \right) = p(y) = \int_{-\infty}^{\infty} p_x(x_1, y x_1) |x_1| \, dx_1 \]
When \( x_1 \) and \( x_2 \) are statistically independent with p.d.f. \( p(x_1) \), \( p(x_2) \), then we can set \( p_4(x_1, x_2) = p(x_1)p(x_2) \).

As a first example in the use of our results, suppose that we want to find the p.d.f. of the product of two correlated, zero-mean, Gaussian r.v.’s \( x_1 \) and \( x_2 \) whose joint p.d.f. is normal. From (1-67) we can write

\[
p_4(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \exp \left[ -\frac{1}{2(1 - \rho^2)} \left( \frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2} - \frac{2\rho x_1 x_2}{\sigma_1\sigma_2} \right) \right]
\]

(1-125)

Then applying (1-124) with \( y = x_1 x_2 \) gives

\[
p(y) = \frac{1}{\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \exp \left[ \frac{\rho y}{\sigma_1\sigma_2(1 - \rho^2)} \right] K_0 \left[ \frac{\sqrt{y}}{\sigma_1\sigma_2(1 - \rho^2)} \right]
\]

(1-126)

where \( K_0(z) \) is the modified Bessel function of the second kind of order zero. Furthermore, the p.d.f. of the ratio \( y = x_2/x_1 \) of the two zero-mean, normally distributed r.v.’s \( x_1 \) and \( x_2 \) is easily obtained from (1-124) as

\[
p(y) = \frac{\sigma_1\sigma_2\sqrt{1 - \rho^2}}{\pi[\sigma_2^2 - 2\rho\sigma_1\sigma_2 + \sigma_1^2\rho^2]}
\]

(1-127)

which is known as the Cauchy p.d.f.

As a final example, let the nonlinear transformation assume the parabolic transfer characteristic (of interest in the analysis of squaring loops and Costas loops)

\[
y(t) = a_1 x(t) + a_2 x^2(t)
\]

(1-128)

and let the input process \( \{x(t)\} \) be the sum of the noise process \( \{n(t)\} \) and the harmonic oscillation

\[
s(t) = A \cos(\omega_n t + \theta)
\]

(1-129)

in which the amplitude \( A \) and the frequency \( \omega_n \) are fixed and \( \theta \) is a uniformly distributed r.v. Assuming that \( \{n(t)\} \) is a zero-mean, stationary, Gaussian process with correlation function \( R_n(\tau) = \sigma^2 \rho_n(\tau) \), we obtain

\[
E(y) = a_2[E(s^2) + E(n^2)]
\]

(1-130)

if the processes \( \{s(t)\} \) and \( \{n(t)\} \) are independent. For \( R_s(\tau) = E(yy_\tau) \), one easily obtains
\[ E(y) = a_1 \left( \sigma^2 + \frac{A^2}{2} \right) \]

\[ R_y(\tau) = a_1^2 \left[ \frac{A^2 \cos \omega_0 \tau}{2} + \sigma^2 \rho_0(\tau) \right] \]

\[ + a_2^2 \left[ 2\sigma^4 \rho_0^2(\tau) + \sigma^2 A^2 + \sigma^4 + 2\sigma^2 A^2 \rho_0^2(\tau) \cos \omega_0 \tau \right. \]

\[ \left. + \frac{A^4}{4} \left( 1 + \frac{\cos 2\omega_0 \tau}{2} \right) \right] \]  

(1-132)

1.12 Characterization of Thermal Noise

The development of an effective design for a communication system requires the coordination of the physical limitations that must be imposed on the equipment—maximum operating range, available transmission power, antenna gains, bandwidths, nature of the channel disturbances, etc.,—with the remaining choice variables, such as modulation system and detection techniques. For this reason one of the first problems facing the communication engineer is that of characterizing the disturbance present in the communication channel.

A number of types of radio noise must usually be considered in any design although, in general, one type will be the predominant type. In broad categories, the noise can be divided into two types: (1) noise internal to the receiving system and (2) noise external to the receiving antenna. The noise internal to the receiving system is usually the controlling disturbance in systems operating above 300 MHz or so. As a rule, the thermal noise, generated in the circuits of the receiver itself, is determined by the engineering design of the input or first stages in the receiver circuitry. For this type of noise, the instantaneous noise voltage has a Gaussian distribution while the noise envelope is Rayleigh distributed.

The second category (external radio noise,) can be subdivided into various types having their own individual characteristics. Examples of natural sources of additive radio noise are (1) atmospheric, (2) galactic, (3) solar noise from antennas pointing at the Sun, (4) corona, and (5) noise re-radiated from any absorbing media through which desired radio signals pass. Various forms of man-made noise sources are (1) power lines and power supplies, (2) automatic ignition systems, (3) switching transients, and so on.
The best known type of interference that manifests itself and limits the minimum signal which may be detected is the so-called thermal or Johnson noise. It has been fashionable to characterize the available thermal noise present at the receiver front end as a stationary Gaussian random process, say \( \{ n(t) \} \). The power spectral density of such a process is given by

\[
S_n(\omega) = \frac{h\omega}{2[\exp(h\omega/kT^o) - 1]} \quad (1-133)
\]

where \( k \) = Boltzmann's constant

\( h \) = Planck's constant

\( T^o \) = noise-source temperature in degrees Kelvin

At radio frequencies, up to approximately 60 GHz, \( h\omega \ll kT^o \), and to good engineering accuracy, the spectral density is effectively flat with a magnitude given by

\[
S_n(\omega) = \frac{N_0}{2} = \frac{kT^o}{2} \text{ watts/Hz} \quad (1-134)
\]

since, at radio and microwave frequencies, the sensitivity of receivers is limited by thermal noise. At these frequencies the quantum effects can be neglected. In what follows, we primarily restrict our attention to communication systems perturbed solely by additive thermal noise that possesses a constant spectral density of \( N_0 = kT^o \) watts/Hz. We shall refer to \( N_0 \) as the one-sided noise spectral density and assume that the random process \( \{ n(t) \} \) is white and Gaussian.

1-13 Characterization of Shot or Impulse Noise

Another very important noise disturbance affecting the efficiency of a synchronous control system is that due to the superposition of random impulses. Thus the statistical characteristics of random processes of such phenomena are important. These characteristics can be determined from the theory of random points. In this theory one deals with a set of points having random positions of occurrence during a given time interval. If one chooses the time interval \( 0 \leq t \leq T \), it is possible to introduce the generating function (Ref.4)

\[
G_T[v(t)] = E\left\{ \prod_{j=1}^{N} [1 + v(t_j)] \right\} \quad (1-135)
\]

where \( N \) is the number of random points lying in the interval \([0, T]\) and the \( t_j \)'s are the position coordinates. The nature of the function \( v(t) \) is restricted by certain requirements that depend on the p.d.f. of the random points and that
will not be discussed here. It is obvious that different systems of random points are characterized by different generating functionals.

A system of random points is said to be a Poisson system if there are no correlations between the points (Ref.4). The generating functions for such a system of points can be shown to be given by

$$G_T(v) = \exp \left[ \int_0^T f(t) v(t) \, dt \right]$$

(1-136)

where $f(t)$ is the distribution function that characterizes the random points. If one defines $P(k)$ as the probability that exactly $k$ points fall in the interval $[0, T]$, then it can be shown on the basis of (1-136) that

$$P(k) = \frac{1}{k!} \left[ \int_0^T f(t) \, dt \right]^k \exp \left[ -\int_0^T f(t) \, dt \right]$$

(1-137)

which is called a Poisson distribution. This distribution is of considerable interest in the development of the theory of optical detection and cycle slipping in phase-locked loops. In this model $P(k)$ is the probability that the number of electrons occurring in the interval $[0, T]$ equals $k$. In the stationary case, the average density of points (rate of occurrence) is constant—that is, $f(t) = f$,

$$P(k) = \frac{(fT)^k}{k!} \exp (-fT)$$

(1-138)

In the characterization of shot noise, one assumes that a single electron, emitted by the cathode of a vacuum tube, reaches the anode and creates a pulse of current $i(t - t_j)$. The density of the current pulses per unit time is assumed to be Poisson and each pulse can be approximated by a delta function of weight $x_j$. That is,

$$i(t - t_j) = x_j \delta(t - t_j)$$

(1-139)

where $x_j$ is the strength of the pulse. Then the total anode current has the form

$$i_a(t) = \sum_j x_j \delta(t - t_j)$$

(1-140)

where the $x_j$ are independent, identically distributed r.v.'s. Thus the name impulse or shot noise is introduced due to the fact that the electrical charge is discrete. The correlation function of shot noise is given by

$$E[i_a(t) i_a(t + \tau)] = f E[x_j^2] \delta(\tau) + \text{constant}$$

(1-141)
The above characterization will be useful when we study the effects of impulse noise on the performance of phase-locked loops.

1-14 Passage of a Random Process Through a Time-Invariant Linear Filter

Suppose that $\mathcal{H}$ is a time-invariant linear filter with impulse response $h(t)$ and transfer function $H(i\omega)$. Then $h(t)$ and $H(i\omega)$ are Fourier transform pairs.

$$H(i\omega) = \int_{-\infty}^{\infty} h(t) \exp(-i\omega t) \, dt$$

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(i\omega) \exp(i\omega t) \, d\omega$$  \hspace{1cm} (1-142)

where, as usual, $\omega = 2\pi f$. Suppose, also, that $\{x(t)\}$ is a zero-mean, stationary, Gaussian process with covariance $k_x(t, t + \tau) = R_x(\tau)$. Let $S_x(\omega)$ be its power spectral density, where $R_x(\tau)$ and $S_x(\omega)$ are a Fourier transform pair.

We are interested in computing the statistics of $\{x(t)\}$ after it has been passed through a time-invariant linear filter. The output process $Z \triangleq \{z(t)\}$ will be given by the convolution integral

$$z(t) = \int_{-\infty}^{\infty} h(t - \lambda) x(\lambda) \, d\lambda$$  \hspace{1cm} (1-143)

after passage through the filter $\mathcal{H}$. The output process is Gaussian, for the output of any linear filter is Gaussian whenever its input is a Gaussian process. Although formal proof of this fact is mathematically involved, we note that the weighted sum of a set of Gaussian r.v.'s is Gaussian. Thus we need only compute the mean and variance of $Z$. By the use of (1-142) in (1-143) we can write $z(t)$ in the alternate but equivalent form

$$z(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} H(i\omega) x(\lambda) \exp[i\omega(t - \lambda)] \, d\omega$$  \hspace{1cm} (1-144)

Since $Z$ has zero mean, the correlation function of the output is

$$R_z(\tau) = E(z(t)z(t - \tau)) \triangleq \int_{-\infty}^{\infty} R_x(\tau + \rho - v) h(\rho)h(v) \, d\rho \, dv$$  \hspace{1cm} (1-145)

and on using (1-142) we can write

$$R_z(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) H(i\omega)H(-i\omega) \exp(i\omega\tau) \, d\omega$$  \hspace{1cm} (1-146)
Taking Fourier transforms of both sides of (1-146) yields

$$S_x(\omega) = |H(i\omega)|^2 S_x(\omega)$$  \hspace{1cm} (1-147)

which relates the output spectrum $S_x(\omega)$ to the input spectrum $S_x(\omega)$. At $\tau = 0$ we have the result that the variance (power) of the output process is

$$\sigma_z^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(i\omega)|^2 S_x(\omega) \, d\omega$$  \hspace{1cm} (1-148)

Notice, also, that if $R_x(\tau) \geq 0$ and $R_x(\tau) \geq 0$ for all $\tau$, then from (1-54) we have

$$\tau_x = \frac{S_x(0)}{2\sigma_x^2} \quad \tau_z = \frac{S_z(0)}{2\sigma_z^2}$$  \hspace{1cm} (1-149)

so that the ratio of the correlation times becomes

$$\frac{\tau_z}{\tau_x} = \frac{|H(0)|^2}{\sigma_z^2} \frac{\sigma_x^2}{\sigma_z^2}$$  \hspace{1cm} (1-150)

where we have made use of (1-147).

Now consider what happens when white Gaussian noise is passed through a linear filter. The output spectral density is

$$S_z(\omega) = \frac{N_0}{2} |H(i\omega)|^2$$  \hspace{1cm} (1-151)

and the variance (power) of the output is

$$\sigma_z^2 = \frac{N_0}{4\pi} \int_{-\infty}^{\infty} |H(i\omega)|^2 \, d\omega$$  \hspace{1cm} (1-152)

Whenever the integral in (1-152) converges, the variance of the output is reduced from an infinite value to a finite one by the action of the filter. The filter has produced the effect of limiting the noise to a band of frequencies. Thus we define the double-sided equivalent noise bandwidth of $\mathcal{F}$ by

$$W_H \triangleq \frac{(1/2\pi) \int_{-\infty}^{\infty} |H(i\omega)|^2 \, d\omega}{\left[H(i\omega)\right]_{\max}^2}$$  \hspace{1cm} (1-153)

In terms of a single-sided equivalent noise bandwidth $B_H$, we have $B_H = W_H/2$. In these terms, the variance of the output process is
\( \sigma_z^2 = \frac{N_0}{2} W_H |H(i\omega)|_{\max}^2 \)  

(1-154)

Viewed graphically (see Fig. 1-1), \( W_H \) is the spectral width of an ideal band-pass filter whose maximum response is the same as that of \( H(i\omega) \) and whose output variance in the presence of white noise is the same as that of \( H(i\omega) \). Frequently, \( |H(i\omega)|_{\max} = 1 \) so that

\[ \sigma_z^2 = \frac{N_0 W_H}{2} = N_0 B_H \]  

(1-155)

The bandwidth \( B_H \) is usually that carried on the nameplate of a filter.

![Graph of Equivalent Noise Bandwidth](image)

**Fig. 1-1.** Illustrating Equivalent Noise Bandwidth.

1-15 Further Studies

The material presented in this chapter has been aimed at the reader with the background offered in a one- or two-semester course in the theory of probability and random processes. For a more detailed treatment, one can find an excellent exposition in the books by Papoulis (Ref. 2), Cramer (Ref. 3), Thomas (Ref. 4), or the first seven chapters of Davenport and Root (Ref. 5) and Middleton (Ref. 6). Further details relative to the theory of random points, probability density, and characteristic functionals can be found in Refs. 1, 7, 8, and 9.

**Problems**

1-1 (a) Verify the transforms given in Table 1-1.
(b) Justify the correlation times given in Table 1-1.
(c) Compute the spectral bandwidth \( B_x \) for the processes defined in Table 1-1.
References

(d) Evaluate the intensity coefficients for the processes defined in Table 1-1.

1-2 (a) Derive the fourth-order moment function given in (1-73) for the stationary, zero-mean Gaussian process \( x(t) \).
(b) When \( y(t) = x^2(t) \), find the correlation function \( R_y(\tau) \) in terms of the correlation function of the process \( x(t) \).

1-3 Evaluate the correlation function for the Brownian motion process defined in Section 1-9.

1-4 Develop the p.d.f.'s for the sum, difference, product, and quotient of two r.v.'s \( x_1 \) and \( x_2 \). [Hint: See Eq. (1-124).]

1-5 Derive Eq. (1-101).

1-6 Verify Eq. (1-132).

References


2

OPTIMUM LINEAR FILTERING
THEORY

2.1 Introduction

The purpose of this chapter is to present a review of that portion of the theory of optimum linear filtering required for the linear analysis of phase-locked loops, two-way tracking, and Doppler measuring systems (Ref. 1). Of course, optimum linear filtering theory has many other applications—for example, in the development of the theory of linear modulation, and so on. In this chapter we shall regard both the signal and the noise as random processes, and the problem of greatest interest to us will be the problem of filtering—that is, extracting the signal as "cleanly" as possible from the "contaminated" combination of the signal plus noise. It is physically clear that complete separation of the signal from the noise can be achieved only if the properties of the signal and the noise are radically different. In general, as we shall see later, even the optimum filter reproduces the signal with some error.

Frequently it is of interest to interpret the problem of filtering more broadly. For example, it is often not the signal itself that is of interest but its derivative (as in frequency demodulation by means of a phase-locked loop) or perhaps its time integral. Then the corresponding filter must be chosen in such a way that it reproduces, with the least possible error, the derivative or the integral of the useful signal. Another problem of frequent interest is to predict
or extrapolate the signal—that is, predict the future values of the signal based on its past behavior and its statistical properties.

2-2 The Mathematical Model

The linear filtering problem can be stated as follows: Let the random process \( \{x(\lambda)\} \) with sample functions \( x(\lambda), t_0 \leq \lambda \leq t \) be applied to the linear filter \( \mathcal{H} \). This filter performs certain mathematical operations on \( \{x(\lambda)\} \) and produces the desired output process \( \{d(\lambda)\} \) of some random process \( \{d(\lambda)\} \) associated with the random process \( \{x(\lambda)\} \). This situation is depicted in Fig. 2-1. The random process \( \{d(\lambda)\} \) may represent an information-bearing message process and \( \{x(\lambda)\} \) may represent a corresponding noise-corrupted version. For the moment we shall take \( \{d(\lambda)\} \) as some linear function of the useful signal process \( \{s(\lambda)\} \). This can be written in the form

\[
d(\lambda) = \mathcal{L}[\lambda, s(\lambda)]
\]

where \( \mathcal{L} \) is a known linear mathematical operator (e.g., an integrator, delay, etc.).

The optimum linear filtering problem is to choose \( \mathcal{H} \) in such a way that it produces \( \hat{d}(\lambda) \), the estimate of the desired signal \( d(\lambda) \), with the minimum possible mean square error

\[
\sigma_\epsilon^2(\lambda) = E[(d(\lambda) - \hat{d}(\lambda))^2] = E[\epsilon^2(\lambda)]
\]

where the difference

\[
\epsilon(\lambda) = d(\lambda) - \hat{d}(\lambda)
\]

is regarded as the instantaneous error of signal reproduction.
Several simple examples follow. In the simple filtering problem, it is the signal $s(\lambda)$ that must be reproduced—that is, $\mathcal{L} = 1$ and $d(\lambda) = s(\lambda)$. Frequently we are interested in the problem of predicting the value of the signal after a time—\(\tau\); $d(\lambda) = s(\lambda + \tau)$. The prediction can be made by using either the infinite past of the function $s(\lambda)$ or only a part of its past. For the case of linear modulation, $s(\lambda) = m(\lambda)c(\lambda)$, where $c(\lambda)$ is a deterministic carrier and $\{m(\lambda)\}$ represents the modulation or message process. A special but very important case occurs when the signal process $\{s(\lambda)\}$ and the noise process $\{n(\lambda)\}$ are additive and independent. That is,

$$x(\lambda) = s(\lambda) + n(\lambda) \quad (2-4)$$

In general, we do not specify the manner in which the noise process $\{n(\lambda)\}$ is involved.

We assume that we know the correlation functions $R_d(t_1, t_2)$ and $R_s(t_1, t_2)$, the cross-correlation function $R_{ds}(t_1, t_2)$, and the moment functions $m_d(t)$ and $m_s(t)$. We exploit certain statistical properties of the random processes $\{d(\lambda)\}$ and $\{n(\lambda)\}$ and thereby apply the statistical concepts covered in Chapter 1.

In what follows, we shall break our summary of optimum linear filtering into two types. A Type I (noncausal) filter works as follows: The input function $x(\lambda)$ is recorded for a certain time interval, theoretically, for $-\infty \leq \lambda \leq \infty$, and then processed via $\mathcal{H}$. Such a filter resembles a computer in that it records the input data and delivers an output in the form of a curve or a table. For Type II (causal) filters, the processes of filtering and delivering the output data are not separated in time but are performed continuously. Consequently, the output function $\hat{d}(\lambda)$ is influenced only by the input data available at the present and in the past—that is, $-\infty \leq \lambda \leq t$. Usually a Type II filter can be realized by an electrical circuit containing resistors, inductors, and capacitors.

Type II filters have the advantage that they are simple to implement and the output data are delivered in real time. On the other hand, Type I filters have the advantage of making complete use of the input signal. Thus to form $\hat{d}(\lambda)$, the Type I filter uses the values $x(\lambda)$ for all possible values of $\lambda$ (that is, $-\infty \leq \lambda \leq \infty$), whereas Type II filters use data from the semi-infinite interval $-\infty \leq \lambda \leq t$.

In the following material we shall be concerned exclusively with linear filters. For linear time-varying filters, the relation between the input and output can be written as

$$\hat{d}(\lambda) = \int_{t_s}^{t} h(\lambda, u)x(u) \, du \quad (2-5)$$

where $h(\lambda, u)$ is the response of the filter at time $\lambda$ to an impulse applied at $u$. If the system is physically realizable, then
\[ h(\lambda, u) = 0, \quad \lambda < u \] (2-6)

for the output cannot preceed the input. If the system is time-invariant, then \( h(\lambda, u) \) depends only on the difference \( \lambda - u \). In the next section we present the filter equation that specifies the filter impulse response \( h(\lambda, u) \).

2-3 The Optimum Linear Filter

In order to incorporate filtering by Type I and Type II filters into our model simultaneously, we assumed that we have available the data \( \{ x(\lambda), t_0 \leq \lambda \leq t \} \). Our problem is to design a filter that is the optimum linear estimate of \( \{ d(\lambda) \} \) in the sense that it minimizes the mean square error (MSE) \( \sigma_\lambda^2(\lambda) \) defined in (2-2) for each point \( \lambda \in [t_0, t] \). Clearly, if we minimize the mean square error for each point \( \lambda \in [t_0, t] \), the MSE over the interval \( t_0 \leq \lambda \leq t \) will be minimized. Using (2-5) in (2-2), we write

\[ \sigma_\lambda^2(\lambda) = E \left\{ \left[ d(\lambda) - \int_{t_0}^{\lambda} h(\lambda, u)x(u) \, du \right]^2 \right\} \] (2-7)

and seek to find the filter function that produces a minimum \( \sigma_\lambda^2(\lambda) \). To solve this minimization problem, we require \( h(\lambda, u) \) to be a continuous function in both variables over \( t_0 \leq (\lambda, u) \leq t \) and we call that \( h(\lambda, u) \), which minimizes \( \sigma_\lambda^2(\lambda) \), the optimum filter \( h_0(\lambda, u) \). Any other function in the allowed class of filter functions can be written as

\[ h(\lambda, u) = h_0(\lambda, u) + \delta h_\delta(\lambda, u) \quad t_0 \leq (\lambda, u) \leq t \] (2-8)

where \( \delta \) is a real parameter and \( h_\delta(\lambda, u) \) is in the allowed class of filters. On substituting (2-8) into (2-7), taking expectations, and collecting terms, it can be shown that the filter function \( h_0(\lambda, u) \) that minimizes the MSE is given by

\[ R_{d_0}(\lambda, u) = \int_{t_0}^{t} h_0(\lambda, v)R_{x}(u, v) \, dv, \quad t_0 < u < t \] (2-9)

The problem therefore reduces to solving (2-9) for \( h_0(\lambda, u) \), given the functions \( R_{d_0}(\lambda, u) \) and \( R_{x}(u, v) \). In general, (2-9) is difficult to solve; however, for the special case when \( x(\lambda) = s(\lambda) + n(\lambda) \) and the additive noise \( n(\lambda) \) is white and the signal and noise are uncorrelated, then

\[ R_{x}(u, v) = R_{s}(u, v) + \frac{N_0}{2} \delta(u - v) \] (2-10)

\[ R_{d_0}(u, v) = R_{s}(u, v) = R_{n}(u, v) \]
and the optimum linear filter is determined from

\[
\frac{N_0}{2} h_0(\lambda, u) + \int_{t_0}^{t} h_0(\lambda, v) R_x(u, v) \, dv = R_x(\lambda, u)
\]  \hspace{1cm} (2-11)

for \( t_0 < u < t \).

The resulting minimum mean square error (MMSE), say \( \sigma_0^2(\lambda) \), for the optimum linear data processor is given by

\[
\sigma_0^2(\lambda) = R_x(\lambda, \lambda) - \int_{t_0}^{t} h_0(\lambda, u) R_{x_0}(\lambda, u) \, du
\]  \hspace{1cm} (2-12)

and, for white noise, the use of (2-11), with \( \lambda = u \), in (2-12) and the fact that \( R_{x_0}(\lambda, u) = R_{x_0}(\lambda, u) \) gives

\[
\sigma_0^2(\lambda) = \frac{N_0}{2} h_0(\lambda, \lambda)
\]  \hspace{1cm} (2-13)

when \( d(t) = s(t) \). We emphasize that we have not invoked the Gaussian assumption. We also note that under the MMSE criterion additional statistical information about the processes cannot be used. All processes, Gaussian or non-Gaussian, with the same \( R_x(u, v) \) and \( R_{x_0}(u, v) \), lead to the same processor \( h_0(\lambda, u) \) and the same MSE.

2-4 Noncausal Filtering of Stationary Random Processes

Let us establish the filtering and performance formulas for the case of stationary statistics and for the case where we do not require the processor to be physically realizable. We set \( t_0 = -\infty \) and \( t = \infty \) so that (2-9) becomes

\[
R_{x_0}(\lambda - u) = \int_{-\infty}^{\infty} h_0(\lambda - v) R_x(u - v) \, dv
\]  \hspace{1cm} (2-14)

since the filter \( h_0(\lambda, v) \) becomes time-invariant for jointly stationary processes. Letting \( \sigma = \lambda - v \) and \( \tau = \lambda - u \), we easily write

\[
R_{x_0}(\tau) = \int_{-\infty}^{\infty} h_0(\sigma) R_x(\sigma - \tau) \, d\sigma
\]  \hspace{1cm} (2-15)

which is the \textit{Wiener-Hopf} equation. Because this equation is valid for all \( \tau \), we can solve for the optimum filter by Fourier transforming both sides. Thus

\[
H_0(i\omega) = \frac{S_{x_0}(\omega)}{S_x(\omega)}
\]  \hspace{1cm} (2-16)
From (2-12) the MMSE becomes independent of $\lambda$; therefore,

$$\sigma_0^2 = R_d(0) - \int_{-\infty}^{\infty} h_0(\sigma)R_{dx}(\sigma) \, d\sigma$$

(2-17)

which, using (2-15), reduces to

$$\sigma_0^2 = R_d(0) - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_0(\sigma)h_0(\mu)R_x(\mu - \sigma) \, d\mu \, d\sigma$$

(2-18)

Since

$$R_d(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_d(\omega) \, d\omega$$

(2-19)

and by relating $R_x(\mu - \sigma)$ to $S_x(\omega)$ through its Fourier transform it follows that

$$\sigma_0^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} [S_d(\omega) - |H_d(i\omega)|^2S_x(\omega)] \, d\omega$$

(2-20)

Then, using (2-16) in (2-20), we obtain

$$\sigma_0^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{S_d(\omega)S_x(\omega) - S_{dx}(\omega)S_{dx}(-\omega)}{S_x(\omega)} \right] \, d\omega$$

(2-21)

where we have used the fact that $S_x(\omega)$ is an even function.

Consider now the special case where $d(t) = s(t)$ and the signal and noise are zero-mean uncorrelated processes and $x(t) = s(t) + n(t)$. Then

$$H_d(i\omega) = \frac{S_d(\omega)}{S_d(\omega) + S_n(\omega)}$$

(2-22)

and (2-21) reduces to

$$\sigma_0^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{S_d(\omega)S_n(\omega)}{S_d(\omega) + S_n(\omega)} \right] \, d\omega$$

(2-23)

If the desired signal $d(\lambda)$ is related to $s(\lambda)$ through the linear integral operator

$$d(\lambda) = \int_{-\infty}^{\infty} l(\lambda')s(\lambda - \lambda') \, d\lambda'$$

(2-24)

then
\[ S_{x\Delta}(\omega) = L(i\omega)S_x(\omega) \quad S_\delta(\omega) = |L(i\omega)|^2S_x(\omega) \]  
\[ (2-25) \]

where we have written

\[ L(i\omega) \triangleq \int_{-\infty}^{\infty} l(\lambda) \exp(-i\omega\lambda) \, d\lambda \]  
\[ (2-26) \]

When we use (2-16) and (2-24), the optimum filter function becomes

\[ H_0(i\omega) = \frac{L(i\omega)S_{x\Delta}(\omega)}{S_x(\omega)} \]  
\[ (2-27) \]

and the MMSE of (2-21) reduces to

\[ \sigma_0^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |L(i\omega)|^2 \left[ \frac{S_x(\omega)S_x(\omega) - |S_{x\Delta}(\omega)|^2}{S_x(\omega)} \right] \, d\omega \]
\[ (2-28) \]

If the signal and noise are uncorrelated and added to produce \( x(t) \), then

\[ H_0(i\omega) = \frac{L(i\omega)S_x(\omega)}{S_x(\omega)} \]  
\[ (2-29) \]

and from (2-28) we have the filter performance formula

\[ \sigma_0^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |L(i\omega)|^2 \left[ \frac{S_x(\omega)S_x(\omega)}{S_x(\omega) + S_x(\omega)} \right] \, d\omega \]  
\[ (2-30) \]

As seen from (2-29), the output of a differentiation filter [i.e., \( L(i\omega) = i\omega \)] is obtained by first reproducing the signal with the least possible error, then differentiating the result. Transfer functions for various linear operations are given in Table 2-1.

### Table 2-1. Transfer Functions \( L(i\omega) \) Corresponding to the Linear Operator \( \mathcal{L} \)

<table>
<thead>
<tr>
<th>( \mathcal{L} )</th>
<th>( d(t) )</th>
<th>( L(i\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identity operator</td>
<td>( s(t) )</td>
<td>1</td>
</tr>
<tr>
<td>Time shift operator</td>
<td>( s(t + \tau) )</td>
<td>( \exp(i\omega t) )</td>
</tr>
<tr>
<td>Differentiation operator</td>
<td>( \frac{ds(t)}{dt} )</td>
<td>( i\omega )</td>
</tr>
<tr>
<td>Integral operator</td>
<td>( \int s(\lambda) , d\lambda )</td>
<td>( \frac{1}{i\omega} )</td>
</tr>
</tbody>
</table>

\( \omega \neq 0 \)
2-5 Causal Filtering of Stationary Processes and the Wiener-Hopf Equation

Here we assume that the initial time \( t_0 = -\infty \) and that the observed data \( x(\lambda) \) is a sample function of a stationary random process, \( -\infty \leq \lambda \leq t \). We also assume that the desired signal and the received signal are jointly stationary and we set \( \lambda = t \). Then from (2-9) we write

\[
R_{d\lambda}(t - u) = \int_{-\infty}^{t} \frac{1}{h_0(t, v)R_x(u - v)} \, dv, \quad -\infty \leq u \leq t \tag{2-31}
\]

Because the processes are stationary, we might expect that the filter \( h_0(t, v) \) be time-invariant. A simple change of variables indicates this fact. Thus we write

\[
R_{d\lambda}(t - u) = \int_{-\infty}^{t} h_0(t - u)R_x(u - v) \, dv \tag{2-32}
\]

Making the substitutions \( \sigma = t - v \) and \( \tau = t - u \), we find that

\[
R_{d\lambda}(\tau) = \int_{0}^{\infty} h_0(\sigma)R_x(\sigma - \tau) \, d\sigma \tag{2-33}
\]

which is the Wiener-Hopf equation derived and solved by Wiener (Ref. 2). If we can find a solution to (2-33), we have a solution to (2-31).

Various methods can be used to solve the Wiener-Hopf equation; however, most are tedious and do not give further insight into the filtering problem. Consequently, we state the solution to (2-33) in the frequency domain and then elaborate on its use. For processes \( \{s(\lambda)\} \) and \( \{n(\lambda)\} \) possessing factorizable rational spectral densities, the optimum filter is given in the frequency domain by (Refs. 3, 9, 19)

\[
H_0(i\omega) = \frac{1}{G^+(\omega)} \left[ \frac{S_{d\lambda}(\omega)}{G^-(\omega)} \right],
\]

where \( S_{d\lambda}(\omega) \) is the cross-spectral density of the processes \( \{d(t)\} \) and \( \{x(t)\} \) and \( G^+ \) and \( G^- \) are certain special factors of the spectral density \( S_x(\omega) \). That is

\[
S_x(\omega) = G^+(\omega)G^-(\omega)
\]

where \( G^+(\omega) \) contains all poles and zeros in the lower half of the \( i\omega \) vs. \( \sigma \) plane and \( G^-(\omega) \) contains all poles and zeros in the upper half of the \( i\omega \) vs. \( \sigma \) plane. The plus (+) superscript implies taking those terms in the partial fraction expansion of \( S_{d\lambda}(\omega)/G^-(\omega) \) that have poles and zeros in the lower half of the \( i\omega \) vs. \( \sigma \) plane. Note that zeros can appear on the real axis. When \( \{d(t)\} \) and \( \{x(t)\} \)
are uncorrelated and \( \{d(t)\} \) is related to \( \{s(t)\} \) through the linear operator \( L(\imath \omega) \), we note that \( S_{ds}(\omega) = L(\imath \omega)S_s(\omega) \).

By using (2-32), the filter performance formula (MMSE) given by (2-12) with \( \lambda = t, t_0 = -\infty \), can be written as

\[
\sigma_0^2 = R_d(0) - \int_{-\infty}^{t} h_0(t - u)R_{dx}(t - u) \, du \\
= R_d(0) - \int_{-\infty}^{t} \int_{-\infty}^{t} h_0(t - u)h_0(t - v)R_x(u - v) \, du \, dv \tag{2-36}
\]

since the processes \( \{x(t)\} \) and \( \{d(t)\} \) are stationary. Introducing the change of variables \( \eta = t - u, \zeta = t - v \) in (2-36), we can write

\[
\sigma_0^2 = R_d(0) - \int_{0}^{\infty} \int_{0}^{\infty} h_0(\eta)h_0(\zeta)R_x(\eta - \zeta) \, d\zeta \, d\eta \tag{2-37}
\]

If we write the correlation functions \( R_d(0) \) and \( R_x(\eta - \zeta) \) in terms of their spectral densities, (2-37) becomes

\[
\sigma_0^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ S_d(\omega) - |H_0(\imath \omega)|^2 S_s(\omega) \right] d\omega \tag{2-38}
\]

when one makes use of the fact that \( h_0(\lambda) = 0 \) for \( \lambda < 0 \). Another useful expression for the MMSE that a causal filter produces can be written when \( d(\lambda) \) and \( s(\lambda) \) are related by (2-24). Writing an expression for the error \( e(\lambda) \) in the frequency domain (assuming the transforms exist) produces

\[
\tilde{e}(\omega) = [L(\imath \omega) - H_0(\imath \omega)]\tilde{s}(\omega) - H_0(\imath \omega)\tilde{n}(\omega) \tag{2-39}
\]

where the tilde \((\sim)\) denotes Fourier transform. Thus

\[
\sigma_0^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} E[\tilde{e}(\omega)\tilde{e}(-\omega)] \, d\omega \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ |L(\imath \omega) - H_0(\imath \omega)|^2 S_s(\omega) + |H_0(\imath \omega)|^2 S_s(\omega) \right] d\omega \tag{2-40}
\]

if the signal and noise processes are uncorrelated.

If the noise is white with \( d(t) = s(t) \), and the signal and noise processes are uncorrelated, the solution to the Wiener-Hopf equation reduces to (see Appendix I)

\[
H_0(\imath \omega) = 1 - \frac{\sqrt{N_0/2}}{[N_0/2 + S_s(\omega)]} \tag{2-41}
\]
with a corresponding MMSE given by (Appendix I)

\[ \sigma_0^2 = \frac{N_0}{2} \int_{-\infty}^{\infty} \ln \left[ 1 + \frac{2S_s(\omega)}{N_0} \right] \frac{d\omega}{2\pi} \]  

(2-42)

Equations (2-41) and (2-42) are known as the Yovits-Jackson formulas. Equation (2-42) is rather remarkable in that the MMSE is determined by the spectral density of the signal; there is no need to evaluate \( H_o(i\omega) \) directly. Frequently, in practice, it is of interest to assess the performance of broad classes of filters rather than evaluate the actual filter transfer function. Simplified expressions like (2-42) are therefore very beneficial to have in applications, for they allow us to explore the MMSE behavior directly.

We now illustrate the mechanics of using causal filtering theory by a few examples.

**Example 1.** As an example of the use of (2-41) we assume that

\[ S_s(\omega) = \frac{A}{\omega^2 + a^2} \]  

(2-43)

so that the denominator in (2-41) becomes

\[ \left[ \frac{N_0}{2} \left( \frac{\omega^2 + b^2}{\omega^2 + a^2} \right) \right]_+ = \sqrt{\frac{N_0}{2}} \frac{i\omega + b}{i\omega + a} \]  

(2-44)

where \( b = \sqrt{a^2 + 2A/N_0} \). Therefore the transfer function of the optimum filter is given by

\[ H_o(i\omega) = \frac{(b - a)}{i\omega + b} \]  

(2-45)

As we shall see later, this example arises in phase demodulation of the process \( s(t) \) in white Gaussian noise by means of a linear PLL.

**Example 2.** Consider the case where

\[ S_s(\omega) = \frac{A}{\omega^2(\omega^2 + a^2)} \]  

(2-46)

then the denominator in (2-41) becomes

\[ \left\{ \frac{N_0}{2} \left[ 1 + \frac{2A/N_0}{\omega^2(\omega^2 + a^2)} \right] \right\}_+ = \sqrt{\frac{N_0}{2}} \left[ \frac{c + (\sqrt{a^2 + 2b})i\omega - \omega^2}{i\omega(i\omega + a)} \right] \]  

(2-47)

so that the optimum filter is specified by
\[ H_0(i\omega) = \frac{c + (\sqrt{a^2 + 2c - a})i\omega}{c + (\sqrt{a^2 + 2c})i\omega - \omega^2} \]  
(2-48)

where \( c = \sqrt{2A/N_0} \). This example arises when one attempts to recover the spectrum \( \omega^2 S_x(\omega) \) by the process of frequency demodulation using a PLL. Other uses of the theory will arise when we attempt to optimize the performance of a PLL in its linear region of operation.

2-6 Optimum Filtering of Polynomial-Type Signals

Later on in the treatment of the linear phase-locked loop theory, we will have occasion to perform filtering of signals possessing a polynomial-type behavior—for example, \( d(t) = a_0 + a_1 t + a_2 t^2 + \cdots \) where the coefficients \( a_0, a_1, a_2 \cdots \) are independent r.v.'s. One can perform the minimization using the method of Lagrange multipliers to show that for white noise the optimum causal filter function is given by

\[ H_0(i\omega) = 1 - \frac{\sqrt{N_0/2}}{[N_0/2 + \Lambda^2 E[D_f(\omega)D_f(-\omega)]]} \]  
(2-49)

where \( D_f(\omega) \) is the Fourier transform of \( d(t) \) and \( \Lambda \) is a Lagrange multiplier, a design parameter. In essence, the theory suggests the replacement of \( S_x(\omega) \) in (2-41) by \( E[\Lambda^2 D_f(\omega)D_f(-\omega)] \). We now illustrate the use of (2-49) by two examples.

**Example 1.** Later in our study of the linear PLL theory we will be concerned with the problem of tracking a random phase offset. For this application \( d(t) = \theta_0 \), where \( \theta_0 \) is a uniformly distributed r.v. on the interval \((-\pi, \pi)\). Therefore the denominator in (2-49) becomes

\[
\left[ \frac{N_0}{2} \left( \frac{\omega^2 + b^2}{\omega^2} \right) \right] = \lim_{\epsilon \to 0} \left[ \frac{N_0}{2} \left( \frac{b - i\omega}{\epsilon - i\omega} \right) \right] 
= \sqrt{\frac{N_0}{2}} \left( \frac{b + i\omega}{i\omega} \right) 
\]  
(2-50)

where \( b^2 = 2\pi^2 \Lambda^2 / 3N_0 \). Thus the optimum filter becomes

\[ H_0(i\omega) = \frac{a}{a + i\omega} \]  
(2-51)

**Example 2.** When one is concerned with the problem of tracking a frequency offset with a random phase, then \( d(t) = \theta_0 + \Omega_0 t \) characterizes the polynomial.
When $\theta_0$ is uniformly distributed as in the preceding example, the denominator in (2-49) becomes

$$\left[ \frac{N_0}{2} \left( 1 + \frac{b}{\omega^2} + \frac{c^2}{\omega^4} \right) \right] = \sqrt{\frac{N_0}{2}} \left[ \frac{c + (b + 2c)i\omega - \omega_i^2}{\omega_i^2} \right]$$

(2-52)

where $b = 2\pi^2 A^2 / 3N_0$ and $c^2 = 2\Omega_0^2 A^2 / N_0$. The optimum filter is then given by

$$H_0(i\omega) = \frac{\sqrt{(b + 2c)i\omega + c}}{c + \sqrt{(b + 2c)i\omega - \omega_i^2}}$$

(2-53)

We shall have occasion to use this result in Chapter 4.

2-7 Applications of the Filter Performance Formulas for White Noise

As an example of the use of the Yovits-Jackson formula (2-42) and (2-30), we presume we have available two classes of stationary processes with spectral densities denoted by $S_i(\omega; k)$ and $S_s(\omega; k)$, $(k = 1, 2, \ldots, \infty)$. Class one is taken to be of the “maximally flat” (Butterworth) form; that is,

$$S_i(\omega; k) = \frac{K_i(k)}{1 + (\omega/a)^{2k}}, \quad k = 1, 2, \ldots$$

(2-54)

where $K_i(k)$ is a constant that is chosen so that the time series, which it represents, has variance

$$\sigma_i^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_i(\omega; k) d\omega = P$$

(2-55)

For the class of “maximally flat” spectra, $K_i(k)$ is given by

$$K_i(k) = \frac{\pi}{a} \left( \text{sinc} \frac{\pi}{2k} \right) P$$

(2-56)

and we have adopted the notation that $\text{sinc} x \triangleq \sin x / x$. This process is both physically reasonable, mathematically convenient, and the integer $k$ is a measure of the rate of spectrum cutoff; for example, $k = 1$ corresponds to a dropoff of 6 dB per octave, $k = 2$ corresponds to 12 dB per octave, and so on. Furthermore, $a/2\pi$ may be considered to be the half-power frequency of $\{s(t)\}$. If $k = 1$, then $S_i(\omega; 1)$ is the spectral density occurring at the output of an RC circuit whose input is white noise. For $k = \infty$, we have
\[ S_1(\omega; \infty) = \begin{cases} \frac{\pi P}{a} & |\omega| \leq a \\ 0 & |\omega| > a \end{cases} \quad (2-57) \]

which is the impulse power response of an ideal low-pass filter of bandwidth \( a/2\pi \) Hz.

Class two processes are taken to be stationary, “asymptotically Gaussian” processes. The spectral density is given by

\[ S_2(\omega; k) = \frac{K_2(k)}{[1 + (\omega/a\sqrt{k})^2]^k}, \quad k = 1, 2, \ldots \quad (2-58) \]

and \( K_2(k) \) is adjusted such that

\[ \sigma_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_2(\omega; k) \, d\omega = P \quad (2-59) \]

Direct substitution of (2-58) into this expression yields the value

\[ K_2(k) = \frac{4\pi P}{a\sqrt{k} B\left(\frac{1}{2}, k - \frac{1}{2}\right)} \quad (2-60) \]

where \( B(u, v) \) is the well-known beta function. If \( k = 1 \), \( S_1(\omega; 1) = S_2(\omega; 1) \); while as \( k \) approaches infinity in (2-58) we have

\[ S_2(\omega; k) = K_2(k) \left[ 1 + \left( \frac{\omega}{a\sqrt{k}} \right)^2 \right]^{-k} \]

\[ = K_2(k) \exp \left[ -k \left( \frac{(\omega/a)^2}{k} - \frac{(\omega/a)^4}{2k^2} + \frac{(\omega/a)^6}{3k^3} - \cdots \right) \right] \quad (2-61) \]

or

\[ \lim_{k \to \infty} S_2(\omega; k) = \frac{2\sqrt{\pi} P}{a} \exp \left[ -\left( \frac{\omega}{a} \right)^2 \right] \quad (2-62) \]

which is the Gaussian spectrum. This process is rather interesting from the physical standpoint in that it may be generated by passing white noise through \( k \) isolated cascaded RC networks. Note that the two random processes possess radically different frequency components as \( k \) becomes large. For \( k = \infty \), the parameter \( a/2\pi \) may be considered to be that frequency at which the spectrum has decayed to \( e^{-1} \) times the value at \( \omega = 0 \). These two classes of random processes are sufficiently general in that they include a broad class of signaling spectra encountered in communication engineering.
For noncausal filtering of the Butterworth spectra in additive white noise, the MMSE is easily found from (2-30), with \( L(\omega) = 1 \), and (2-54) to be

\[
\sigma_i^2(k) = P \left[ 1 + \frac{2PK_c(k)}{N_0} \right]^{1/2k-1}
\] (2-63)

Defining the signal-to-noise ratio (SNR) \( \rho \) as the ratio of the mean square value of the signal power \( P \) to the MMSE, we have

\[
\rho_i \triangleq \frac{P}{\sigma_i^2(k)} = \left[ 1 + 2\pi R \text{sinc} \left( \frac{\pi}{2k} \right) \right]^{1/2k}
\] (2-64)

where \( R \triangleq P/aN_0 \). For the "maximally flat" case, \( k = \infty \), and we have

\[
\rho_i = 1 + 2\pi R
\] (2-65)

which for large values of the parameter \( R \), (2-63) becomes

\[
\rho_i \approx \left[ 2\pi R \text{sinc} \left( \frac{\pi}{2k} \right) \right]^{1/2k}
\] (2-66)

The MMSE for causal filters operating on signal class one it is easily shown, using (2-42) and (2-54), that

\[
\rho_{II} = \frac{\pi R \text{sinc} (\pi/2k)}{k[1 + 2\pi R \text{sinc} (\pi/2k)]^{1/2k} - 1}
\] (2-67)

which for large \( R \) is asymptotic to

\[
\rho_{II} \approx \frac{\pi R \text{sinc} (\pi/2k)}{k[2\pi R \text{sinc} (\pi/2k)]^{1/2k} - 1}
\] (2-68)

Of special interest is the case where \( k = \infty \). It can be shown, using (2-67), that

\[
\rho_{II} = \frac{2\pi R}{\ln (1 + 2\pi R)}
\] (2-69)

Comparison of (2-66) with (2-68) shows that, for large \( R \) and small \( k \), noncausal filters have a SNR of approximately 2\( k \) times the SNR of causal filters. As \( k \) approaches infinity, (2-68) and (2-72) show that the performance of noncausal filtering becomes inferior to causal filtering by a factor of \( \ln (2\pi R) \). For \( k = 1 \) and large \( R \), noncausal filters outperform causal filters by a factor of approximately 3 dB.
Similarly, for noncausal filtering of the "asymptotically Gaussian" processes, we find from (2-30) and (2-62) that

$$\rho_1 \triangleq \frac{P}{\sigma_1^2(k)} = \frac{\sqrt{\pi}}{2} \left[ \int_0^{\infty} \frac{\exp(-x^2)}{1 + 4\sqrt{\pi} R \exp(-x^2)} \right]^{-1}$$  \hspace{1cm} (2-70)

while $\rho_{II}$ can be deduced from (2-42) and (2-62), giving

Fig. 2-2. Filter Performance Characteristics.
\[
\rho_{II} = \frac{2\pi R}{\int_0^\infty \ln [1 + 4\sqrt{\pi R} \exp (-x^2)] \, dx} \quad (2-71)
\]

Plotted in Figs. 2-2 and 2-3 is the SNR \( \rho \) vs. \( R \). These curves represent the performance of noncausal and causal filters. They are also applicable to the problem of determining the performance of coherent communication systems employing linear (amplitude) modulation.

![Graph](image)

**Fig. 2-3. Filter Performance Characteristics.**
2-8 Filtering to Maximize the Output Signal-to-Noise Ratio

The Wiener optimum filter provides that linear estimate of a random signal in noise which reduces the mean-squared error between filter output and desired output to its minimum value. There always seems to be confusion as to whether the Wiener filter also maximizes the output signal-to-noise ratio (SNR). Not that any filter which maximizes output SNR is a Wiener filter (to the contrary, it will be only when properly normalized) but the converse (that a Wiener filter maximizes output SNR) is true, and this section provides a simple proof.

Let \( \{s(t)\} \) and \( \{n(t)\} \) be two independent stationary random processes. Suppose that the sample function \( x(t) = s(t) + n(t) \) is applied to the linear filter \( H(i\omega) \). The filter output we shall denote by \( y(t) \). Let us now suppose that there is some zero-mean, unit-variance stationary process \( \{z(t)\} \), possibly scaled down, and possibly not even present at all, which we desire to see in the output process \( \{y(t)\} \). Everything else in the output is noise, as far as we are concerned. That is, we wish to express \( y(t) \) as

\[
y(t) = \mu z(t) + N(t)
\]

(2-72)

where \( N(t) \) and \( z(t) \) are uncorrelated. It is then clear that

\[
\mu = E[y(t)z(t)]
\]

(2-73)

when \( N(t) \) has zero mean.

It is worthwhile to point out that the output of the filter is often expressed by \( y(t) = z(t) + \epsilon(t) \) in which \( z(t) \) and the error \( \epsilon(t) \) are not necessarily independent. From (2-72) the variance of the output is

\[
\sigma_y^2 = \mu^2 + \sigma_N^2
\]

(2-74)

and it follows that the ratio of powers in the signal and noise components of the output is

\[
\rho_y = \frac{\mu^2}{\sigma_N^2} = \frac{1}{\left(\frac{\sigma_y}{\mu}\right)^2 - 1}
\]

(2-75)

By including a gain \( K \) in \( H(i\omega) \), the level of \( y(t) \) is changed, and this affects the error \( \epsilon(t) \) but not the output SNR, \( \rho_y \). By choice of \( K \), the level of \( \mu z(t) \) can be made anything desired; hence, for convenience, we shall always choose \( K \) so that

\[
y(t) = z(t) + N(t)
\]

(2-76)
which says that \( \mu = 1 \). It now appears as if \( N(t) \) would play the same role as \( \epsilon(t) \) in any analysis based on minimizing \( \sigma^2_z \). But such is clearly not the case, for \( \epsilon(t) \) and \( z(t) \) are not always uncorrelated, whereas \( N(t) \) and \( z(t) \) are. We have chosen, purely for convenience, to make \( \epsilon(t) = N(t) \), which places a constraint on the gain of the filter.

To maximize \( \rho \), we minimize \( \sigma^2\mu^2 \) or, equivalently, we minimize \( \sigma^2_z \) under the constraint \( \mu = 1 \). Define

\[
\Sigma \triangleq \sigma^2_z + \lambda(\mu - 1)
\]

(2-77)

where \( \lambda \) is a constant (the Lagrange multiplier). The values of \( \sigma^2_z \) and \( \mu \) are given by

\[
\begin{align*}
\sigma^2_z &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_x(u - v)h(u)h(v) \, du \, dv \\
\mu &= \int_{-\infty}^{\infty} R_{zx}(u)h(u) \, du
\end{align*}
\]

(2-78)

where \( h(\tau) \) is the unit-impulse response of the filter \( H(i\omega) \). Set \( h(u) = h_0(u) + \eta g(u) \), where \( h_0(u) \) is the (gain-constrained) filter that maximizes the output SNR and \( g(\tau) \) is any function which is zero prior to \( \tau = 0 \). Then

\[
\frac{\partial \Sigma}{\partial \eta}|_{\eta=0} = \int_{-\infty}^{\infty} \left[ 2 \int_{-\infty}^{\infty} R_x(u - v)h_0(v) \, dv + \lambda R_{zx}(u) \right] g(u) \, du
\]

(2-79)

and since \( R_x(\tau) \) is a positive-definite function (in the nonsingular case)

\[
\frac{\partial^2 \Sigma}{\partial \eta^2}|_{\eta=0} = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(u)R_x(u - v)g(v) \, du \, dv > 0
\]

(2-80)

For \( h_0(v) \) to be optimum, it is necessary that the first derivative be zero for all \( g(u) \); this is possible only when

\[
\int_{-\infty}^{\infty} R_x(u - v)h_0(v) \, dv = -\frac{\lambda}{2} R_{zx}(u)
\]

(2-81)

The positive second derivative of \( \Sigma \) ensures that the preceding equation is both necessary and sufficient for \( \Sigma \) to be minimized. Notice that (2-81) is the Wiener-Hopf equation when \( \lambda = -2 \). We evaluate \( \lambda \) by multiplying (2-81) by \( h_0(\tau) \) and integrating. Thus, using this procedure and (2-78)

\[
\sigma^2_{sz} = -\frac{1}{2} \lambda \mu = -\frac{1}{2} \lambda
\]

(2-82)
Therefore the filter \( h'_0(\tau) = h_0(\tau)/\sigma^2_\nu \) is the Wiener optimum filter. Since gain constants do not affect the output SNR, we conclude that the linear filter which maximizes output SNR is equivalent to a Wiener optimum filter followed by an arbitrary gain.

When the input signal \( s(t) \) is a deterministic time function, the output signal-to-noise ratio at a specified time \( T \) is defined as

\[
\rho_0 \triangleq \frac{z^2(T)}{\sigma^2_\nu} \tag{2-83}
\]

We consider the case where \( s(t) \) exists only in the interval \( 0 \leq t \leq T \). If we assume that the input noise is white Gaussian noise, then

\[
\rho_0 = \frac{2 \left| \int_0^\infty s(T - \lambda)h(\lambda) \, d\lambda \right|^2}{N_0 \int_0^\infty \hat{h}(\lambda) \, d\lambda} \tag{2-84}
\]

Using Schwartz's inequality, (2-84) can be written as

\[
\rho_0 \leq \frac{2 \left[ \int_0^\infty |s(T - \lambda)|^2 \, d\lambda \right] \left[ \int_0^\infty |\hat{h}(\lambda)| \, d\lambda \right]}{N_0 \int_0^\infty \hat{h}(\lambda) \, d\lambda} \tag{2-85}
\]

The equality in this equation holds when

\[ h(\lambda) = h_0(\lambda) = s(T - \lambda), \quad 0 \leq \lambda \leq T \]
\[ = 0, \quad \lambda > T \tag{2-86} \]

Such a filter is known as a matched filter. In fact, the maximum output signal-to-noise ratio can be determined from (2-86) and (2-84) to be

\[
\rho_0 = \frac{2E}{N_0} \tag{2-87}
\]

where \( E \) is the energy contained in the signal \( s(t) \). It is also known that the matched filter operation is the same operation as the operation of cross-correlation. That is,

\[
\int_0^T x(t)s(t) \, dt = \int_0^T x(\lambda)h_0(T - \lambda) \, d\lambda \tag{2-88}
\]
The integral on the left is a cross-correlation operation, whereas the operation on the right is a matched filter operation; \( h_0(\lambda) \) is defined in (2-86).

2-9 Further Studies

In the early 1940s Wiener (Ref. 2) and Kolmogorov (Ref. 3) first discussed problems of least-squares estimation for random processes, but by entirely different methods. Kolmogorov (Ref. 3) studied only discrete-time problems using a method suggested by Wold (Ref. 4). Although the original papers of Kolmogorov and Wold are quite readable, a more accessible and very readable reference is the monograph by Whittle (Ref. 5).

Wiener, on the other hand, took an almost completely nonprobabilistic approach (Refs. 2, 6). He mainly studied continuous-time problems and reduced them to the problem of solving the Wiener-Hopf equation. Although Wiener undertook this work in response to an engineering problem, his solution was beyond the reach of his engineering colleagues, and the work soon came to be labeled the "Yellow Peril" among engineers. In 1950 Bode and Shannon (Ref. 7) published a derivation of Wiener's work that was intended to make it more accessible to engineers.

The results in Refs. 3 to 7 were all obtained for stationary processes with infinite or semi-infinite observation intervals. The paper by Zadeh and Ragazzini (Ref. 8) was the first significant attempt to extend the theory. Over the past two decades various extensions and generalizations have been obtained, and many of these have been documented in textbooks—for example, Davenport and Root (Ref. 9), Laning and Battin (Ref. 10), Doob (Ref. 11), Pugachev (Ref. 12), Lee (Ref. 13), Yaglom (Ref. 14), Whittle (Ref. 5), Deutch (Ref. 15), Liebelt (Ref. 16), Balakrishnan (Ref. 17), Bryson and Ho (Ref. 18), and Van Trees (Ref. 19). The original work due to Yovits and Jackson can be found in Ref. 20.

Gradually the importance of methods that treat the estimation and filtering problem in the time domain, and thereby do not lose its dynamical features, was recognized. An important step in this direction was the filtering and prediction theory (1960–61) of Kalman and Bucy (Refs. 21, 22). In their model the signal \( s(t) \) is a Gaussian random vector that evolves according to linear stochastic differential equations driven by white noise. (A concise reference to the subject of stochastic differential equations is given in Ref. 23.) At each time \( t \) a white noise-corrupted linear observation of \( s(t) \) is made. It turns out that not only \( s(t) \) but also its conditional mean \( \hat{s}(t) \) is Gaussian and evolves according to linear stochastic differential equations [\( \hat{s}(t) \) equals the mean-square optimal estimate for \( s(t) \) given the observations up to time \( t \)]. The conditional covariance matrices evolve according to Riccati-type ordinary differential equations.
The success of the Kalman-Bucy linear model was followed by the work of several authors on mean-square optimal filtering for systems that evolve according to nonlinear stochastic differential equations. In the nonlinear case one still wishes to calculate the conditional mean $\hat{s}(t)$, given the observed data, but in order to do so the conditional distribution of $s(t)$ is needed. This evolves according to a stochastic differential equation in a function space, first derived formally by Stratonovich and Kushner and later treated rigorously. (See Refs. 4, 5, 6, 7, and 8.) This equation is generally intractible for actual computation; however, methods have been proposed for finding the first few moments of the conditional distribution (Refs. 9, 30). A simple derivation of the Kalman filter has been given by Kailath (Ref. 31).

Finally, an alternate approach to least-squares filtering theory is given by Yao in Refs. 32 and 33.
APPENDIX I
DEVELOPMENT OF THE YOVITS-JACKSON FORMULAS

From (2-34) and (2-35) we have, for $d(\lambda) = s(\lambda)$,

$$H_0(i\omega) = \frac{1}{[S_x(\omega)]^+} \left\{ \frac{S_x(\omega)}{[S_x(\omega)]^+} \right\}^+ \quad (I-1)$$

where

$$S_x(\omega) = S_0(\omega) + \frac{N_0}{2} = G^+(\omega)G^-(\omega) \quad (I-2)$$

Now (A-1) can be rewritten as

$$H_0(i\omega) = \frac{1}{G^+(\omega)} \left[ \frac{S_x(\omega) + N_0/2}{G^-(\omega)} - \frac{N_0/2}{G^-(\omega)} \right]^+ \quad (I-3)$$

and since the first term in the bracket is just $G^+(\omega)$, we have
\[ H_0(i\omega) = 1 - \frac{\sqrt{N_0/2}}{G^+(\omega)} \left\{ \frac{1}{G^-(\omega)} \right\}_{+} \]

\[ = 1 - \frac{\sqrt{N_0/2}}{G^+(\omega)} \left\{ \frac{1}{\left[ \frac{S_x(\omega)}{\sqrt{N_0/2}} \right]^-} \right\}_{+} \quad (I-4) \]

and we have taken \( \sqrt{N_0/2} \) out of the brace and put the remaining \( \sqrt{N_0/2} \) inside the bracket \([\quad]\)\(^-\). Let \( S_x(\omega) \) assume the rational spectrum expansion

\[ S_x(\omega) = \frac{N(\omega^2)}{D(\omega^2)} \quad (I-5) \]

where the denominator is a polynomial in \( \omega^2 \) whose order is at least one higher than the numerator polynomial. Thus

\[ \frac{S_x(\omega)}{N_0/2} = \frac{D(\omega^2) + (2/N_0)N(\omega^2)}{D(\omega^2)} = \prod_{k=1}^{N} \frac{\omega^2 + \alpha_k^2}{\omega^2 + \beta_k^2} \quad (I-6) \]

and

\[ \frac{1}{[S_x(\omega)/N_0/2]^+] = \prod_{k=1}^{N} \frac{\beta_k - i\omega}{\alpha_k - i\omega} = \prod_{k=1}^{N} \left( 1 - \frac{\beta_k - \alpha_k}{\alpha_k - i\omega} \right) \quad (I-7) \]

Substituting (A-7) into (A-4) and taking those terms in the partial fraction expansion that have poles in the lower half-plane yields (2-41),

\[ H_0(i\omega) = 1 - \frac{1}{\left[ 2S_x(\omega)/N_0 \right]_+} \quad (I-8) \]

To arrive at (2-42) we note from (2-13) that

\[ \sigma_0^2 = \frac{N_0}{2} h_0(0) = \frac{N_0}{4\pi} \int_{-\infty}^{\infty} H_0(i\omega) \, d\omega \quad (I-9) \]

which reduces to

\[ \sigma_0^2 = \frac{N_0}{4\pi} \int_{-\infty}^{\infty} \left[ 1 - \frac{1}{[2S_x(\omega)/N_0]_+} \right] \, d\omega \quad (I-10) \]

and from (A-6) we write (A-10) as
\[ \sigma_0^2 = \frac{N_0}{4\pi} \int_{-\infty}^{\infty} \left[ 1 - \prod_{k=1}^{N} \frac{(i\omega + \beta_k)}{(i\omega + \alpha_k)} \right] d\omega \]

\[ = \frac{N_0}{4\pi} \int_{-\infty}^{\infty} \left[ 1 - \prod_{k=1}^{N} \left( 1 + \frac{\beta_k - \alpha_k}{i\omega + \alpha_k} \right) \right] d\omega \]  

(I-11)

Expanding the product in (A-11), we write

\[ \sigma_0^2 = \frac{N_0}{4\pi} \int_{-\infty}^{\infty} \sum_{k=1}^{N} \frac{\alpha_k - \beta_k}{i\omega + \alpha_k} d\omega \]

\[ - \int_{-\infty}^{\infty} \left[ \sum_{k=1}^{N} \sum_{j=1}^{N} \frac{\gamma_{kj}}{(i\omega + \alpha_k)(i\omega + \alpha_j)} + \sum \sum \sum + \cdots \right] d\omega \]  

(I-12)

The integral of the first term is just \(2\pi\) times the sum of the residues, whereas the second is zero, for the integral is analytic in the upper half of the \(i\omega\) vs. \(\sigma\) plane. That is, the integral from \(-\infty\) to \(\infty\) equals the integral around a semicircle with infinite radius. In particular, these remaining terms vanish at least as rapidly as \(|\omega|^2\) for large \(|\omega|\). Therefore the integral around the semicircle is zero, which implies that

\[ \sigma_0^2 = \frac{N_0}{2} \sum_{k=1}^{N} (\beta_k - \alpha_k) \]  

(I-12)

Finally, from (A-6), we note that

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \ln \left( 1 + \frac{2S_\sigma(\omega)}{N_0} \right) d\omega = \sum_{k=1}^{N} \int_{-\infty}^{\infty} \ln \left( \frac{\omega^2 + \alpha_k^2}{\omega^2 + \beta_k^2} \right) d\omega \]

\[ = \sum_{k=1}^{N} (\beta_k - \alpha_k) \]  

(I-13)

It is then clear from (A-12) and (A-13) that

\[ \sigma_0^2 = \frac{N_0}{4\pi} \int_{-\infty}^{\infty} \ln \left[ 1 + \frac{2S_\sigma(\omega)}{N_0} \right] d\omega \]  

(I-14)

which is (2-42) with \(d(t) = s(t)\).

Problems

2-1 Assume that the signal process \([s(t)]\) and the noise process \([n(t)]\) are statistically independent and have rational spectral densities \(S_s(\omega)\) and \(S_n(\omega)\) respectively.

(a) Show that the optimum realizable filter is given by

\[ H_0(i\omega) = 1 - \frac{1}{G^+(\omega)} \left[ \frac{S_s(\omega)}{G^-(\omega)} \right] \]

with \(G^+(\omega)G^-(\omega) = S_s(\omega) + S_n(\omega)\).
(b) If $S_n(\omega) = N_0/2$, prove that the optimum realizable filter is given by

$$H_0(i\omega) = 1 - \frac{\sqrt{N_0/2}}{G^+(\omega)}$$

if the signal power is finite.

2-2 Let $x(t) = s(t) + n(t)$ and assume that the processes $\{s(t)\}$ and $\{n(t)\}$ are uncorrelated with

$$S_s(\omega) = \frac{2k\sigma_s^2}{\omega^2 + k^2}, \quad S_n(\omega) = \frac{N_0}{2}$$

(a) The desired signal is $s(t)$. Find the optimum realizable linear filter that produces the MMSE.
(b) Find the optimum noncausal filter that minimizes the mean-squared error.
(c) Evaluate the mean-squared error for both (a) and (b). Such a problem arises in the demodulation of phase modulation $\{s(t)\}$ by means of a phase-locked loop.

2-3 Find the transfer function $H_0(i\omega)$ of the optimum realizable (causal) filter when

$$S_s(\omega) = \frac{A}{\omega^2(\omega^2 + a^2)}, \quad S_n(\omega) = \frac{N_0}{2}$$

Compare this transfer function to that of the optimum noncausal filter. Assume that the desired signal is $s(t)$. Such a problem arises in the demodulation of frequency modulation by means of a phase-locked loop.

2-4 Let $x(t) = s(t) + n(t)$, where $\{s(t)\}$ and $\{n(t)\}$ are independent random processes with rational spectra $S_s(\omega)$ and $S_n(\omega)$. Let $H_1(i\omega)$ and $H_2(i\omega)$ be the optimum realizable filters for $\{s(t)\}$ and $\{n(t)\}$, respectively—that is, $H_1(i\omega)$ is designed as though $s(t)$ were the desired signal and $H_2(i\omega)$ as though the desired signal were $n(i)$. What is the output of the following filter network?

Fig. P2-4.

2-5 Let

$$S_s(\omega) = \frac{1}{(1 + \omega^2)^2}, \quad S_n(\omega) = \frac{N_0}{2}$$

and assume that the processes $\{s(t)\}$ and $\{n(t)\}$ are independent.
(a) Find the (MMSE) causal and noncausal filters.
(b) Repeat (a) if
\[ S_{r}(\omega) = \frac{1 + \omega^2}{1 + \omega^4} \]
(c) Find the variance of the mean-squared error for (a) and (b).

2-6  The received waveform is of the form \( x(t) = s(t) + n(t) \), where \( [s(t)] \) and \( [n(t)] \) are uncorrelated random processes with spectral as given in Prob. 2-3. Let \( d(t) = ds(t)/dt \).
(a) Find the optimum (MMSE) causal and noncausal filters.
(b) Evaluate the performance of both filters.
(c) Repeat (a) if \( d(t) = \int_{-\alpha}^{t} s(\lambda) \, d\lambda \) with \( \alpha > 0 \).

2-7  A polynomial-type signal \( d(t) = a_1 t \) is to be separated from additive white Gaussian noise. The coefficient \( a_1 \) is a zero-mean Gaussian r.v.
(a) Find the optimum causal filter.
(b) How can you eliminate the Lagrange multiplier in your solution? Such a problem arises in the linear tracking theory to be discussed later.

2-8  Verify (2-64) and (2-67) in the text.

References


References


3

SYSTEMS EMPLOYING THE PHASE-LOCK OR ENTRAINMENT PRINCIPLE

3-1 Introduction and Motivation

Before proceeding we pause to introduce the phase-lock principle and to discuss many of its applications to the fields of telecommunication and power engineering, biology, biophysics, and physiology. Our discussion is both qualitative and quantitative. For simplicity and clarity, we first derive the loop equation of operation in the absence of noise. From this equation we point out fundamental concepts that serve to motivate the study of nonlinearity and solutions to nonlinear differential equations. Secondly, we develop the loop equation of operation in the presence of noise and oscillator instabilities. This equation, which is a nonlinear, stochastic, integrodifferential equation, is given an interpretation in the form of a block diagram—the baseband loop model. The statistical characteristics of the equivalent phase-noise process are then developed.

From this point we proceed to develop the theory as it applies to coherent communications (digital and/or analog). Included in the discussion are carrier and subcarrier demodulators, Costas and squaring loops, decision-directed methods, and the data-aided and hybrid loop techniques that arise in demodu-
lation, tracking, and synchronization theory. Motivated by these various loops and their applications, we show that a broad class of symbol synchronizers, coherent demodulators, tracking loops and their interconnected rhythms can be studied and represented *en masse* via a single stochastic, nonlinear integro-differential equation. This equation is further used to motivate the necessary theoretical material pertaining to nonlinear oscillations, rhythms, Markov and diffusion processes, Fokker-Planck equations, and solutions of integropartial differential equations occurring in nonlinear filtering. This material is discussed in Chapters, 5, 6, 7, and 8.

3-2 The Phase-Locked Loop Mechanization

The essentials of a phase-locked loop (PLL) are a phase detector (PD), a loop filter, and a voltage-controlled oscillator (VCO); the generator to be synchronized. The circuit configuration shown in Fig. 3-1 is the same regardless of whether it is being used for (1) tracking, (2) synchronization, (3) linear demodulation, (4) phase or frequency demodulation of analog or digital signals, or (5) amplitude detection. In practice, more elaborate systems frequently use double-heterodyne techniques, IF limiters, and signal acquisition aids; however,

![Diagram of a Phase-Locked Loop](image)

*Fig. 3-1. Basic Phase-Locked Loop Indicating Signal Extraction Points for Various Telecommunication Applications.*
insofar as analysis of the behavior is concerned, most systems reduce to this basic model. We elaborate on other mechanizations much later. By proper selection of the loop filter time constants, the PD, the VCO, and the loop gain, the circuit can be used as a versatile and powerful tool in modern communications, radar, telemetry, command and control and ranging systems.

The effectiveness of a PLL circuit is contingent on the amount of a priori information that the communication engineer can design into the circuit, both in its configuration and in the values of its components. This information usually is dictated by the particular application and takes the form of predictions of what the incoming signal shape resembles. In turn, this information is based on (1) what has been observed in the past and (2) all frequency or phase variations that it could exhibit in the future. The PLL provides a method whereby we can create in the receiver a signal that is an estimate of what the incoming signal will be. By a process of comparison, or correlation, the loop continually modifies the local reference (VCO output) to reflect the latest changes in the actual input signal.

When the circuit is designed for use as a narrowband tracking filter, an error signal is produced at the output of the PD if the input signal tends to "lead-ahead-of" or "lag-behind" the locally generated reference signal. Low-frequency components at the filter output produce a signal that changes the frequency of the VCO in such a way as to move it toward the frequency of the received signal. This effect manifests itself as a tracking (by phase locking) of the input signal, provided, of course, the noise is not too large. In effect, the signal into the VCO acts as a governor that controls the "speed" of the VCO in much the same way as a governor controls the rotor speed of a power generator.

3.2.1 Applications in Telecommunication and Power Engineering

Before continuing let us motivate our investigation further by quantitatively presenting examples that illustrate the practical utilization of the PLL illustrated in Fig. 3-1.

In detecting frequency modulation (FM), such as in FM/FM telemetry, one must recover the entire signal spectrum extending from a few Hertz up to several thousand Hertz. For this application the loop time constants are adjusted so that the circuit can follow the instantaneous frequency of the input signal as it deviates over a range of several thousand hertz. The loop filter output produces a control signal that forces the VCO to follow the input signal. When the loop is properly designed, the modulation appears at the post-loop filter output.

In phase-modulation (PM) detection by means of a PLL, we are interested in producing an output signal that varies in accordance with the deliberate shifts in phase of the phase-modulated input signal. When the loop is properly
designed, this is precisely the information we get from the output of the PD or loop filter. These phase shifts have associated with them the corresponding concept of signal bandwidth (rise time), which must be compatible with the rate of data transmission.

In using a PLL for amplitude detection, the physical basis for separation of the amplitude component from the phase or frequency components lies in the quadrature relationship between the sidebands in a linear-modulated wave and in a phase or frequency-modulated wave. In PM or FM we detect the sideband present in quadrature with the carrier. In linear modulation we detect sidebands whose resultant is always in phase with the carrier.

In various scientific measurements (e.g., Faraday rotation, occultation experiments, rotation rate of a planet, range to a planet, etc.), frequent use is made of phase methods in which some physical quantity is estimated by measuring the phase difference between two oscillations. This method is used to measure accurately the propagation speed of electromagnetic energy. The same principle is used in coherent ranging systems and phase-coherent communication receivers of the double-superheterodyne type.

Investigations of nuclear reactions by means of cyclotrons presuppose high stability of the parameters of the accelerated-ion beam and of the frequency of the accelerating voltage. Frequency deviations of the resonant system lead to a considerable reduction in the voltage in the dee circuits. To eliminate this reduction it is necessary either to adjust the frequency of the master generator automatically to the natural frequency of the dee circuit or to adjust the dee circuit to the frequency of the generator. The control system frequently incorporates automatic phase control of the frequency by means of a PLL.

Such techniques may also be used to detect and track phase-coherent vibrations in missile systems. Most inflight vibration-producing sources are random in character; however, coupling mechanisms often serve to produce phase-coherent sources of quasisinusoidal vibration. These latter sources generally lie in the air frame and the propulsion system. In the air frame, the mechanism is flutter; in the propulsion system, it is motor chamber pressure oscillations. When phase-coherent vibration is found to exist, it is extremely important, from the point of view of the air frame designer and the propulsion unit designer, to be able to detect the existence of any coherent vibration originating in his portion of the missile system, because this information may be a good clue to other undesirable behavior characteristics.

A very important factor in the launching of artificial satellites, rockets, and deep-space vehicles or probes is knowledge of their orbit parameters. These parameters can be measured by various methods (interference, Doppler, etc.), but all are based on the use of a highly stabilized rhythmic source synchronized to an autonomous generator. Among the major elements of such devices are automatic phase-tracking systems.
In order to solve many problems in radio engineering, it is necessary to generate highly stabilized microwave oscillations. The existing klystron and magnetron generators, as a rule, do not have enough phase or frequency stability. To increase their frequency stability, such generators are synchronized by means of low-power oscillations from highly stabilized standard generators, like quartz-controlled or quantum generators, and other atomic standards, such as cesium beam, rubidium vapor, and hydrogen masers.

The synchronization or phase locking of several synchronous generators occurs for 60-cycle electric power generation. Essentially these generators are nonlinear oscillators connected in parallel. Each has its own governor and fairly good regulation, but properly connecting together a large number of these generators produces tremendous stability independent of randomly imposed loads. If one generator tends to go a little faster than the others, or leads in phase, it will experience a retarding force from the others when they are phase locked. This interaction is a consequence of nonlinearity of the generator response. In fact, the interaction causes regularity of power to appear that manifests itself as a virtual governor. It does not have a physical existence in the usual sense but a virtual existence that emerges from the interaction of many components. This multiple-component system acts as a single unit—an example of self-organization. In fact, we shall later see that the phase error process in a PLL can be used to describe the hunting phenomena occurring in synchronous motors and generators due to randomly time-varying loads (Ref. 1). However, this kind of phase locking can also break down. The disastrous power failure on the East Coast of the United States and Canada in November 1966 was due to the disconnection of a large relay in the trunk line going to Canada. A sudden drop in the load left the energy being developed with no place to go; the generators began to become asynchronous (out of phase) with one another, thereby producing massive instability. As a result, the circuit breakers sequentially removed themselves from the power network. Synchronization of several oscillators also occurs in interconnected communication systems (see Probs. 3-15, 3-17, 3-19, and 3-20).

In television receivers are located horizontal and vertical oscillators positioning the fluorescent dot that fleetingly forms the picture as it scans. These oscillators must be operating at exactly the same frequency and phase as the signal being transmitted; otherwise the picture will move sideways or vertically. The vertical and horizontal knobs can be used to adjust the frequency of the oscillators so that they are nearly the same frequency as that transmitted. Then they phase-lock or synchronize to a constant phase relation. In color television systems, synchronization is essential to restore subcarrier frequencies in the receiver.

Among other applications of the PLL we point to time-service systems, multiple-access satellite systems, synchronous radio broadcasting, coherent
radar, two-way Doppler systems, some types of phase-controlled navigation systems, such as OMEGA and DECCA. The synchronization of a rhythmic source to an autonomous oscillator allows for the possibility of establishing a worldwide time synchronization system capable of synchronizing clocks at spatially remote locations with a master clock to within a few microseconds by means of radar reflections or reflections relayed through satellites. The principle is also applicable to the demodulation of pulse-position telemetry, frequency- and linear-modulated signals.

In command and telemetry systems, coherent detection has come into extensive use recently. Word, symbol, subcarrier, and carrier synchronization references are usually established by means of the phase-lock principle. Promising radio communications systems with single-sideband modulation or with double-sideband, suppressed carrier modulation call for the presence of the closed-loop system illustrated in Fig. 3-1.

### 3.2.2 Rhythms, Entrainment, and the Phase-Lock Principle

Outside the field of communications and power-generating systems, the phase-lock principle has been noted and referred to as the principle of entrainment. Some physicists are beginning to believe that this technique may be useful in stabilizing plasmas. In the fields of biophysics and physiology, in particular, those concerned with all types of rhythmical phenomena entrainment is at the heart of the matter in the investigations of biological rhythms (oscillations) because the internal organs of organisms synchronize with each other while the organisms themselves synchronize with their environments. Almost every living organism becomes entrained or locked in step to its natural environment. Consider work schedules, the paper boy’s route, routine coffee breaks, and so forth, in lock with the rotation of the Earth. Entrainment is also of interest to atomic physicists, students of hydrodynamics, meteorologists, and oceanographers.

Studies of circadian rhythms have established light as a major entraining agent for most living organisms. If one inverts a man’s cycle by placing him in an artificial environment that is exactly 180 degrees out of phase (place him on the other side of the world instantaneously) with its normal phase, it takes about a week for his temperature cycle to invert. This gives some idea of how long it takes to become re-entrained or resynchronized. Some adjustment of synchronization for jet travelers is noted in traveling from the East Coast to Europe, for it involves approximately a 90 degree shift in phase.

Bees and starlings use their circadian rhythms as clocks for navigation by the Sun. They use their circadian rhythms as a chronometer to make navigation possible. If the Sun is in a certain position, they do not know whether it is East or West unless they know the time of day. The peak abilities of people also become entrained to the time of day. For crucial tasks, such as
that carried out by a fighter pilot or an athlete, peak performances can be achieved by gearing activities to the optimum time of day.

The fertility cycles of many coastal marine animals (e.g., the grunion) are phase locked to the full Moon cycle. Gruinion lay their eggs during the few highest waves and then the tide pulls out. We note that the tides are phase locked to the Moon's cycle about the Earth. The incubation period is about 2 weeks, so that when the next highest monthly tide comes in, these eggs hatch on contact with the water. Light also affects the ovulation of chickens and hence their productivity. If one leaves the light on all night in the chicken house, chickens lay eggs at night as well as during the day. The gonads of a male hamster subjected to one hour of light per day will shrink to one-fourth their previous weight.

Aristotle knew that sea urchin's gonads were largest at the full Moon. He did not know, however, that it was the light of the Moon that was phase locked to the fertility cycle. Then, possibly, such entrainment of artificial light can ensure a firmly predictable natural rhythm method of birth control. Thus the population size can be controlled by phase locking ovulation to a predictable light source.

In the days of propeller-driven aircraft, sunlight reflecting off the propeller blades or through them would sometimes give the pilots the feeling that they were flying upside down. These delusions can be explained by the phase lock principle in that photostimulation can act so as to entrain certain rhythms produced by the brain. It has been noted that rotating beacons on flying aircraft could cause trouble at night when it is raining. The light from these beacons could be reflected into the pilot's eyes from the water particles at such a frequency and phase as to cause entrainment with certain rhythms produced by the brain. The result could cause trouble if the pilot is fatigued. Henri Gastaut, the famous French neurophysiologist, explained that crashing of helicopter pilots was due to the blades of the propeller periodically interrupting the Sun reaching their eyes, thus inducing disorientation and a possible loss of consciousness. In considering the design of the channel tunnel between England and France, there was concern about the spacing of the lights inside; periodic photostimulation might cause loss of consciousness or disorientation due to entrainment of certain brain waves.

Aside from power generation, the phase locking of many oscillators occurs in nature. If one were to carry out the simple experiment of catching fireflies that are flashing randomly on a dark night and placing them in a jar for observation, one would note that after a period of time the firefly flashes become entrained or phase locked. The internal flashing mechanism is evidently light sensitive in that they occur only at night, indicating that it is locked in step with the fireflies' environment—namely, day and night.

When the individual cells of the heart become phase locked in the embryonic stages of fetal development, they combine in such a way as to produce
the beat clearly recognizable by the obstetrician. Sometimes a heart gets something like "power failure" and the cells get out of synchronism. The heart stops beating, and unless it is defibrillated the person will die. The process of defibrillation—that is, shocking the heart—can resynchronize the cells such that they again beat synchronously. The phase-lock principle is also useful for explaining certain types of epileptic seizures observed via electroencephalograms or brain waves. In fact, one can flicker a lamp at 10 Hz/sec into the eye and note that the rhythm in the brain will become phase locked to this input signal.

Spectral analysis of stocks listed on the New York Stock Exchange has shown a periodic structure in the Brownian motion of stock prices. This has been inferred on the basis of periodic behavior in the volume for intervals of a day, week, 3 months, and one year. The ability to be able to predict and track such periodic oscillations is of considerable interest to the economist who wishes to predict cycles in the economy or to those who wish to exploit such for profit.

In the field of music, the phase-lock principle is in effect at many levels. For instance, various instrument sounds are combined in such a way as to produce rhythms pleasing to the ear; when those rhythms become de-entrained, the conductor or band director's responsibility is to see that they become re-entrained. In fact, when sound from a particular instrument becomes de-entrained with those emitted from the others, a noticeable unpleasant effect in the music is observable. It is also known that loud, vigorous musical sounds can increase the normal heartbeat rate, whereas soft, slow, smooth musical sounds can cause the normal heartbeat to be reduced. This is a good example of the case where internal organs of organisms become locked in step to the environmental conditions. Another obvious example appears when one dances to rock music—muscles of the body become phase locked to the rhythms of the band. Certain progressions and combinations of musical rhythms can be used to entrain the minds of crowds so as to induce riots, fights, marches, and so on. Cheers, yells, etc., at a football game can induce rhythms into an unenthusiastic crowd to the extent that they become emotionally involved.

In football, a large measure of success of the offensive team has to do with timing—that is, the ability of the teammates to become entrained with the cadence calls of the quarterback. It is important to the defense to be able to break up the entrainment of the offense—for example, cause a lineman or back to move prior to the snap from center. The quarterback frequently uses very short or very long counts in the hope that the defensive team will not be ready for action or that it will become so anxious that offside occur. A similar phenomenon occurs between the hitter and pitcher in baseball; entrainment of the swing with the pitcher's rhythm is a must to becoming a successful hitter.

This list could continue at some length; however, let us proceed with the development of the loop equation.
3.3 Loop Equation in the Absence of Noise

Once a mathematical description of the phase-detector characteristic is given, loop analysis can begin. Consider the basic loop shown in Fig. 3-2, where, for the moment, the phase detector has been replaced by a multiplier.\(^*\)

\[
s(t, \Phi) = \sqrt{2} A \sin \Phi
\]

(3-1)

in which \(A(t)\) is the amplitude modulation, produced at the transmitter and/or due to the time-varying multipath, \(\Phi(t) = \omega_0 t + \theta(t)\), where \(\omega_0/2\pi\) is the center frequency of the process \(\{s(t, \Phi)\}\) and \(\theta(t)\) is the input phase modulation produced at the transmitter and/or due to time-varying multipath associated with the communication channel. The signal \(s(t, \Phi)\) is multiplied by the VCO output

\[
r(t, \hat{\Phi}) = \sqrt{2} K_1 \cos \hat{\Phi}
\]

(3-2)

In the absence of noise, the term \(\hat{\Phi}(t) = \omega_0 t + \hat{\theta}(t)\) is the loop estimate of \(\Phi(t)\), \(\hat{\theta}(t)\) is the loop estimate of \(\theta(t)\), and \(K_1\) is the root mean square amplitude of the VCO output.

\(^*\)In order to write the differential equations that follow in compact form, we introduce the Heaviside operator \(p \triangleq d/dt\). In general, if the input to a linear filter with transfer function \(F(s)\) is \(e(t)\), then the time domain representation of the differential equation which relates the output \(z(t)\) to the input is written compactly as \(z(t) = F(p)e(t)\). With \(p^k = d^k/dt^k\) then \(F(p)\) has the representation

\[
F(p) = \left[ \sum_{k=1}^{N} a_k (d^k/dt^k) \right] / \left[ \sum_{k=1}^{N+1} b_k (d^k/dt^k) \right]
\]

where the \(a_k\)'s and \(b_k\)'s are constants. Thus the filter transfer function is represented in Laplace transform notation by \(F(s) = F(p)|_{p=s}\), or in Fourier transform notation by \(F(j\omega) = F(p)|_{p=j\omega}\). When \(F(p)\) appears in a figure by itself we mean \(F(s) = F(p)|_{p=s}\), or \(F(j\omega) = F(p)|_{p=j\omega}\). This will avoid changes in notation.
In practice, the multiplication is accomplished by a device unable to respond to the double-frequency terms present; hence the actual output of the multiplier is taken (double-frequency terms neglected) to be

\[ \varepsilon = AK_1 K_m \sin \varphi \quad (3-3) \]

where \( K_m \) is the multiplier gain. The quantity

\[ \varphi \triangleq \Phi - \hat{\Phi} = \theta - \hat{\theta} \quad (3-4) \]

is defined to be the total loop phase error. In practice, the signal \( \varepsilon(t) \) is referred to as the dynamic phase error and \( z(t) \) is frequently referred to as the static phase error. In operator notation \( z(t) \triangleq F(p)\varepsilon(t) \), where \( p \triangleq d/dt \) is the Heaviside operator. The acquisition voltaghe \( e(t) \) is used, in practice, to expedite entrainment of the oscillations of the input with those produced by the VCO. Acquisition of the incoming oscillations by the VCO is analogous to the problem of starting a synchronous motor. The starting of the motor requires an auxiliary device that will bring the motor to synchronous speed. The average torque produced is zero until synchronous speed is reached and its two fields are phase locked. In a PLL this is analogous to tuning the VCO until its frequency is sufficiently close to that of the arriving oscillations so that phase acquisition or pull-in occurs.

The instantaneous VCO output phase \( \hat{\theta} \), referenced to zero, is taken to be a linear function of the control signal which appears at the VCO input. The input control signal \( e(t) + z(t) \) to the VCO acts much like a governor used on power generators for speed control. When the control signal \( e(t) + z(t) \) to the VCO is removed, the VCO generates a constant frequency sinusoid of \( \omega_0 \text{ rad/ sec} \) and we shall refer to its frequency as the rest or quiescent frequency of the VCO. When the control signal is applied, the VCO output frequency becomes \( \omega_0 + K_\nu [z(t) + e(t)] \), where \( K_\nu \) is the VCO gain in radians per second per volt. Consequently, the phase estimate \( \hat{\theta} \), developed by the loop, is related to its input in operator form through

\[ \hat{\theta} = \frac{K_\nu}{p} (z + e) \quad (3-5) \]

where \( z = F(p)e \). Substituting for \( z \) into (3-5) and using (3-3), we obtain

\[ \hat{\theta} = \frac{AKF(p)}{p} \sin \varphi + \frac{K_\nu e}{p} \quad (3-6) \]

where \( K \triangleq K_1 K_m K_\nu \) is defined as the open-loop gain. Since \( \hat{\theta} = \theta - \varphi \), (3-6)
Loop Equation in the Absence of Noise

becomes

$$\varphi = \theta - \frac{AKF(p)}{p} \sin \varphi - \frac{K_v e}{p}$$  \hspace{1cm} (3-7)

which is the equation of operation in the absence of noise.* We note that (3-7) is a nonlinear integrodifferential equation when $F(p) \neq 0$ and represents a mathematical model of the loop.

For the moment, let us assume that $e(t) = 0$. Then when $F(p) = 1$, (3-7) implies

$$\frac{d\varphi}{dt} = \frac{d\theta}{dt} - AK \sin \varphi$$  \hspace{1cm} (3-8)

is a first-order nonlinear differential equation; hence the configuration of Fig. 3-2 is called a first-order PLL. When $F(p) = 1/p$ and $\theta = 0$, then (3-7) implies

$$\frac{d^2 \varphi}{dt^2} + AK \sin \varphi = 0$$  \hspace{1cm} (3-9)

is the pendulum equation familiar to us from physics. Since it is a second-order nonlinear differential equation, (3-9) represents the behavior of a second-order loop. In general, when $F(p)$ has $n$ finite poles, the loop equation is an $(n + 1)$th-order, nonlinear, integrodifferential equation and the device in Fig. 3-2 is said to be an $(n + 1)$th-order loop.

Now that we have a mathematical model, we are under obligation to give an explanation of loop operation. Several questions can be asked here: How does one solve (3-7) for various deterministic or nondeterministic modulation inputs and various loop filters? How does one optimize the loop? Will the oscillations of the VCO output entrain or synchronize to the oscillations of the input signal? If so, how long does it take? The answer to these rather complicated questions will be considered after establishing a base of knowledge pertaining to “solutions” of nonlinear differential equations. The purpose, then, of Chapters 5, 6, 7, and 8 is to present theoretical developments that can be used to aid in the understanding of the nonlinear behavior of a PLL. Only through a buildup of working with nonlinearity can one effect a thorough understanding of the phase-lock principle, the principle of synchronization or entrainment.

*The form of (3-7) implies the equation

$$\frac{d\varphi(t)}{dt} = \frac{d\theta(t)}{dt} - \int_0^t f(t - \lambda)[A(\lambda) \sin \varphi(\lambda)]d\lambda - K_v e$$

where $f(x)$ is the impulse response of the loop filter.
3-4 Loop Equation in the Presence of Noise

In proceeding let us focus our attention on the communication system model depicted in Fig. 3-3. As we shall see later, this model is sufficiently general to include the use of a PLL as (1) a demodulator of analog or digital signals (FM or PM), (2) a reference extractor for purposes of linear demodulation and amplitude detection, (3) a carrier or subcarrier tracking system, or (4) a synchronizer for telemetry, command and radar applications, and so on. Of course, for each application loop design proceeds in a different way even though a broad class of communication system problems can be studied at the outset from the same equation of operation. One of the major goals of later chapters is to consider the problem of loop design and analysis for the various applications.

The operation of the system in Fig. 3-3 can be described as follows: The harmonic oscillations \( s(t, \Phi) \) of the transmitter’s signal generator (SG), plus the additive channel noise \( n(t) \) and the reference signal \( r(t, \Phi) \) of the voltage controlled oscillator (VCO) (oscillator to be synchronized), are applied to the phase detector (mechanized as a multiplier) to produce an output \( \epsilon(t) \) that depends on the total phase error \( \phi(t) = \Phi(t) - \hat{\Phi}(t) \) and noise. The signal \( \epsilon(t) \), frequently referred to as the dynamic phase error, is passed through the loop filter \( F(p) \) and produces the output \( z(t) \). This signal, called the static phase error, alters the frequency and phase of the VCO in such a way that it coincides with the frequency and phase of the transmitter oscillator. In practice, internal fluctuations that are inherent in the circuit components arise at the transmitter and reference oscillators. These instabilities, as well as the noise \( n(t) \), cause fluctuations in the total phase error \( \phi(t) \). As we shall see, the effect of the noise and the internal fluctuation on the operation of a PLL poses a very complex mathematical problem owing to the specific nonlinearity of the system.
The transmitted signal and the output of the synchronized oscillator, VCO, are of the form:

\[ s(t, \Phi) = \sqrt{2} A \sin \Phi = \sqrt{2} A(t) \sin [\omega_0 t + \theta(t)] \]
\[ r(t, \hat{\Phi}) = \sqrt{2} K \cos \hat{\Phi} = \sqrt{2} K \cos [\omega_0 t + \Theta(t)] \] (3-10)

where \( \theta(t) \) and \( \Theta(t) \) are random phase-modulation signals to be subsequently characterized.

As was shown in Chapter 1, a narrowband, Gaussian random process \( \{n_i(t)\} \) with a symmetrical spectral density has the sample function representation at time \( t \)

\[ n_i = \sqrt{2} [n_e \cos \omega_0 t - n_s \sin \omega_0 t] \]
\[ = \sqrt{2} N_i \cos \Phi_i = \sqrt{2} N_i(t) \cos [\omega_0 t + \theta_i(t)] \] (3-11)

where \( N_i(t) \) is the envelope of the noise and \( \theta_i(t) \) characterizes the random phase. From (3-11) we have, respectively, the cosine and sine components

\[ n_c = N_i \cos \theta_i \quad n_s = N_i \sin \theta_i \] (3-12)

with correlation functions (see Chapter 1)

\[ r(\tau) = \overline{n_e n_{e\tau}} = \overline{n_s n_{s\tau}} \]
\[ \overline{n_e n_{e\tau}} = \overline{n_s n_{s\tau}} = 0 \] (3-13)

where we reiterate that \( n_e = n_e(t) \) and \( n_{e\tau} = n_e(t + \tau) \). The correlation function of the additive noise is represented by

\[ R_n(\tau) = 2r(\tau) \cos \omega_0 \tau \] (3-14)

A simple trigonometric identity shows that the sum of the transmitter output and additive noise can be written as follows:

\[ x = \sqrt{2} \left[ (A - N_i) \sin \Phi + N_e \cos \Phi \right] \] (3-15)

where

*In the presence of reference oscillator instabilities, the phase function \( \hat{\Phi} \) is represented by \( \hat{\Phi}(t) = \omega_0 t + \Theta(t) \) where \( \Theta(t) \) is the sum of \( \hat{\theta}(t) \) plus any reference oscillator instabilities. When the oscillator instabilities are absent, then \( \Theta(t) = \hat{\theta}(t) \).
\[ N_c \triangleq N_i \cos (\theta_i - \theta) \quad N_s \triangleq N_i \sin (\theta_i - \theta) \] (3-16)

and we have suppressed the time variable.

Expanding the sine and cosine terms in (3-16) and using (3-12) we get

\[ N_c = n_c \cos \theta + n_s \sin \theta \]
\[ N_s = n_s \cos \theta - n_c \sin \theta \] (3-17)

so that the correlation functions of \( N_c(t) \) and \( N_s(t) \) are

\[ \overline{N_c N_{cc}} = \overline{N_s N_{ss}} = r(\tau) \cos \Delta \theta_c \]
\[ \overline{N_c N_{cs}} = -\overline{N_s N_{sc}} = r(\tau) \sin \Delta \theta_c \] (3-18)

where \( \Delta \theta_c \triangleq \theta(t) - \theta(t + \tau) \) and we have made use of (3-13).

Assuming that the phase detector possesses the gain \( K_m \), the signal \( \epsilon(t) \) (see Fig. 3-3), becomes

\[ \epsilon = K_i K_m [(A - N_i) \sin \varphi + N_c \cos \varphi] \]
\[ \varphi \triangleq \Phi - \hat{\Phi} = \theta - \Theta \] (3-19)

and we have neglected the double-frequency terms due to the fact that the loop will not respond to them. The instantaneous frequency of the VCO output, \( \Theta \), referenced to zero, is related to its input through

\[ \dot{\Theta} = \frac{K_r}{p} z + \frac{K_v}{p} \epsilon + \frac{\psi_v}{p} \] (3-20)

where \( z(t) = F(p)\epsilon(t) \) and \( K_v \) is the VCO gain. In (3-20), and hereafter, dots over a variable denote operation on the variable by \( p \)—that is \( px = dx/dt \triangleq \dot{x} \). Therefore

\[ \Theta = \frac{K_r}{p} [F(p)\epsilon] + \frac{K_v}{p} \epsilon + \psi_v \] (3-21)

which becomes, upon using (3-19),

\[ \Theta = \frac{KF(p)}{p} [A \sin \varphi + N(t, \varphi)] + \frac{K_v}{p} \epsilon + \psi_v \] (3-22)

where the equivalent phase noise is defined by
\[ N(t, \varphi) \triangleq N_e \cos \varphi - N_s \sin \varphi \] (3-23)

and \( K = K_r K_m K_v \) is the open-loop gain. Since \( \varphi = \theta - \Theta \), we have the following nonlinear stochastic integrodifferential equation of system operation.

\[ \varphi = \theta - \frac{KF(p)}{p} [A \sin \varphi + N(t, \varphi)] - \frac{K_v e}{p} - \psi_2 \] (3-24)

In general, the signal phase modulation \( \theta(t) \) consists of three terms

\[ \theta(t) = \underbrace{d(t)}_{\text{Input doppler}} + \underbrace{M(t)}_{\text{Digital or analog modulation}} + \underbrace{\psi_1(t)}_{\text{Transmitter oscillator instabilities}} \] (3-25)

so that the shortened notation of (3-24) and (3-25) implies the equation

\[ \dot{\theta} = d + M - \int_0^t Kf(t - \lambda) [A(\lambda) \sin \varphi(\lambda) + N(\lambda, \varphi)] d\lambda + \Delta \psi - K_v e \] (3-26)

where \( f(x) \) is the impulse response of the loop filter and \( \Delta \psi \triangleq \psi_1 - \psi_2 \). Equation (3-24) or (3-26) represents a mathematical model of the system illustrated in Fig. 3-3. The mathematically equivalent loop model of the system defined by (3-24) is depicted in Fig. 3-4 and will be referred to as the baseband loop model.

**Fig. 3-4.** Baseband Loop Model when \( \theta(t) \) is Being Tracked by the Loop.

*In practice, there is frequently another constant in \( K \) which is due to frequency multiplication.*
When $F(p)$ is a constant, (3-24) is a first-order stochastic differential equation; hence the loop is called a first-order PLL. Similarly, when $F(p)$ has $n$ finite poles, the system equation is an $(n + 1)$th-order stochastic integro-differential equation and the loop is said to be an $(n + 1)$th-order PLL.

Notice that when $F(p) = 1$ and $A = e = \psi_2 = \theta = 0$, the loop equation (3-24) reduces to

$$\dot{\phi} = -KN(t, \varphi) \quad (3-27)$$

and if $N(t, \varphi)$ is a white process, $\varphi$ becomes the Wiener process (Brownian motion) discussed in Chapter 1. On the other hand, when $F(p) = 1/p$, $e = \psi_2 = \theta = 0$, then (3-24) becomes

$$\ddot{\phi} = -AK \sin \varphi - KN(t, \varphi) \quad (3-28)$$

which is the pendulum equation forced by noise.

The sinusoidal nonlinearity, sometimes called the loop $S$ curve, is due to the particular type of signal transmitted and to the particular phase-detection scheme used. It is possible, as we shall see, to have a phase-detector characteristic that is not sinusoidal and very much dependent on the type of signal waveform and preprocessing used. A number of S curves have been used in communications—switched phase detectors, pseudo-noise modulation trackers, and so on. Later on we explore the effects on loop performance due to various phase-detector mechanizations.

What, then, is the problem in the presence of noise? If the loop equation were a simple linear function of the phase error involved, there would be only a few problems to solve. But the fact is that the precise relationship between the input and response function is a nonlinear, stochastic integro-differential equation. Because of this unfortunate circumstance, a melange of problems must be solved: What is the meaning of a solution to a differential equation driven by noise? What is the meaning of phase-lock in the presence of noise? If the loop is initially out of lock, will the oscillations in $r(t, \hat{\Phi})$ synchronize to the oscillations in $x(t)$? (Chapter 5 discusses the theory needed to answer this question in the absence of noise.) Will it false-lock to some internally generated VCO frequency rather than the input? How can the in-lock characteristics be analyzed? How does one select the loop filter and how can loop parameters be optimized? What is the signal acquisition time and range? Needless to say, the list of questions could go on at some length.

In fact, Chapters 2, 5, 6, 7, and 8 are written for the purpose of providing the necessary theoretical developments needed in obtaining solutions to various aspects of the PLL problem. This buildup of knowledge is fundamental to understanding how the PLL works as well as representing the mathematical background required in developing the nonlinear analysis of PLLs. It is the
author's opinion that the reader who grasps these fundamental principles will have no problem in thoroughly understanding and analyzing most problems of interest that pertain to the PLL. Chapter 4 presents a treatment of linear PLL theory.

3-5 Statistical Properties of the Phase-Noise Process \( N(t, \varphi) \)

Later on in our study of the nonlinear theory we will have to make use of the statistics of the process \( N(t, \varphi) \), in particular, the correlation function \( R_N(\tau) \). Consequently, the purpose of this section is to characterize this correlation function for use later. In general, the correlation function of \( \{N(t, \varphi)\} \) is difficult to evaluate because of the correlation that exists between \( N_s(t) \), \( N_c(t) \), and \( \varphi(t) \). However, in principle, this factor can be taken into account, for in a number of practical cases (e.g., tracking situations and synchronization loops) one can assume that the random functions \( N_s(t) \) and \( N_c(t) \) vary much more rapidly than the function \( \varphi(t) \). Stated another way, the correlation times of the processes \( \{N_s(t)\} \) and \( \{N_c(t)\} \) are short in comparison with the correlation time of \( \{\varphi(t)\} \). In what follows, we shall be primarily concerned with this case. Consequently, one can evaluate the correlation function by first holding the slowly varying quantities fixed and averaging over the rapidly varying quantities. Next, one averages over the slowly varying quantities separately.

Applying this technique to the problem at hand requires that we hold \( \varphi(t) \) fixed and average over the fast-varying quantities \( N_s(t) \) and \( N_c(t) \). Thus

\[
R_N(\tau) = E[R_N(\tau|\varphi)]
\]  \hspace{1cm} (3-29)

which can be written as

\[
R_N(\tau) = E[E[N(t, \varphi)N(t + \tau, \varphi)|\varphi]]
\]  \hspace{1cm} (3-30)

and from (3-23) this becomes

\[
R_N(\tau) = E[N_s N_{cr}\cos^2 \varphi + N_c N_{sr}\sin^2 \varphi \\
- (N_c N_{sr} + N_s N_{cr}) \sin \varphi \cos \varphi]
\]  \hspace{1cm} (3-31)

which reduces to

\[
R_N(\tau) = \overline{N_s N_{cr}} = r(\tau) \overline{\Delta \theta_c}
\]  \hspace{1cm} (3-32)

when we make use of (3-18). This says that, under the foregoing assumptions, the correlation function of the effective phase-noise process is independent of the phase error process. The intensity coefficient
\[ K_N \triangleq \int_{-\infty}^{\infty} r(\tau) \cos \Delta \theta, d\tau \] (3-33)

will play a major role when we develop the nonlinear theory of PLLs. We therefore see that the correlation function of \( N(t, \varphi) \) is related to the correlation functions of \( n_e \) and \( n_t \). For example, if we assume that

\[ r(\tau) = \frac{N_0 B_i}{2} \left( \frac{\sin \pi B_i \tau}{\pi B_i \tau} \right) \] (3-34)

then when the correlation time \( \tau_n \ll \tau_\varphi \)—that is, \( B_i \gg B_\varphi \)—then \( R_N(\tau) \approx r(\tau) \).

Thus for large \( B_i < \omega_0/\pi \),

\[ R_N(\tau) \approx \frac{N_0}{2} \delta(\tau) \] (3-35)

and \( K_N \approx N_0/2 \) from (3-33). Under these assumptions, this says that the baseband noise process \( \{N(t, \varphi)\} \) is approximately white. In what follows, we shall treat the process \( \{N(t, \varphi)\} \) as though it were white so that the total phase error embedded in \( N(t, \varphi) \) does not enter into loop analysis, linear or nonlinear.

### 3.6 Applications to Coherent Communications

As discussed earlier, by proper design a PLL can be used to track the carrier component of an observed signal in background noise, to perform phase or frequency demodulation of analog or digital signals, to perform a Doppler measurement, or several of these functions simultaneously. We now discuss, without loss in generality, the loop equation for these applications when the signal amplitude is constant. Later we shall discuss the problem when \( A(t) \) represents digital or analog modulation.

#### 3.6.1 Carrier Tracking

Since \( \varphi \) represents the total loop phase error, we must account for that portion which is due to the carrier modulation we wish to track. We note that for any application the phase estimate \( \hat{\Theta} \) represents the VCO estimate of the signal to be tracked by the loop. For this application \( \hat{\Theta} \) represents the loop estimate of \( d + \psi_1 \). Since \( \Theta = \hat{\Theta} + \psi_2 \) and \( \varphi = \theta - \Theta = d + \psi_1 + M - \hat{\Theta} - \psi_2 \), then \( \varphi = \varphi_c + M - \psi_2 \), where \( \varphi_c \) is the carrier-tracking phase error defined by

\[ \varphi_c \triangleq d + \psi_1 - \hat{\Theta} \] (3-36)

Use of these facts in the loop equation (3-24) produces the loop equation for carrier tracking—that is,
\[ \varphi_e = d + \psi_1 - \frac{K_F(p)}{p} [A \sin (\varphi_e + M - \psi_2) + N(t, \varphi)] - \frac{K_v e}{p} \] (3-37)

Notice how the VCO instabilities enter into the loop equation. The baseband loop model is illustrated in Fig. 3-5a. For this application the output filter is not required.

(a) Model when \( d + \psi_1 \) is being tracked by the loop

(b) Model when \( d \) is being tracked by the loop

(c) Model when \( d + M \) is being tracked by the loop

Fig. 3-5. Various Baseband Loop Models.
3-6.2 Doppler Tracking*

In this case we require the loop to estimate the signal $d$; that is, $\hat{\theta}$ is the loop estimate of $d$, so $\hat{\theta} = \hat{d}$. When the Doppler error $\varphi_d = d - \hat{d}$ is inserted into (3-24) and (3-25) is used, we obtain the equation of operation for Doppler tracking,

$$\varphi_d = d - \frac{KF(p)}{p} [A \sin (\varphi_d + M + \Delta\psi) + N(t, \varphi)] - \frac{K_re}{p} \quad (3-38)$$

Notice how the transceiver instabilities, $\Delta\psi$, enter into the loop equation and how Doppler tracking differs from carrier tracking only in the presence of transmitter instabilities; that is, $\varphi_d = \varphi_c$ when $\psi_1 = 0$. The baseband loop model is illustrated in Fig. 3-5b. For this application the output filter is not required.

3-6.3 Angle Demodulation or Modulation Tracking

In this case $\hat{\theta}$ represents the loop estimate of $M + d$, since for practical reasons the loop must also track out the Doppler. When the phase error $\varphi_M = d + M - \hat{\theta}$ is introduced into (3-24), we obtain

$$\varphi_M = d + M - \frac{KF(p)}{p} [A \sin (\varphi_M + \Delta\psi) + N(t, \varphi)] - \frac{K_re}{p} \quad (3-39)$$

$$\hat{M} = K_iK_mF_0(p)F(p)[A \sin (\varphi_M + \Delta\psi) + N(t, \varphi)]$$

for the equation of loop operation. For this application the loop filter bandwidth is usually much wider than for the case of carrier and Doppler tracking. The baseband loop model is illustrated in Fig. 3-5c. Notice that the output filter is included in the model.

*The Doppler effect was discovered in 1842 by an Austrian physicist and mathematician, Christian Johann Doppler. It is an effect on the observed frequency of any periodic oscillation—for example, electromagnetic or acoustic—emitted by a source when there is relative motion between the source and an observer. When the source moves toward an observer, the observed frequency of the oscillation is increased; and when it moves away from the observer, this frequency decreases. The Doppler effect is used in a myriad of practical applications on land, on sea, and in space. Some of these would probably make Christian Doppler's head spin if he were alive today. For example, police speed traps are set using electronic equipment based on the Doppler effect. Surveying of large land masses, sonar employed by submarines, tracking satellites and deep-space probes, monitoring the motion of distant stars, and tracking weather balloons are but a few of the very different uses of the Doppler effect. Probably one of the most widely publicized applications of the Doppler effect was in the Doppler radar system used in the Apollo program. This system was used to guide five Surveyor units, as well as the Eagle and Intrepid lunar modules. Soft Moon landings and space dockings were accomplished using Doppler measuring instruments.
3-6.4 Carrier and Doppler Tracking in the Presence of Angle Modulation

For convenience, we set $\psi_1 = \psi_2 = 0$ in (3-37) and (3-38) so that $\varphi_c = \varphi_d$. Then expanding the sinusoidal nonlinearity in (3-38) produces

$$\varphi_c = \varphi_d = d - \frac{KF(p)}{p} [A \sin \varphi_c \cos M + A \cos \varphi_c \sin M + N(t, \varphi)] - \frac{K_v e}{p}$$

(3-40)

as the loop equation. When the information-bearing signal is suitably (Ref. 2) modulated onto data subcarriers, the demodulation term $(A \cos \varphi_c \sin M)$ lies outside the bandwidth of the loop filter. Then, for most practical purposes, this term can be neglected when studying the problem of carrier tracking. Thus the equation of operation reduces to

$$\varphi_c = \varphi_d = d - \frac{AK(\cos M)F(p)}{p} \left[ \sin \varphi_c + \frac{N(t, \varphi)}{A(\cos M)} \right] - \frac{K_v e}{p}$$

(3-41)

when we assume that the loop filter responds only to the dc component $\cos M$ in $\cos M$. In practice, this can be ensured by proper design of $s(t, \Phi)$ and $F(p)$. The baseband loop model for tracking $d(t)$ in the absence of transceiver instabilities is shown in Fig. 3-6. As before the output filter is not required for this application.

![Fig. 3-6. Baseband Loop Model when $d$ is Being Tracked in the Presence of Angle Modulation, $\Delta\psi = 0$.](image)

3-6.5 Angle Modulation Extraction in the Presence of Carrier Tracking

As we shall presently see, the phase-detector output (3-19) can be used for purposes of extracting the modulation $M(t)$. Since $\varphi = \varphi_c + M$ when $\psi_1 = \psi_2 = 0$, the phase-detector output $\epsilon(t)$ given in (3-19) can be written as
\[ \varepsilon = K_1 K_m \left[ A \sin (\varphi_c + M) + N(t, \varphi) \right] \]  

(3-42)

where we have made use of (3-23). When the signal modulation is placed on unmodulated data subcarriers (e.g., see Fig. 3-7) such that the resulting data spectrum lies outside the bandwidth of the loop filter, then a bandpass filter (the subcarrier(s) extraction filter(s)) can be used to extract the particular modulated data subcarrier of interest (see Figs. 3-7 and 3-8).

When \( M(t) \) is the bi-phase modulated, square-wave subcarrier [the case of digital communications or ranging (Ref. 2)]

---

(a) Frequency spectrum of two unmodulated data subcarriers

(b) Frequency spectrum of two bi-phase modulated data subcarriers

(c) Frequency spectrum of the corresponding phase modulated carrier \( s(t, \Phi) \); Low modulation index is assumed

Fig. 3-7. Signal Frequency Spectra for a Typical Application.
Fig. 3-8. Carrier Tracking, Doppler Measurement and Angle Modulation Extraction, \( \Delta \psi = 0 \).

\[
M(t) = (\cos^{-1} m)X_d(t) \cdot \begin{cases} \text{Data sequence} & \text{Square-wave} \\ \text{subcarrier} & \end{cases} S(t) \tag{3-43}
\]

then

\[
\sin(\varphi_c + M) = m \sin \varphi_c + \sqrt{1 - m^2}X_dS \cos \varphi_c \tag{3-44}
\]

since \( \varphi_d = \varphi_c \), and the product \( X_dS \) is a sequence of \( \pm 1 \). Assuming \( A \) is constant for all \( t \) the parameter \( m \) in the modulation index, \( (\cos^{-1} m) \), serves to apportion the total power \( A^2 \) between the carrier and sidebands. In fact, the power remaining in this carrier component of \( s(t, \Phi) \) is \( P_c = m^2 A^2 \) and that in the sidebands is \( S = (1 - m^2)A^2 \). If the subcarrier frequency is chosen such that the spectrum of the signal \( X_dS \) lies outside the bandwidth of the loop filter, then the carrier tracking loop of Fig. 3-8 operates in accordance with

\[
\varphi_c = d - \frac{KF(p)}{p} \left[ \sqrt{P_c} \sin \varphi_c + N(t, \varphi) \right] \tag{3-45}
\]

Essentially the carrier tracking loop operates as though the modulation side-
bands of Fig. 3-7c are not present. If we pass \( e(t) \) through the bandpass filter \( H(p)/K_1K_m \), then the input to the subcarrier demodulator becomes

\[
u = H(p)[\sqrt{S}X_sS \cos \varphi_c + N(t, \varphi)]
\]  
(3-46)

![Diagram](image)

Fig. 3-9. Carrier Tracking, Doppler, and Digital Modulation Extraction.

since the bandpass filter in Fig. 3-8 rejects the baseband term \( \sqrt{P_s} \sin \varphi_c \). Notice how the loop phase error affects the amplitude of the demodulated signal \( u \). Figure 3-9 illustrates a typical receiver mechanization for this particular case, while Fig. 3-10 shows typical frequency spectra located at various points in the loop. The data demodulator is mechanized as a cross-correlator (see Chapter 2) matched in frequency and time to the data symbols of duration \( T \) seconds. Later we discuss how the subcarrier \( S(t) \) can be recovered from the extracted modulated data subcarrier given in (3-46).

Frequently \( N \) phase-modulated (digital or analog) sinusoidal subcarriers and/or \( L \) binary-valued signals are used to phase-modulate a carrier. In this case

\[
M(t) = \sum_{n=1}^{N} \sqrt{2} p_n \cos [\omega_n t + (\cos^{-1} m_n)x_n(t)] + \sum_{i=1}^{L} T_i(t)
\]  
(3-47)
Fig. 3-10. Spectrum of Signals Appearing at Various Points in the Receiver of Fig. 3-9.

with the parameter \( p_n^2 \) representing the average power in the \( n \)th modulated data subcarrier, \( m_n \) representing the modulation factor for the \( n \)th data subcarrier, and \( x_n(t) \) representing the \( n \)th data signal (analog or digital). For convenience, it is assumed, without loss in generality, that

\[
T_i(t) = \tau_i b_i(t), \quad i = 1, \ldots, L
\]  

(3-48)

where \( b_i(t) \) is a sequence of \( \pm 1 \)s and \( \tau_i \) is the modulation index for the \( i \)th sequence. It is further assumed that the sequence \( b_i(t) \) has a digit period of \( T_i \) seconds.
In general, some of the $T_i(t)$ signals in (3-48) could be used to transmit data. In this case the $T_i(t)$'s become the product of a data sequence and an unmodulated square-wave subcarrier. The so-called single-channel mechanization is a good example (Ref. 2) of a system that combines the data and synchronizing signals into one composite signal—that is, $N = 0, L = 1$. In the case of communication satellites with $N_i$ data sources, one would need to phase-modulate the carrier with only one signal from the set $\{T_i(t)\}$—that is, $N = N_i, L = 1$. This signal could be used as a timing signal for purposes of synchronizing each receiver's data demodulator. For purposes of generality, however, we treat the case of $L$ binary-valued signals and let the design engineer select the signaling configurations for his particular application.

When the subcarrier frequencies are chosen such that the spectrum of $M(t)$ lies outside the bandwidth of the loop filter, then the loop equation (3-40) reduces to (Ref. 2)

$$\varphi = d - \frac{KF(p)}{p} \left[ \sqrt{\prod_{n=1}^{N} J_0(\sqrt{2p_n^2}) \prod_{i=1}^{L} \cos^2 \tau_i} A \sin \varphi + N(t, \varphi) \right]$$  (3-49)

where $J_0(x)$ is the Bessel function of zero order and argument $x$. Appearing at the phase-detector output are the demodulated subcarriers. The extracted subcarrier signals can be separated by means of bandpass subcarrier extraction filters, and further applied to their respective data demodulators—for example, matched filters if the signals are digital or angle demodulators of the PLL type if the modulation is analog. In fact, for demodulation of these analog-modulated subcarriers by means of a PLL, the loop equation of the $j$th subcarrier demodulator, $j = 1, 2, \ldots, N$, is well approximated by

$$\phi_j = \dot{x}_j - K_j F_j(p)[A_j \sin \varphi_j + N_j(t, \varphi)]$$  (3-50)

where (Ref. 2)

$$A_j = \sqrt{2J_1(\sqrt{2p_j^2}) \prod_{n \neq j} J_0(\sqrt{2p_n^2}) \prod_{i=1}^{L} \cos^2 \tau_i (\cos \varphi_j) A}$$  (3-51)

and we have neglected all intermodulation products, for the subcarrier extraction filter attached to the phase-detector output of the carrier tracking loop will not respond significantly to these terms if the subcarrier frequencies are appropriately chosen. In (3-51) $J_0(x)$ and $J_1(x)$ are Bessel functions. For the square-wave subcarriers, the input to the $l$th data demodulation of the cross-correlator type is given by

$$y_l = H_l(p)[A_l \cos \varphi_z T_i + N(t, \varphi)]$$  (3-52)

where
$A_i^2 = \left[ \sqrt{\sin^2 \tau_i \prod_{n=1}^{N} J_n^2(\sqrt{2p_n^2}) \prod_{k=1}^{L} \cos^2 \tau_k} \cos \varphi_c A \right]^2$ (3-53)

is the power in the $l$th extracted modulated square-wave subcarrier. In later chapters we shall consider the design and performance of PLL type circuits for these various applications.

3-7 Suppressed Carrier (Subcarrier) and Modulation Tracking Loops

Since the phase-lock principle does not distinguish between a carrier and a subcarrier, the methods of tracking and demodulation processes to be discussed in this section apply to either case. In fact, one major purpose of this section is to show how the phase-lock principle can be used to reconstruct a carrier in a suppressed carrier signal. In various applications the amplitude modulation $A(t)$ and/or the phase modulation $\theta(t)$ may be sufficient to suppress the residual spectral component that existed for tracking by means of the sinusoidal PLL discussed in the previous section. In what follows, we also assume that $A(t)$ possesses no dc component in its power spectrum. A typical suppressed carrier frequency spectrum in background noise is shown in Fig. 3-11.

A number of loops have been proposed for generating a carrier reference from a suppressed carrier signal. For the most part, these loops are motivated by one aspect of generalized harmonic analysis (Ref. 3), normally referred to as the cross-spectrum. In the usual form of generalized harmonic analysis, the physical significance of the cross-spectrum consists of the spectrum of mutual power. In other words, consider two signals, possibly immersed in noise, coming from two different channels. The cross-spectrum represents the spectral

![Fig. 3-11. Typical Suppressed Carrier Frequency Spectrum.](image-url)
distribution of power that is mutually shared between the two signals in a phase-coherent manner. In order to have a mutual power, the signal must be phase coherent.

In the field of optics, the cross-spectrum is related to the coherency of the fields. In nonlinear systems, however, mutual power may often be shared at other than the same frequency. In a nonlinear system there may be coherency between one frequency in one channel and some multiple of that frequency in a separate channel. This power may, in fact, occur in any combination of multiples, sums, and differences of frequencies. Such interactions frequently remain undetected by the usual linear schemes; however, by utilizing nonlinear methods, coherency of suppressed carrier signals can be exploited at twice the carrier frequency. A coherent reference is then obtained by dividing the reference frequency by two.

The purpose of this section is to discuss these methods for later analysis. Moreover, they serve to motivate and illustrate how powerful the phase-lock principle is in telecommunication applications.

### 3-7.1 The Squaring Loop

The mechanization of a typical squaring loop is illustrated in Fig. 3-12.

![Fig. 3-12. The Squaring Loop.](image)

The received signal is bandpassed filtered by $H(p)$ to produce $x(t)$, given in (3-15). The signal $x(t)$, being noncoherent with respect to a sinusoid at frequency $\omega_0$, is then squared to produce a double-frequency component that is coherent with a sinusoid at frequency $2\omega_0$. The baseband terms in this product are assumed to be ideally removed by the bandpass filter (BPF) preceding the loop multiplier. This component is tracked in a PLL operating at a quiescent frequency of $2\omega_0$. Thus a coherent reference of fundamental frequency $2\omega_0$ is provided by the VCO. By dividing its frequency by two, we have established a coherent reference, say $c(t)$, at $\omega_0$.

In this case the phase of the VCO output is related to the signals illustrated in Fig. 3-12 through
\[ 2\Theta - \frac{K_v e}{p} = \frac{K_v}{p} z + 2\psi_2 = \frac{K_v F(p)}{p} (x^2 r_{eq}) + 2\psi_2 \quad (3-54) \]

where \(\psi_2\) is the VCO instabilities. Letting \(r_{eq}(t, \hat{\Phi}) = 2K_1 \sin 2\hat{\Phi}\) and selecting only the double-frequency terms in \(x^2\), we have

\[ \epsilon = x^2 r_{eq} = K_1 K_m [A_x^2 \sin 2\varphi + N_e(t, \varphi)] \quad (3-55) \]

where the equivalent phase noise in the loop is defined by

\[ N_e(t, \varphi) \triangleq (N_x^2 - N_e^2 - 2AN_e) \sin 2\varphi + 2(A - N_e)N_e \cos 2\varphi \quad (3-56) \]

Since \(\varphi = \Theta - \Theta\), we have from (3-54) and (3-55) that

\[ \phi = \Theta - \frac{KF(p)}{2p} [A_x^2 \sin 2\varphi + N_e(t, \varphi)] - \psi_2 - \frac{K_v e}{2p} \quad (3-57) \]

as the equation of loop operation. Assuming that \(A(t) = A(t + \tau)\) for all \(\tau\) of interest, the correlation function of the equivalent phase-noise process \([N_e(t, \varphi)]\)

\[ R_{N_e}(\tau) = A_x^2 R_{N_e}(\tau) + R_{N_e}^2(\tau) \quad (3-58) \]

will be of considerable interest later. As in the case of a PLL, one can discuss the various applications of the loop to coherent communication problems; however, we postpone such a discussion until we have derived the loop equation for the Costas or In Phase-Quadrature (I-Q) Loop.

### 3.7.2 The Costas or In Phase-Quadrature (I-Q) Loop

The Costas loop is illustrated in Fig. 3-13. Coherency of a carrier component at \(2\omega_0\) is established in the following manner. With reference signals

\[ r_s(t, \hat{\Phi}) = 2\sqrt{K_1} \cos \hat{\Phi} \quad r_s(t, \hat{\Phi}) = 2\sqrt{K_1} \sin \hat{\Phi} \quad (3-59) \]

the outputs \(z_e\) and \(z_s\) of \(G(p)\) can be combined to give

\[ z_0 = \frac{z_e z_s}{K_m} = K_m G(p)(x_{r_e})G(p)(x_{r_s}) \quad (3-60) \]

where \(K_m^{-1}\) is the gain assumed for the multiplier preceding the loop filter and \(K_m\) is the gain of the upper and lower phase detectors. Now the phase of the VCO output can be written in the form
Fig. 3-13. The Costas Loop.

\[ 2\Theta - \frac{K_v e}{p} = \frac{K_v z_o}{p} + 2\psi_2 = \frac{K_v K_m}{p} F(p)(G(p)x_r)(G(p)x_s) + 2\psi_2 \]  

(3-61)

Assuming that \( G(p) \) rejects the double-frequency terms in the products \( x_r \) and \( x_s \) and that \( G(p) \) passes undistorted the baseband terms in the products, then

\[ z_c = K_m G(p)x_r = \sqrt{2K_i} K_m [A - N_s] \sin \phi + N_c \cos \phi \]
\[ z_s = K_m G(p)x_s = \sqrt{2K_i} K_m [A - N_s] \cos \phi - N_c \sin \phi \]  

(3-62)

This is equivalent to assuming that the low-pass equivalent of the bandpass filter \( H(p) \) preceding the squaring loop is equal to \( G(p) \). Substituting these equations into (3-61) and noting that \( \phi = \theta - \Theta \), we have that

\[ \phi = \theta - \frac{KF(p)}{2p} [A^2 \sin 2\phi + N_s(t, \phi)] - \psi_2 - \frac{K_v e}{2p} \]  

(3-63)

for the equation of operation. From this we can conclude that when the low-pass equivalent of the presquaring bandpass filter \( H(p) \) is equal to the low-pass filter characteristic \( G(p) \), the Costas loop and the squaring loop have the same stochastic integrodifferential equations of operations. Thus the loops are said to be stochastically equivalent.

In passing, we note that by proper design the Costas loop can be used for linear-modulation extraction, tracking suppressed carrier signals, angle
demodulation, and establishing coherent subcarrier references for use in the data demodulator circuit illustrated in Fig. 3-9. In the later case one usually has to resolve the 180 degree phase ambiguity arising as a result of squaring \(x\). There are various methods for accomplishing this in practice, e.g., employ differential encoding of the data sequence (Ref. 2).

### 3.7.3 Carrier and Doppler Tracking of Suppressed Carrier Signals with Angle Modulation Present

For the case where the signal \(A(t)\) is a sequence of pulses of duration \(T\) and amplitude \(\pm \sqrt{S}\) and we wish to track the carrier, then \(\hat{\theta}\) is the loop estimate of \(d + \psi_1\) and \(\varphi = \varphi_c + M - \psi_2\). Thus (3-63) reduces to (\(e = 0\))

\[
\varphi_c = d + \psi_1 - \frac{KF(p)}{2p} \left[ S \sin \left( 2(\varphi_c + M - \psi_2) \right) + N_c(t, \varphi) \right] - \frac{K_v e}{2p} \tag{3-64}
\]

where \(\varphi_c = d + \psi_1 - \hat{\theta}\). We also note that the reference signal \(r_s(t, \hat{\Phi})\), appearing at the input to the lower phase detector of Fig. 3-13, can be used as the subcarrier reference needed in the demodulation of the modulated data subcarrier \(M = (\cos^{-1} m)X_4S\) discussed in Section 3-6.5.

For Doppler tracking, the loop equation is given by

\[
\varphi_d = d - \frac{KF(p)}{2p} \left[ S \sin \left( 2(\varphi_d + M + \Delta\psi) \right) + N_c(t, \varphi) \right] - \frac{K_v e}{p} \tag{3-65}
\]

where \(\varphi_d = d - \hat{d}\). Notice that \(\varphi_d = \varphi_c\) only when \(\psi_1 = 0\).

### 3.7.4 Linear Modulation Extraction

If \(A(t)\) is analog modulation, we can write

\[
A^2(t) = A_1(t) + \bar{A}^2 \tag{3-66}
\]

where \(A_1(t)\) is a zero-mean process. We also assume that \(M = \psi_1 = \psi_2 = 0\) without loss in generality. When the frequency spectrum of the process \(A_1(t)\) lies outside the bandwidth of the loop filter—that is, \(A_1(t)\) is a high-pass process—then the loop can be used to provide a coherent reference for extraction of the linear modulation. In fact, the signal \(z_s\) appearing at the output of \(G(p)\) in Fig. 3-13 can be passed through a Wiener filter (see Chapter 2) to produce the least-squares estimate of \(A_1\), say \(\hat{A}\). The loop equation is obtained by replacing \(A^2(t)\) in (3-63) by its mean square value \(\bar{A}^2\). When \(A(t)\) is digital modulation, we note that the data demodulator in Fig. 3-13 becomes a cross-correlator with readout every \(T\) seconds. Detailed analysis for these situations are given in Ref. 2.
Since any discussion pertaining to angle demodulation (digital or analog) or angle modulation extraction (digital or analog) by means of a Costas loop would closely parallel that in Section 3-6, we leave this as an exercise for the reader.

3-7.5 The Decision-Directed Feedback Loop

A third method for deriving a coherent reference from a suppressed carrier signal produced by digital modulation makes use of the principle of decision-directed feedback. The input signal $x(t)$ is multiplied by the reference signal $r_{0}(t, \hat{\Phi}) = \sqrt{2} K_{1} \cos \hat{\Phi}$. By inspection of Fig. 3-14, we can write

$$z = K_{m} F(p)[\hat{A}(t - T)e^{-pT}x_{r}(t, \hat{\Phi})]$$  \hspace{1cm} (3-67)

![Diagram](image)

**Fig. 3-14.** The Decision-Directed Feedback Loop.

where $T$ is the time duration of the binary signaling elements.* Carrying out the indicated multiplication by using the expressions for $x$, $r_{0}$ and delaying the result by $T$ seconds produces

$$z = K_{1}K_{m}F(p)[\hat{A}(t - T)A(t - T) \sin \phi_{T} + N[t - T, \phi(t - T)]]$$  \hspace{1cm} (3-68)

When $\hat{A}(t)$ is a sequence of pulses of amplitude $\sqrt{S}$ or $-\sqrt{S}$ and the bandwidth of the loop filter is much smaller than $1/T$, then the phase estimate can

*We assume that the relationship between the loop bandwidth and $T$ are such that the factor $\exp(i\omega T)$ is approximately unity for all $\omega$ within this loop bandwidth. There is no loss in generality in setting the acquisition voltage $e(t) = 0$ here.
be written (when the loop is tracking) as

$$\hat{\theta} = \frac{KF(p)}{p} \left\{ \sqrt{S} \left[ 1 - 2P_E(\phi) \right] \sin \phi + N_{eq}(t, \phi) \right\} \quad (3-69)$$

where $N_{eq}(t, \phi) \triangleq N[t - T, \phi(t - T)]$. Here we have neglected oscillator instabilities and assumed that the phase error is constant for several symbol times. This allows us to replace $A\hat{A}$ by its average value $E(A\hat{A}) = 1 - 2P_E(\phi)$. In (3-69), $P_E(\phi)$ is the symbol error probability given by (Ref. 2)

$$P_E(\phi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left( -\frac{x^2}{2} \right) dx \quad (3-70)$$

for phase-shift keyed (PSK) modulation. In (3-70), $R = ST/N_0$. We reiterate that this loop only responds to digital modulation and that an estimate of the carrier (subcarrier) is available for use in coherent demodulation of suppressed carrier signals at the input to the lower phase detector. The lower loop in Fig. 3-14 is used as a cross-correlation detector to produce $\hat{A}$.

### 3-7.6 Baseband Modulation Reconstruction Loops

One disadvantage of implementing the decision-directed PLL is that one must provide a delay in the loop. Moreover, symbol synchronization—that is, timing for the integrate and dump circuit—is also necessary, which in turn may require a considerable amount of time before carrier synchronization is achieved. To alleviate these two problems, one can provide fast carrier acquisition by means of the loop illustrated in Fig. 3-15. The loop operates as

---

![Diagram](attachment:baseband_modulation_reconstruction_loop.png)

Fig. 3-15. Baseband Modulation Reconstruction Loop.
follows: The incoming oscillations \( x \), given in (3-15), are multiplied by the reference

\[
q(t, \Phi) = \sqrt{2} \sin \Phi
\]  

(3-71)

to produce the signal

\[
\epsilon_1 = A \cos \varphi - [N_x \cos \varphi + N_c \sin \varphi]
\]  

(3-72)

which is applied to the ideal limiter. When the amplitude of the transmitted signal \( s(t, \Phi) \) assumes the phase-shift keyed form \( A = \sqrt{S} X_d \), where \( X_d = \pm 1 \) for \( 0 \leq t \leq T \), then the limiter output represents an estimate \( \hat{X}_d(t) \) of the modulation—that is, a sequence of \( \pm 1 \)s. The signal \( y \) is then characterized by \( u = \hat{X}_d x \) and the output of the PLL's phase detector is given by

\[
\epsilon = K_1 K_2 \hat{X}_d [\sqrt{S} \ X_d \sin \varphi + N(t, \varphi)]
\]  

(3-73)

so that the loop equation of operation is

\[
\varphi = d - \frac{K_F(p)}{p} \left[ \hat{X}_d X_d \sqrt{S} \sin \varphi + N(t, \varphi) \right] - \frac{K_v e}{p}
\]  

(3-74)

if we neglect the transreceiver oscillator instabilities. If the loop phase error is constant for several symbol times, then one can replace \( X_d \hat{X}_d \) by its average value. Thus (3-74) reduces to

\[
\varphi = d - \frac{K_F(p)}{p} \left\{ \sqrt{S} \left[ 1 - 2P_x^*(\varphi) \right] \sin \varphi + \hat{X}_d N(t, \varphi) \right\} - \frac{K_v e}{p}
\]  

(3-75)

Fig. 3-16. Signal-Modulation Reconstruction Loop.
where $P^c_e(\varphi)$ is the conditional probability of error occurring in the reconstruction of the digital data sequence. When symbol synchronization is available and carrier synchronization is achieved, one can produce the matched filter estimate $\hat{X}_d$ of the data (see Fig. 3-15).

An alternate approach is to reconstruct the modulation at baseband and remodulate the VCO reference as illustrated in Fig. 3-16. We do not pursue the development of the loop equation, for it is straightforward and left as an exercise for the reader. The main point to note is that the loop regenerates the estimate $s(t, \hat{\Phi})$ of the signal $s(t, \Phi)$.

3-8 Coherent Receivers that Exploit Coherency of the Signal's Sidebands at the Carrier Frequency for Tracking

The loops discussed in this section are motivated by the cross-spectrum concept of generalized harmonic analysis discussed in Section 3-7 and 3-7.1. In what follows, we assume, without loss in generality, that $\psi_1 = \psi_2 = 0$. For simplicity, we consider first the case of digital modulation for which $M(t)$ is given by (3-43). We generalize at a later time; however, for the single channel case, the transmitted signal assumes the form

$$s(t, \Phi_c) = \sqrt{2P_c} \sin \Phi_c + X_2 S \sqrt{2S} \cos \Phi_c$$  \hspace{1cm} (3-76)

where $P_c = m^2 A^2$, $S = (1 - m^2) A^2$, and $\Phi_c(t) = \omega_0 t + d(t)$. In the previous section we discussed various loops that are capable of tracking the phase and frequency of the sideband component. It is intuitively clear that an optimum coherent receiver, which is to coherently extract the modulation and track the carrier, should make effective use of the power (when available) in both components of (3-76); that is, it should exploit at the carrier frequency any mutual coherency power present in the sideband component of the observed signal. This requires removing the digital modulation from the second term of (3-76) in such a way that the resulting signal is phase coherent with the carrier component. There are various methods for achieving the desired coherency. In fact, all methods discussed in Section 3-7 can be used to estimate the modulation $X_d$. One method that we shall discuss is to estimate the modulation and multiply the generated sideband component by this estimate as was done in the decision-directed loop. The second method to be discussed removes the modulation by means of a nonlinear operation analogous to the Costas loop. We presently discuss these two methods of providing coherency of the sideband component at the carrier frequency. In what follows we set $e(t) = 0$ without any loss in generality.
3.8.1 Data-Aided Loops

The data-aided coherent receiver structure is illustrated in Fig. 3-17. The upper loop of the carrier tracking loop is a standard PLL (SPLL), whereas the lower loop is used to remove the modulation from the sideband \( x_d S \cos \Phi \), and inject a signal into the VCO whose strength is proportional to \( S \). In practice, the estimate \( \hat{x}_d \) of the data signal and the suppressed subcarrier estimate \( \hat{S} \) are usually available for use in the receiver. The loop equation can be written by inspection.

\[
\dot{\theta} = \frac{K_v}{p} (z_i + z_u) = \frac{K_v}{p} \left[ F_s(p)x_r + F_s(p)\hat{x}_d(t-T) \exp(-pT)\hat{S}x_r \right]
\]

(3-77)

where \( \dot{\theta} \) is the VCO phase estimate of \( \theta \) referenced to zero frequency. Characterizing the reference signals by
Coherent Receivers that Exploit Coherency of the Signal's Sidebands

\[ r_u(t, \hat{\Phi}_c) = \sqrt{2} K_1 \cos \hat{\Phi}_c \quad r_i(t, \hat{\Phi}_c) = -\sqrt{2} K_2 \sin \hat{\Phi}_c \quad (3-78) \]

and noting from (3-15) and (3-76) that

\[ x = \sqrt{2} \left[ (\sqrt{P_c} - N_i) \sin \varphi_c + (\sqrt{S} X_d S + N_c) \cos \varphi_c \right] \quad (3-79) \]

then the output \( z_u \) of the upper loop filter is given by

\[ z_u = K_1 K_{m1} F_u(p) [\sqrt{P_c} \sin \varphi_c + N(t, \varphi_c)] \quad (3-80) \]

where we have neglected the double-frequency terms and assumed that the upper loop filter \( F_u(p) \) rejects the sideband signal. Similarly, the output \( z_i \) is given by

\[ z_i = K_2 K_{m2} F_l(p) \exp(-pT) [\sqrt{S} X_d \hat{X}_d \hat{S} \sin \varphi_c + \hat{X}_d \hat{S} N_l(t, \varphi_c)] \quad (3-81) \]

where

\[ N_l(t, \varphi_c) = N_c \cos \varphi_c + N_i \sin \varphi_c \quad (3-82) \]

is the phase noise affecting the lower loop. Since \( r_i \) and \( r_u \) are orthogonal, we note that \( N_l(t, \varphi_c) \) is orthogonal to \( N(t, \varphi_c) \). In writing (3-81) we have neglected those signal terms that appear at the subcarrier frequency. This is justifiable on the basis that the filter \( F_l(p) \) will not respond to these terms. Substitution of (3-80) and (3-81) into (3-77) leads to the VCO estimate

\[ \hat{\theta} = \frac{K F_u(p)}{p} [\sqrt{P_c} \sin \varphi_c + N(t, \varphi_c)] \]

\[ + \frac{K_l F_l(p)}{p} \exp(-pT) [\sqrt{S} X_d \hat{X}_d \hat{S} \sin \varphi_c + n_l(t, \varphi_c)] \quad (3-83) \]

where \( K = K_1 K_{m1} K_r \) and \( K_r = K_2 K_{m2} K_r \) is the effective gain of the lower loop and \( n_l(t, \varphi_c) \) is the effective lower-loop baseband noise

\[ n_l(t, \varphi_c) \triangleq \hat{X}_d \hat{S} N_l(t, \varphi_c) \quad (3-84) \]

which, for all practical purposes, is statistically independent from the process that appears at the output of \( F_u(p) \). Certainly when \( n_l \) is passed through the delay, it is independent of \( N(t, \varphi_c) \). Noting that \( \varphi_c = \hat{\Phi}_c - \hat{\Phi}_c = d - \hat{d} \), then the loop equation of operation is given by
\[
\varphi_e = d - \frac{KF_e(p)}{p} \left[ \sqrt{P_e} \sin \varphi_e + N(t, \varphi_e) \right] \\
- \frac{K_i F_i(p)}{p} \exp(-pT) \left[ \sqrt{\mathcal{S}} X_d \hat{X}_d \mathbf{S} \mathbf{S} \sin \varphi_e + n_l(t, \varphi_e) \right] \quad (3-85)
\]

In practice, the bandwidth of loop filter \( F_i(p) \) is usually small in comparison to \( 1/T \) and the subcarrier estimate is nearly perfect—that is, \( \hat{\mathbf{S}} \approx \mathbf{S} \). When this is true, the phase error can be assumed to be constant for several symbols and the product \( X_d \hat{X}_d \) can be replaced by its average value \( E(X_d \hat{X}_d) = 1 - 2P_e(\varphi_e) \). Here \( P_e(\varphi_e) \) is the conditional symbol error probability given by (3-70) for phase-shift keying. Consequently, (3-85) can be approximated by

\[
\varphi_e = d - \frac{KF_e(p)}{p} \left[ \sqrt{P_e} \sin \varphi_e + N(t, \varphi_e) \right] \\
- \frac{K_i F_i(p)}{p} \left[ \sqrt{\mathcal{S}} \left[ 1 - 2P_e(\varphi_e) \right] \sin \varphi_e + n_l(t - T, \varphi_e) \right] \quad (3-86)
\]

This result is valid when the loop is locked. When \( F_i(p) = F_e(p) = F(p) \), then

\[
\varphi_e = d - \frac{F(p)}{p} \left( \left[ K \sqrt{P_e} + K_i \sqrt{\mathcal{S}} \right] \left[ 1 - 2P_e(\varphi_e) \right] \right) \sin \varphi_e + n_e \quad (3-87)
\]

where \( n_e \triangleq KN(t, \varphi_e) + K_i n_l(t - T, \varphi_e) \) is the equivalent loop phase noise. When the correlation time of \( \varphi_e \) is small, compared with that of the channel noise process, then \( n_e \) is well approximated by a “white” Gaussian noise process with spectral density

\[
S_n(\omega) \approx \frac{N_0}{2} (K^2 + K_i^2) \quad (3-88)
\]

Note now that the equivalent loop \( S \)-curve

\[
S(\varphi_e) \triangleq \left( K \sqrt{P_e} + K_i \sqrt{\mathcal{S}} \right) \left[ 1 - 2P_e(\varphi_e) \right] \sin \varphi_e \quad (3-89)
\]

consists of two terms; the first term is of the sinusoidal PLL type, while the second term is identical with that obtained in a decision-directed feedback loop. The gain \( K \) should be adjusted in direct proportion to \( \sqrt{P_e} \) while \( K_i \) is made proportional to \( \sqrt{\mathcal{S}} \). Thus when \( P_e = 0 \), the PLL portion of the data-aided loop opens up; and when \( \sqrt{\mathcal{S}} = 0 \), the decision-directed branch opens. Consequently, as \( m \) is varied, the loop operates as a linear weighting of a standard PLL and a decision-directed loop. When \( M(i) \) consists of multiple data-modulated subcarriers [e.g., (3-47)], coherency of multiple sidebands at the carrier frequency can be exploited by means of multiple loops.
3-8.2 Hybrid Carrier and Modulation Tracking Loops

The hybrid loop establishes coherency of the sidebands at the carrier frequency by employing the Costas loop principle at low SNR and the baseband modulation reconstruction principle at high SNR. This loop is illustrated in Fig. 3-18. The operator $Q(\cdot)$, linear or nonlinear, is inserted for the sake of generality. When $Q(x) = x$, the outer loop performs operations on the sideband component of (3-76) much like the Costas loop of Fig. 3-13. When $Q(x) = \text{sgn} x$—that is, $Q(x)$ is an ideal limiter—then the outer loop reconstructs the modulation $X_d(t)$ at baseband much like the loop in Fig. 3-15. In general, one might set

$$Q(x) = \tanh x \approx \begin{cases} \text{sgn} x & |x| \gg 1 \\ x & |x| \ll 1 \end{cases}$$  \hspace{1cm} (3-90)$$

to approximate a “soft-limiter” characteristic. In what follows, we develop the equation of operation when $Q(x) = x$. 

Fig. 3-18. The Hyrid Loop.
By inspection of Fig. 3-18, we see that the phase estimate $\hat{\theta}$ of the VCO output is related to $z$ and $z_0$ through

$$\hat{\theta} = \frac{K\gamma}{p} (z + z_0)$$

$$= \frac{K\gamma}{p} (K_{1m}F(p)x_{ru} + K_{1m}K_{2m}F_1(p)G(p)\hat{S}x_{ru}[G(p)\hat{S}x_{ri}]) \quad (3-91)$$

where the "m" subscript on the $K$'s denotes multiplier gains. The first term in (3-91) is due to inner PLL loop, whereas the second term is attributable to the squaring operation familiar in the Costas loop. Now

$$x_{ru} = K_1K_{1m}[\sqrt{\frac{P_c}{C}} - N_i] \sin \varphi_c + (\sqrt{\frac{S}{2}} X_d S + N_r) \cos \varphi_c$$

$$x_{ri} = K_2K_{2m}[-(\sqrt{\frac{P_c}{C}} - N_i) \cos \varphi_c + (\sqrt{\frac{S}{2}} X_d S + N_r) \sin \varphi_c] \quad (3-92)$$

Assuming that $\hat{S} \approx S$ and using (3-92) in (3-91) gives

$$\hat{\theta} = \frac{KF(p)}{p} (\sqrt{\frac{P_c}{C}} \sin \varphi_c + N(t, \varphi_c))$$

$$+ \frac{K_1F_1(p)}{p} \left[ \frac{S}{2} \sin 2\varphi_c + N'(t, \varphi_c) \right] \quad (3-93)$$

where $K_1 = K_2K_{2m}K$ and

$$N'(t, \varphi_c) = n_1n_2 + X_d\sqrt{S} (n_1 \sin \varphi_c + n_2 \cos \varphi_c) \quad (3-94)$$

Here, $n_1$ and $n_2$ are orthogonal, band-limited Gaussian noise processes with spectral densities well approximated for carrier-tracking applications by

$$S_n(\omega) = S_m(\omega) = \frac{N_0}{2} |G(i\omega)|^2 \quad (3-95)$$

In arriving at (3-93), we have assumed that $\varphi_c$ is slowly varying relative to subcarrier frequency. We also note that $N'(t, \varphi_c)$ is spectrally equivalent to the equivalent phase noise that arises in the Costas and squaring loops. Furthermore, the processes $N(t, \varphi_c)$ and $N'(t, \varphi_c)$ are essentially statistically independent, for they correspond to different frequency bands of the input wideband noise spectrum. Assuming that $F(p) = F_1(p)$ and noting that $\varphi_c = \theta - \hat{\theta} = d - \hat{\theta}$, we have

$$\varphi_c = d - \frac{F(p)}{p} \left[ K \sqrt{\frac{P_c}{C}} \sin \varphi_c + \frac{GS}{2} \sin 2\varphi_c \right] + n_e(t, \varphi_c) \quad (3-96)$$
where \( G \triangleq K_t/K \) and

\[
\eta_r(t, \varphi_r) = K[N(t, \varphi_r) + GN(t, \varphi_r)]
\]

One principal point to note is that the shape of the equivalent loop \( S \)-curve [the term in braces of (3-96)] is modified by the addition of a double-harmonic term that serves to make the \( S \)-curve more “rectangular.” As we shall see, the optimum loop \( S \)-curve, for zero loop detuning, is one that is “rectangular” when the additive noise is white and Gaussian. We also note that the \( S \)-curve for the hybrid loop is analogous to the electromagnetic power developed by a salient-pole synchronous machine (Ref. 1). As we shall later see, the phase error process is generated by motor hunting due to a randomly varying load. In such a motor, the power generated by the term \( \sin 2\varphi_r \) is due to the fact that the direct axis of in-phase synchronous reactance differs from the quadrature-axis synchronous reactance.

As before, the gain \( K \) of the inner loop and \( K_t \) of the outer loop should be made proportional to the carrier and sideband powers respectively. Thus as

Fig. 3-19. Combined Data-Aided and Hybrid Loop.
m varies between zero and one, the hybrid loop operates as a conventional PLL when m = 0 and a Costas loop at m = 1. It is for this reason that the loop has been referred to as the hybrid loop.

Finally, we note that the hybrid loop and data-aided loop can be combined to produce the loop of Fig. 3-19. Due to the delay and the low-pass filters, the three noise components appearing at the input to the VCO are independent. Development of the loop equation is left as an exercise for the reader.

3-8.3 Elimination of the Subcarrier Reference in Hybrid Loops

Development of the loop equation (3-96) has assumed a perfect subcarrier reference in the receiver. Inclusion of the subcarrier reference jitter can easily be carried out in order to see how this jitter would ultimately affect loop performance. More important is the possibility of eliminating the provision for a subcarrier reference. This simplifies the loop mechanization as well as the signal acquisition procedure; that is, the signal can be acquired without opening the lower loop until the radio frequency (RF) carrier is acquired; acquiring the subcarrier and then closing the lower loop.

When the subcarrier references are omitted, the corresponding simplified loop is shown in Fig. 3-20. In this figure the filters \(H(p)\) represent bandpass subcarrier extraction filters. The loop equation is now given by (3-96);
however, \( N'(t, \varphi_c) \) must be redefined as

\[
N'(t, \varphi_c) \triangleq 2n_1n_2 + X_d\sqrt{S} \left( n_1 \sin \varphi_c + n_2 \cos \varphi_c \right)
\]  

and loop performance is obviously inferior, since we get twice as much noise appearing in the cross-product term.

3.9 Pseudo-Noise Tracking Receivers and Coherent Transponders

As already pointed out, one of the basic problems encountered in various telecommunication systems is tracking. There are two aspects to tracking. One is concerned with keeping directional antennas pointed and the receiver tuned. The other has to do with obtaining information from which present and future positions of the space vehicle can be determined. In deep-space communications this information is typically some combination of angle, Doppler, and range data. On the other hand, the standard techniques for determining the orbits of satellites use one or more of three major data types—topocentric range, topocentric range rate, and topocentric angle data. Topocentric range is determined by measuring the round trip time of a radio signal to and from the satellite. Topocentric range rate is determined by measuring the Doppler shift in the signal. Finally, topocentric angles are measured optically or, somewhat more crudely, by measuring antenna pointing angles at the receiving antenna.

Phase-coherent tracking and communication systems often have to contend with interference and multipath problems, as well as additive noise. In certain cases the transmission of a large amount of power can be used to overcome the problem; however, there are certain practical situations (e.g., Tracking Delay Relay Satellites Systems) where increased power does not overcome the deleterious effects due to multipath. Consequently, signal design must be considered for possible solutions to these problems.

In the past, pseudo-noise (PN) systems employing signal design theory have been used successfully to (1) combat interference, (2) combat fading due to multipath as in Rake system, (3) provide ranging, (4) provide symbol synchronization, (5) provide addressing, (6) perform radar and sonar measurements, (7) provide interferometry measurements, and (8) provide secure links. Thus we shall discuss PN systems since they have such a broad range of applications. A pseudo-noise system makes use of a shift register that is connected so that the outputs of its last stage and one or more of its previous stages are modulo 2 added and inputted to the first stage (see Fig. 3-21). We note that the minus one should be considered to be zero for the modulo 2 addition. Under certain conditions (Refs. 4 and 30) the PN generator will provide a sequence of ±1’s of length \( L = 2^n - 1 \) before repeating itself. For all practical purposes, this
sequence will be a Markov process of arbitrary plus and minus ones having much the same characteristics as wideband noise; hence the name pseudo-noise generator (PNG). This generator, being driven by a clock (CL), oscillates freely except when the \( n \)-stage shift register is initially filled with all zeros. Now the sequence of \( \pm 1 \)'s that the PNG provides is cyclic and is periodic with period \( L = (2^n - 1) \) symbols. This gives rise to the normalized autocorrelation function of the PNG output shown in Fig. 3-22. The spectral density, being the Fourier transform, of this autocorrelation function is given by

\[
S_{PN}(\omega) = \frac{L + 1}{L^2} \left( \frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}} \right)^2 \sum_{n=-\infty}^{\infty} \delta \left( \omega - \frac{2\pi n}{LT} \right) + \frac{1}{L^2} \delta(\omega) \quad (3-98)
\]

where \( L \) is the period of the PN sequence, and \( 2T \) is the clock period and \( \text{sinc} x \triangleq \sin x/x \).

![Fig. 3-22. Autocorrelation Function of a PN Sequence.](image)

There are three points to be noted about such a spectrum. First, it is a line spectrum with frequencies at multiples of the fundamental frequency and
the power in the dc component goes to $1/L^2$. Second, because the PN waveform is a constant amplitude square wave with constant power, there is a scale factor inversely proportional to the period of the sequence. Thus if the period of the sequence is doubled, the lines in the spectrum become twice as dense and the power in each is reduced by a factor of 2. Third, the envelope of the spectrum is determined by the clock period. This means that the bandwidth required to transmit the PN sequence is determined solely by the clock time $T$. In fact, the first null in the spectrum occurs at $f = 1/T$. For more details pertaining to the properties and generation of PN sequences, the reader is referred to Refs. 4 and 30.

We now present the algebra required for operations of PN sequences, clocks (CL) and square-wave carriers (CA). It is obvious that the multiplication (denoted $\otimes$) of two synchronized PN sequences, clocks, or carriers is equal to unity or

\[
\begin{align*}
\text{PN} \otimes \text{PN} &= 1 \\
\text{CL} \otimes \text{CL} &= 1 \\
\text{CA} \otimes \text{CA} &= 1
\end{align*}
\]

(3-99)

for all time. The fundamentals have now been laid to demonstrate PN system synchronization characteristics. This will be done first by discussing the PN system synchronization without a carrier.

### 3-9.1 The Double Loop PN Tracking System

Since replicas of the PN sequence can be generated at spatially remote locations, it is of interest to be able to phase lock or synchronize the two generators for the various applications mentioned earlier. Consider the transmitter-receiver pair illustrated in Figs. 3-23, 3-24, and 3-25. At the transmitter, the PN is multiplied by the clock that drives the PN generator (see Fig. 3-23). The output PN $\otimes$ CL still has the properties of a PN sequence. At the receiver,
the first operation is to multiply the received signal by the locally generated PN sequence (code), as illustrated in Fig. 3-24. If the received signal is synchronous with the locally generated PN code, then the multiplier output becomes

\[
\text{PN} \times \text{CL} \times \text{PN} = \text{CL} \quad (3-100)
\]

Thus the receiver clock output is in synchronization with the clock at the transmitter. Since the clock signal is periodic, a PLL may be locked to this signal and used to drive the PNG at the receiver. Since the synchronized oscillator in a PLL is 90 degrees out of phase with the synchronizing signal, a 90-degrees phase shift must be used in the receiver so that the PNG is driven with the proper phase. The complete PN synchronization system is illustrated in Fig. 3-25. The open-loop S-curve seen at the first multiplier output of the receiver is shown in Fig. 3-26. Two different polarity S-curves are seen to be generated, only one of which corresponds to the correct feedback polarity (stable point). The unstable point arises because the PN code length is an odd number of clock cycles. In spatially remote operating systems, this second unstable point causes difficulties in phase locking but in practice can be removed by code switching methods (Refs. 2, 4). The method used switches the PN code polarity between alternate PN cycles. The resulting code is twice the length of the original PN code although it has lock points separated by twice the original length.
It is clear from Fig. 3-25 that the inner loop (closed through the 90-deg shift of the clock VCO) is a PLL which is designed to track the clock. The outer loop (closed via the PNG) forces the locally generated code to follow the clock. If the two codes are exactly in phase and if the VCO output is exactly in phase with the incoming clock, then there is no dc signal at the output of the second receiver multiplier. If the incoming code and clock tend to shift with respect to the local code and clock, then there is a dc signal at the output of the second multiplier that causes the clock VCO to shift frequency and brings the local clock, and hence the local code, back into phase with the incoming signal. This two-loop tracking device is analogous to a situation where an internal organ in an organism is locked to internal generated rhythms (the inner loop) and the outer loop is locked to the environment. The system of Fig. 3-25 can be combined with a phase-coherent transponder (discussed in Chapter 4) to produce a ranging receiver at radio frequencies; it can also provide for Doppler measurements while simultaneously extracting commands or telemetry and voice.

3.9.2 The Delay-Locked Loop

A block diagram of the delay-locked loop (DLL) that employs the phase-lock principle is illustrated in Fig. 3-27. Applications of the DLL include active ranging (radar and sonar), angle tracking (interferometry), the tracking of a research submersible vehicle exploring the ocean floor, and as a synchronization (subcarrier and symbol) device in telecommunication systems.

![Diagram of the Delay-Locked Loop (DLL)](image-url)
The essentials of a DLL is a local \( n \)-stage shift register that generates time-displaced versions of a PN sequence. Each of the reference versions \( PN(t - \hat{t} - T) \) and \( PN(t - \hat{t} + T) \), which have amplitudes \( \pm 1 \), is multiplied by received signal \( APN(t) + n(t) \), where \( A^2 \) is the power in the received pseudo-noise sequence. The difference between the multiplier output signals is obtained by a subtraction device that, together with the multipliers, forms part of the phase detector or cross-correlation network. Using the properties of PN sequences, it can easily be shown (see Fig. 3-27) that the contribution of these sequences to the long time average of the output of the cross-correlation network is proportional to

\[
S(\tau) = R_{PN}(\tau - T) - R_{PN}(\tau + T) \tag{3-101}
\]

which is the loop phase-detector characteristic or \( S \)-curve.

3-10 The Digital Phase-Locked Loop (DPLL)

The development in the last few years of low power, medium scale integration (MSI) and large scale integration (LSI) digital logic elements has made digital mechanization of a PLL technically and economically feasible in many applications. Also, the availability of a low cost PLL system in a monolithic circuit package is now emerging as a new and versatile building block, similar to the monolithic operational amplifier in the diversity of its applications (Ref. 3-32). Therefore, the essential functional blocks of phase detection, loop filter, and VCO in a PLL (see Fig. 3-1) may each be performed in a strictly digital manner while analog-digital hybrid combinations may be applicable to satisfy special constraints. Many configurations may arise in a practical DPLL depending upon the applied signal characteristics and applications. References (3-32) to (3-43) serve as an initial step towards a more comprehensive study of the DPLLS than that given here.

The usual advantages of digital mechanization which may be ascribed to a DPLL are (1) parameter stability and accuracy, (2) improved reliability, and (3) compatibility with a growing number of interfacing digital subsystems. In addition, initial phase and frequency estimates can be inserted into the loop so that one loop could conceivably be time-shared for several inputs. Reduced size and cost are not automatically achieved by the use of digital techniques, but the possibility of LSI implementation holds the promise of these further benefits.

A DPLL may be designed using digital circuitry, digital multipliers, adders, digital filters, and core memory. Although some of the problems with analog domain implementation that results due to VCO nonlinearity, phase detector inaccuracy, saturation of components, and unstability of higher-order systems can be eliminated, new phenomena common to digital systems become
apparent. These include quantization noise, round-off error, and overflow errors (Ref. 3-33).

3-10.1 Digital Techniques for Mechanizing Loop Functional Blocks

(a) Voltage Controlled Oscillator. The PLL requires a functional block providing periodic properties modifiable by an active control input. A phase vector must be generated with a known rate of change (frequency) and a known absolute phase at some instant. An accumulator (accum) register, equivalent to a continuous integrator at sample instants, is a digitally realizable “oscillator” function.

The accumulator generates a monotonically increasing vector (modulo 2, meaning we ignore the count of integer cycles, just as we measure phase modulo $2\pi$ radians) as the center frequency constant is added at each clock time; the overflow rate is the output frequency, and the content of the register represents the phase state of the “oscillator.” The phase is controlled by an additional additive input, which is used to linearly increment or decrement the phase state at each clock time. An alternative or extension to the incrementable (decrementable) divider is a selectable-tap delay line. This digital “oscillator” technique inherently is limited to a finite phase resolution set by the clocking rate (proportional to $2^{-k}$ where $k$ is the number of bits). This quantization noise is usually designed to be small compared with other noise sources.

A hybrid alternative is to use a digital-to-analog (D/A) converter and analog VCO, or a digitally controlled frequency synthesizer. The former may be adequate in many simple signal-processing applications while the latter is particularly applicable to computer-implemented hybrid systems.

(b) Phase Detector. The error-summing junction of a DPLL is the phase detector, which provides a measure of the phase error between input and output oscillator. There are three primary parameters that control phase detector design: (1) signal waveform, for example, code or sinusoid, cw or modulated, (2) received signal bandwidth, and (3) dynamic range and signal-to-noise ratio.

The signal definition determines the basic structure, for example, a sinusoidal correlator for a sinewave signal, or an early-late correlator for bilevel coded signals, or an I-Q demodulator for suppressed-carrier signals. Any of these structures can be implemented digitally with sampling at a rate satisfying the Nyquist criterion. Wideband signals require extremely high sampling rates, and consequently many bits of processing logic to achieve high processing gain. Cascaded stages of processing at successively lower sample rates may be used in some wideband systems. The sampling in the wideband preprocessor may be at a constant clock rate using semi-classical correlation techniques; however, in the narrowband case, the phase detector sampling may be preformed at the signal frequency. The dynamic range and signal to noise ratio influence
the selection of quantization granularity; generally, the quantization is chosen to be negligible compared to all other noise effects. The degradation with additive random Gaussian noise is only 1.96 dB for the limiting case of one-bit analog-to-digital (A/D) conversion, that is, hard limiting for a single channel; this simple digitization is adequate in many applications. A hybrid alternative for wideband systems is to perform analog correlation processes and digitize the narrowband results.

(c) Loop Filter. In a DPLL, the loop filter conditions the phase error signal (by scaling, low pass filtering, and integration) to form the oscillator control voltage. For a first-order DPLL, simple scaling (right shift for attenuation in powers of two) is required. A digital, second-order loop filter is implemented as a sum of a direct scaled term and an accumulator term (the latter is analogous to the integral term of the active analog high gain integrator).

### 3-10.2 Example Mechanization of a DPLL

An example of an all-digital PLL is the configuration indicated in Fig. 3-28. This is a direct digital analogue of a second-order loop. The “controlled oscillator” function is implemented by a phase accumulator, where the content, \( P \), represents the phase vector of center frequency \( 1/K_3 T \) cycles per second. Phase detection is performed by simple sampling of the narrowband noise corrupted input sinusoid, using and A/D converter strobed once per cycle of the phase accumulator. Loop filtering is effected by a proportional-plus-accumulated operation. The loop parameters are controlled in the loop filter while

---

**Fig. 3-28.** Mechanization of a Digital Phase-Locked Loop.
digital mechanization permits setting the scaling by shifting the digital words or altering clock rates.

A number of digital, symbol (bit) synchronization PLLs have been described in literature for various applications, (Refs. 3-36 to 3-39). Notable among these are digital-data transition tracking by means of a decision directed, phase-tracking loop by Anderson and Lindsey (Ref. 3-36), a digital symbol synchronizer for low signal-to-noise ratio coded systems by Hurd and Anderson (Ref. 3-37) and a digital, bit synchronization PLL employing binary phase error quantization and sequential loop filtering by Cessna (Ref. 3-38). References (3-40)–(3-41) give theoretical and experimental results for carrier synchronization of PSK modems for TDMA Satellite Communication Systems. Pasternack and Whalin (Ref. 3-42) give the analysis and synthesis of a DPLL for FM demodulation application in low speed data set. Schilling, et. al., Ref. 3-33, have preliminary analysis for demodulating several FM signals simultaneously. Holmes (Ref. 3-43) also gives an analysis of a DPLL for use in sub-carrier tracking.

With the advent of monolithic integrated circuit (IC) technology, the digital mechanization of the PLL should find further applications in synchronization, demodulation, frequency synthesis functions and in many digital system designs.

3-11 Unified Loop Equation for Synchronous Control Systems

It is difficult to give a general theoretical analysis of nonlinear systems because separate categories of nonlinear systems must be studied by special methods that are frequently applicable in restricted regions. Here we have seen that: (1) various loops give rise to various nonlinearities (S-curves or phase-detector characteristics), (2) the equation of each loop is characterized by a nonlinear stochastic integrodifferential equation, (3) the noise source appearing in each equation depends on the loop configuration and application, and (4) the loop nonlinearity is periodic and continuous. Table 3-1 summarizes the various nonlinearities and the equivalent phase noise source $n_p(t)$ which have been discussed thus far. We note that we have assumed a common loop filter for the data-aided and hybrid loops and have dropped the subscript on the phase error variable. This table is certainly not exhaustive for it should be pointed out that most symbol (bit) synchronization systems (Ref. 2) which use the phase-lock principle can also be characterized by a nonlinear, stochastic integrodifferential equation possessing the above properties. Consequently, this suggests that we can characterize the behavior of a broad class of carrier (subcarrier) tracking loops, suppressed carrier (subcarrier) tracking loops, phase-coherent demodulators of analog and digital signals, and symbol synchronization systems (Ref. 2) via the unified stochastic integrodifferential equation.
of operation written in the shortened form

\[
\phi = \theta - \frac{KF(p)}{p} [Ag(\phi) + n_g(t)] - \frac{K_v e}{p} - \psi_2
\]  

(3-102)

Here \(g(\phi)\) is the system’s S-curve or phase-detector characteristic and \([n_g(t)]\) represents the equivalent phase noise process which, in general, may depend upon \(g(\phi)\). Literally (3-102) implies the equation

\[
\frac{d\phi(t)}{dt} = \frac{d\theta(t)}{dt} - K \int_0^t f(t - \lambda)[A(\lambda)g(\phi(\lambda)) + n_g(\lambda)]d\lambda - K_v e - \frac{d\psi_3(t)}{dt}
\]

(3-103)

where \(f(x)\) is the impulse response of the loop filter. The mathematically equivalent loop model, that is, the baseband loop model, is illustrated in Fig. 3-29. Notice also that \(y = F_0(p)z\) where \(F_0(p)\) is the transfer function of the output filter. From Fig. 3-29 we write

\[
y = K_1 K_m F_0(p) F(p) [Ag(\phi) + n_g]
\]

(3-104)

which relates the output \(y(t)\) to other system parameters. We shall refer to

<table>
<thead>
<tr>
<th>Mechanization</th>
<th>(g(\phi))</th>
<th>(n_g(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sinusoidal PLL</td>
<td>(\sin \phi)</td>
<td>(\frac{N(t, \phi)}{A})</td>
</tr>
<tr>
<td>Costas loop or squaring loops</td>
<td>(\sin 2\phi)</td>
<td>(\frac{N_e(t, \phi)}{A^2})</td>
</tr>
<tr>
<td>Decision-directed feedback loops</td>
<td>([1 - 2P_{E}(\phi)] \sin \phi)</td>
<td>(\frac{N_{ee}(t, \phi)}{\sqrt{S}})</td>
</tr>
<tr>
<td>Data-aided loops</td>
<td>(1 + \frac{K_i}{K} \sqrt{\frac{1 - m^2}{m^2} [1 - 2P_{E}(\phi)]}) (\sin \phi)</td>
<td>(\frac{KN(t, \phi) + K_i n_i(t, \phi)}{K \sqrt{P_c}})</td>
</tr>
<tr>
<td>Hybrid loops</td>
<td>(\sin \phi + \frac{K_i}{K} \frac{S}{\sqrt{P_c}} \sin 2\phi)</td>
<td>(\frac{KN(t, \phi) + K_i n_i(t, \phi)}{K \sqrt{P_c}})</td>
</tr>
<tr>
<td>Double-loop tracking device</td>
<td>See Fig. 3-26.</td>
<td>(\frac{N(t, \phi)}{A}) (L \gg 1)</td>
</tr>
<tr>
<td>Delay-locked loop</td>
<td>(R_{PN}(\tau - T) - R_{PN}(\tau + T))</td>
<td>(\frac{N(t, \phi)}{A}) (L \gg 1)</td>
</tr>
<tr>
<td>(n)th-order Tanlock</td>
<td>(\left[rac{(1 + k)^n \sin \phi}{(1 + k \cos \phi)^n}\right] 0 \leq k &lt; 1)</td>
<td></td>
</tr>
</tbody>
</table>
Further Studies

(3-102) or (3-103) and (3-104) as the equations of operation of a *Synchronous Control System* (SCS).

3-12 Further Studies

The history of the phase-lock principle is long and interesting. Consequently, no attempt is made here to cite the several hundred papers where various results appeared for the first time. Any attempt to make systematic references to sources would be likely to entail errors and omissions, unless one wishes to become involved in special bibliographic research and name-calling contests. The books by Gardner (Ref. 5) and Viterbi (Ref. 6) contain a rather thorough list of the work produced by American researchers up to 1966. Unfortunately, reference to most of the excellent work accomplished in the Russian literature is omitted there; however, some of the major contributors and references are cited below and in the book by the Russian authors Skakhgilidian and Lyakhovkin (Ref. 7). Since it is not our intent to cite here all the work in this field, we tabulate at this point some of the major efforts that, in the author’s opinion, have lead to the development of the theory.

As far as the author can tell, the first observations of the phase-lock principle were made by Huygens in 1665 when he noticed two pendulum clocks in long-term synchrony far beyond their capacity, as mechanisms, to be matched in accuracy. He assumed a sort of “sympathy” between the two clocks side by side on the wall and pursued experiments to determine how interaction took place. He found out that it was through the mutual support (i.e., the walls on which the clocks hung), and for the first time a physical explanation for mutual phase locking of two oscillators emerged. It seems highly probable that Huygens did not realize the important role that phase locking would play in future biological theory and physiology. Nor did he realize how important the phase locking of oscillations would become in telecommunication and power-distribution systems.
Mathematically speaking, the origins of the phase-lock principle date back to the 1920s and 1930s, for it was in the early 1930s that the fundamentals of synchronization theory were established. This early work was carried out by Van der Pol (Ref. 8) and by Andronov and Witt (Ref. 9). Subsequently, a major contribution to the development was made by the Soviet scientists L. I. Mandel'shtam, N. D. Papaleksi, N. N. Bogolyubov, S. M. Rytov, V. V. Migulin, Yu B. Kobzarev, S. I. Evtyanov, R. V. Khokhlov, and others. A complete review of this early research is given in Ref. 10, while the original work due to Van der Pol will be discussed in Chapter 5. A direct continuation of the earlier work due to Andronov, Pontryagin, and Witt (Ref. 11) was a series of theoretical and experimental papers by I. H. Bershtein, discussed in Ref. 12.

Apparently the first description of a PLL was published by de Bellescize (Ref. 13) in 1932 where he treated the problem of coherent reception of radio signals. Superheterodyne receivers had come into use during the early 1920s, but there was a continual search for simpler techniques. The first serious application of the concept began as a horizontal-line synchronizing device for television in the 1940s (Refs. 14, 15, 16). Shortly thereafter, Jaffe and Rechtin (Ref. 17) showed how a PLL could be used as a tracking filter for a missile beacon. The first analysis (linear) including the effects of noise appeared in a paper published by Jaffe and Rechtin (Ref. 17) in 1955. The baseband loop model in Fig. 3-4 is essentially due to Develet (Ref. 18) although the statistical properties of $N(t, \varphi)$ were not fully explored and understood. In the absence of noise and oscillator instabilities, the model had been used by several authors, notably Gruen (Ref. 19).

Since development of the linear PLL theory did not adequately characterize the behavior of a PLL, a continual effort was expended on determining its nonlinear behavior. Viterbi (Ref. 20) was apparently the first American author to consider the problem in the absence of noise, while the Russian authors Stratunovich (Ref. 21) and Tikhonov (Refs. 22, 23) worked on determining the nonlinear behavior of a first-order loop in the presence of noise. The results of Stratunovich (Ref. 21) were substantially expanded and experimentally verified by I. G. Akopyan (Ref. 10). Subsequently, Viterbi (Ref. 24) extended and complemented Tikhonov's and Stratunovich's earlier work for a first-order loop. Lindsey and Charles (Ref. 25) extended, with some success, the earlier work on a first-order loop in noise to that of second-order loop. These extensions were complemented with a body of experimental results. With regard to the mathematical formalism employed by these authors, the main papers can be divided into two groups. One group used the Fokker-Planck-Kolmogorov equation and the other used the small-parameter method. Since the occurrence of the Jaffe–Rechtin paper, various approximate new theories, refinements, extensions, and new applications were made. These analyses have not, however,
been of mere academic interest, for they have paved the way toward explaining
the behavior of PLLs as well as for building the most sensitive and flexible
receivers in the world today. It appears that the Russian A. A. Pistol'kors was
the first to suggest the extraction of a sync reference from the information
bearing signal (Author’s Certificate 34039, April 24, 1932).

In passing, we mention the approximate theories, aside from the linear
theory, of Develet (Ref. 18) based on Booton’s (Ref. 26) describing function
method, Van Trees perturbation method using Volterra kernels (Ref. 27), the
work of Margolis (Ref. 28), and a later development given by Tausworthe
(Ref. 29). These theories were tailored to the problem of computing the steady-
state variance of the phase error and will be discussed in Chapters 9 and 10.

Finally, we mention the book by Stiffler (Ref. 31), which discusses
various method for establishing and maintaining synchronization. Moreover,
the text by Lindsey and Simon (Ref. 2) applies the theory developed in this
book to analyze and characterize the performance of the various synchronous
control systems discussed in this chapter.

Problems

3-1 (a) Using (3-17) and (3-23), show that an equivalent expression for the
additive phase noise is given by

\[ N(t, \varphi) \triangleq n'(t) = n_e(t) \cos \Theta(t) + n_v(t) \sin \Theta(t) \]

(b) Compute the correlation function \( R_n(\tau) \) based on (3-13) and (3-34).
(c) Devise an argument and conditions whereby \( \{n'(t)\} \) is approximately
white.

This noise model is used by Viterbi, Ref. 6.

3-2 Frequently, in the practical design of a PLL to be used as a tracking loop in
the phase-coherent receiver of a spacecraft, a configuration different from the
model of Fig. 3-2 is used. The question immediately arises as to the equiva-
ience in performance among the various topologies. Consider first the PLL
illustrated in Fig. P3-2.1. It should be intuitively obvious (and can be shown)
that if the external reference generator at frequency \( \omega_0 - \omega_1 \) is noise free, the
additional demodulation done at this frequency should in no way affect loop
performance in the presence of noise. Perhaps not so obvious is the configu-
ration shown in Fig. P3-2.2, where the additional demodulation reference is
derived from the VCO output. Since the two demodulation references—that
is, the frequencies \( \omega_1 \) and \( \omega_0 - \omega_1 \)—are coherent, show that the stochastic
differential equations of operation of Fig. P3-2.2 is identical to Fig. P3-2.1 if
the loop parameters are defined appropriately. The advantage of the configu-
nation in Fig. P3-2.2 over that of Fig. P3-2.1 is the saving in an external fre-
quency synthesizer at the expense of a frequency multiplier. Neglect any

3-3 Suppose that the input to a PLL is of the form

\[ x(t) = \sqrt{2} \left[ A_1(t) \sin \Phi_1(t) + A_2(t) \sin \Phi_2(t) \right] + n(t) \]

where

\[ \Phi_1(t) = \omega_0 t + \theta_1(t) \quad \text{and} \quad \Phi_2(t) = \omega_0 t + \theta_2(t) \]

(a) Find the loop equation of operation.
(b) Where might such a situation arise in practice?

3-4 When \( \theta_1(t) = d(t) + M(t) \), of Prob. 3-3, is the desired signal and \( A_1(t) \) and
$A_2(t)$ are both constant, define the loop phase error for the case of (a) carrier tracking, (b) Doppler tracking, and (c) modulation tracking. Write the resulting loop equation of operation for these three cases.

3-5 If the loop, in Probs. 3-3 and 3-4, is used for modulation extraction and

$$M(t) = (\cos^{-1} m)X_a(t)S(t)$$

with $X_a$ and $S(t)$ are sequences of $\pm 1$. Develop the expression for the signal that feeds the data demodulator of Fig. 3-9.

3-6 Derive the expressions (3-58) for $R_{X_a}(\tau)$ in the Costas and squaring loops. (*Hint: See Prob. 1-2.)*

3-7 For the decision-directed feedback loop, show that $E[AA^\dagger] = S[1 - 2P_e(\varphi)]$ when $A(t) = \sqrt{S}$ or $A(t) = -\sqrt{S}$ for $0 \leq t \leq T$. Assume that the binary symbols are equiprobable, that the loop is tracking and the phase error is constant for several symbol times.

3-8 Derive the equation of operation for the signal-modulation reconstruction loop of Fig. 3-16. Compare this equation with that of the baseband modulation reconstruction loop of Fig. 3-15.

3-9 Show, by means of a block diagram, how a decision-directed feedback loop can be used in a data-aided loop mechanization when $\theta(t) = d(t) + M(t)$ is defined in (3-47). Assume that $N = 0$ and $L = 2$.

3-10 Show, by means of a block diagram, how the hybrid loop becomes a multiple-loop device when $M(t)$ consists of multiple subcarriers given in (3-47). Assume that $N = 2$ and $L = 0$.

3-11 By means of block diagrams, construct loops that recover the cross-modulation term when $N = 0$ and $L = 2$ in (3-47).

3-12 Develop the equation of operation for the combined data-aided and hybrid loop of Fig. 3-19 if $F_a(p) = F_i(p)$ and

(a) $Q(x) = \text{sgn } x = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

(b) $Q(x) = x$

3-13 A tan-lock loop has a nonlinearity

$$g(\varphi) = \frac{(1 + k) \sin \varphi}{1 + k \cos \varphi}$$

for $0 \leq k < 1$. Show that the loop mechanization illustrated in Fig. P3-13 realizes this nonlinearity. Notice that as $k \to 1$, $g(\varphi) \to 2 \tan (\varphi/2)$. 
Devise a carrier-recovery circuit to coherently detect four-phase (quadriphase) PSK transmissions. (See Prob. 3-16).

Two PLLs connected in parallel are shown in Fig. P3-15. The input $x$ is characterized by

$$x = \sqrt{2} \ (A_1 \sin \Phi_1 + A_2 \sin \Phi_2) + n_i$$

where $\{n_i(t)\}$ is given in (3-11) through (3-17).

(a) Develop expressions that relate the instantaneous phase of each VCO (referenced to zero frequency) to other loop parameters and filters.
(b) When $\varphi_m \triangleq \theta_m - \Theta_m$, $m = 1, 2$, write the stochastic differential equations of system operation.

During $T$ seconds a polyphase signal of the form is transmitted.
\[ s(t) = \sqrt{2S} \sin \left( \Phi(t) + \frac{(2k + 1)\pi}{N} \right); \quad k = 0, 1, 2, \ldots, N - 1 \]

Here \( \Phi(t) = \omega_0 t + \theta_0 \). For \( N = 2 \) the above signaling format represents phase-shift keying while \( N = 4 \) corresponds to quadruphase signaling, that is, for \( N = 4 \)

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**Fig. P3-16.1**

---

**Fig. P3-16.2**
where \( d_1(t) \) and \( d_2(t) \) are \( \pm 1 \) digital waveforms. For \( N = 8 \) one obtains octaphase modulation. Assume that the observed data is bandpassed filtered by \( H_i(s) \) and the resultant is of the form \( x(t) = s(t) + n(t) \) where \( \{n_i(t)\} \) is a Gaussian process with a narrowband expansion about \( \Phi(t) \).

(a) For \( N = 4 \) developed the stochastic differential equation of system operation for the 4th power loop of Fig. P3-16.
(b) Generalize the result in (a) for \( N \) arbitrary and the \( N \)th power loop illustrated in Fig. P3-16.1.
(c) Repeat (a) for the generalized Costas (I-Q) loop of Fig. P3-16.2.
(d) When are the two circuits, that is the \( N \)th power loop and the generalized Costas loop, equivalent?

3-17 Two first-order PLL systems are connected in cascade (the problem of two-way tracking and Doppler measurements in deep-space communications) as shown below. The input signal is characterized by \( s_1(t) = \sqrt{2} A_1 \sin(\omega_0 t + \theta_1) \), and the reference signals \( s_2(t) = \sqrt{2} A_2 \sin(\omega_0 t + \theta_2(t)), k = 1, 2 \). The signal \( s_2(t) \) is \( \sqrt{2} A_2 \sin(\omega_0 t + \hat{\theta}_2(t)) \) is \( r_1(t) \) amplified by \( A_2 \) and phase-shifted by \( \pi/2 \). Here \( \hat{\theta}_1(t) \) is PLL_1's estimate of \( \theta_1 \) and \( \hat{\theta}_2(t) \) is PLL_2 estimate of \( \theta_2(t) \). Denote the loop phase errors by \( \varphi_1(t) = \theta_1 - \hat{\theta}_1(t) \) and \( \varphi_2(t) = \theta_2(t) - \hat{\theta}_2(t) \). It is desired to phase lock the two loops. Derive the stochastic differential equations of loop operation if the input narrowband, white Gaussian noise processes \( \{n_{ik}(t), k = 1, 2\} \) are statistically independent.

3-18 Two PLLs are to be connected together in order to estimate the ratio, \( k \), of two frequencies. Such a problem arises in military applications.
(a) If the observed data are of the form \( x(t) = \sqrt{2} A_1 \sin(\omega_1 t + \theta_1) + \sqrt{2} A_2 \sin(\omega_2 t + \hat{\theta}_2) + n(t) \), show how the two loops should be connected. Assume that \( \omega_2 = k \omega_1 \) with \( k \geq 10 \).
(b) Find the equation of operation of the individual loops.

3-19 Frequently, in the synchronization of large digital communication and computer-controlled networks, it is of interest to phase-lock several oscillators and produce mutual synchronization among the oscillators. One method is to phase-lock all oscillators in a common VCO, as shown in Fig. P3-19. Find the equation of operation of the loop if the inputs to the oscillations are characterized by
\[ x_m(t) = \sqrt{2} \ A_m \sin \Phi_m(t) + n_{im}(t) \]

where \( \Phi_m(t) = \omega_m t + \theta_m, \ m = 1, 2, \ldots, M \) and \( \theta_m \) are constants.

**Fig. P3-19**

3-20 The input to the double loop tracking system illustrated in Fig. P3-20 is characterized by

\[ x(t) = \sqrt{2} \ A \sin \Phi_1(t) \sin \Phi_2(t) + n_i(t) \]

**Fig. P3-20. Double-Loop Tracking System.**
\[ \varphi_1 = \theta_1 - \frac{K_1 F_1(p)}{2p} [A \cos \varphi_2 \sin \varphi_1 + n_1] \]
\[ \varphi_2 = \theta_2 - \frac{K_2 F_2(p)}{2p} [A \cos \varphi_1 \sin \varphi_2 + n_2] \]

With \( \varphi_1 \triangleq \Phi_1 - \hat{\Phi}_1 \) and \( \varphi_2 \triangleq \Phi_2 - \hat{\Phi}_2 \) show that the system equations of operation are of the form

Certain modifications of this loop make it attractive in ranging systems that employ pseudo-noise sequences and in communication between low altitude satellites and a satellite at synchronous altitude.

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LINEAR THEORY OF SINUSOIDAL PHASE-LOCKED LOOPS WITH APPLICATIONS

4-1 Introduction

In this chapter we present a concise treatment of linear PLL theory for the sinusoidal PLL.* Extension of the linear theory to include other loops of Table 3-1 follow in an analogous way. The reason for this presentation is that the linear PLL theory for sinusoidal PLLs has been extensively used in the design of many tracking and communication networks. Moreover, any book on the subject that omitted this theory would necessarily be incomplete. Furthermore, many of the system parameters that arise in the linear theory will occur later

*Strictly speaking, it is the linear PLL model (to be presented here) from which this theory is developed. As such it is not the theory which is linear or nonlinear for that matter; however, we find it convenient to talk about the linear theory which is developed from the linear model and the nonlinear theory which is developed using the nonlinear model. Also the term sinusoidal PLL is used here to imply the PLL mechanization in which \( g(\varphi) = \sin \varphi \). This distinguishes it from the many other possibilities discussed in Chapter 3. To modify the theory for these possibilities, one need only replace \( N_0 \) in all formulas which follow by the value of the spectral density of the equivalent phase noise at \( \omega = 0 \), while the factor \( A \) or \( \sqrt{P_c} \) must be replaced by the "equivalent slope" of \( g(\varphi) \) at \( \varphi = 0 \). This of course, assumes narrowband loops.
when we study the nonlinear theory. At the outset, their effects on loop performance are best understood by first considering them in the context of the linear model.

The chapter consists of two main parts. The first six sections deal primarily with the optimum design of Doppler and carrier tracking loops based on the linear PLL theory and the Wiener filtering theory given in Chapter 2. The theory of bandpass limiters, developed in Appendix I, is combined with the linear PLL theory in such a way as to account for the design and performance of coherent receivers that employ bandpass limiting in the earlier stages of the receiver.

Sections 4-7 through 4-10 contain the theory pertaining to part two. This part considers the optimum design of loops that are to be used as analog demodulators of angle modulation; in particular, phase and frequency modulation (PM and FM). These sections explore demodulator performance in the linear region of system operation and allude several times to the material discussed in Chapters 1, 2, and 3.

Finally, Section 4-11 discusses practical loop mechanizations for carrier tracking—for example, coherent transponders, double-superheterodyne PLL receivers. Section 4-12 presents a sample design of a superheterodyne PLL receiver that incorporates intermediate frequency (IF) limiting. This example exercises the theory presented in the first six sections. The chapter concludes by discussing where other approximation techniques to loop behavior can be found and where the subject of optimum analog demodulation has been treated from the point of view of estimation theory.

4-2 The Linear PLL Model when the Loop is Designed to Track $\theta(t)$

If a PLL system is to be of use in the reconstruction of the phase process $\{\theta(t)\}$, one would anticipate that the total loop phase error $\varphi(t)$ must be small. This forms the basis of the linear PLL theory—namely, if the loop is capable of reducing the phase error to a small value, say $|\varphi| < \pi/6$, we can use the first term in a Taylor series expansion for $\sin \varphi$ and write $\sin \varphi \approx \varphi$. Then from (3-24) the equation of operation becomes

$$\varphi = \theta - \frac{K_F(p)}{p} (A\varphi + N_\varphi) - \psi_2 - \frac{K_F e}{p}$$

(4-1)

and we have assumed that $N_\varphi(t) \gg N_\varphi(t)\varphi(t)$. Since

$$\theta = \underbrace{d}_{\text{Doppler input}} + \underbrace{M}_{\text{Angle modulation}} + \underbrace{\psi_1}_{\text{Transmitter instabilities}}$$

(4-2)
(4-1) reduces to
\[ \dot{\phi} = \dot{d} + \dot{M} - KF(p)[A\phi + N_c] - K_v e + \Delta \psi \]  \hspace{1cm} (4-3)

where \( \Delta \psi = \psi_1 - \psi_2 \) and, for simplicity of notation, we have omitted the dependence on \( t \) in all variables that are functions of time. One of the main objections of using the linear PLL theory (or any perturbation of this theory) to describe loop behavior is that it ignores the fact that the loop slips cycles. This is analogous, as we shall see, to ignoring the fact that a pendulum can rotate through multiples of \( 2\pi \) radians. Much like the phase error process in a PLL corresponds to hunting in synchronous machines, cycle slipping is analogous to the "skip-a-pole" or "fall-out-of-step" phenomena occurring in alternating current motors. In induction motors this is due to an excessive applied torque and corresponds to the slip of the squirrel cage winding relative to the rotating electric field. It is also analogous to the most common disorder of the heart's rhythm, the so-called extrasystole or premature beat. This is a hesitation of the heart that amounts to a skipping or dropping out of a heart beat. When a series of premature beats (analogous to several cycles slipped in succession in a loop) get started, all strung together at a rapid rate, it is called paroxysmal tachycardia.

Thus, within the linear approximation, nothing at all can be said about the many problems arising due to cycle slipping. These effects can only be understood in the context of the nonlinear PLL theory to be given later. The baseband equivalent loop model of Fig. 3-4 reduces to that illustrated in Fig. 4-1. This model is called the linear model. Since in the linear region

![Fig. 4-1. Linear Baseband Equivalent Loop Model when \( \theta(t) \) Is Being Tracked.](image)

\( N(t, \phi) = N_c(t) \), the effective phase noise that affects loop performance is approximately white (see Chapter 3, Section 3-5). In fact, if the spectral density of the input noise \( \{n_i(t)\} \) has the spectrum illustrated in Fig. 4-2 and if \( B_\theta \ll B_i \) then
Fig. 4-2. Spectral Density of the Input Noise Process, $\pi B_i < \omega_0$.

$$\lim_{B_i \to \infty} S_n(\omega) \to \frac{N_0}{2} \quad (4-4)$$

This says that the equivalent loop phase noise is white and Gaussian. In what follows, we assume that the physical situation is such that (4-4) holds.

4-3 Linear PLL Theory when $\theta(t)$ is Tracked by the Loop

If we solve (4-3) for $\varphi$, we find in operator form that

$$\varphi = \left[ \frac{p}{p + AKF(p)} \right] \left( d + M + \Delta \psi - \frac{K_p e}{p} \right) - \left[ \frac{AKF(p)}{p + AKF(p)} \right] \frac{N_e}{A} \quad (4-5)$$

The closed-loop transfer function of the linearized loop model illustrated in Fig. 4-1 is defined by

$$H_{\varphi}(s) \triangleq \frac{\bar{\Theta}(s)}{\bar{\hat{\theta}}(s)} = \frac{AKF(s)}{s + AKF(s)} \quad (4-6)$$

where we have introduced the Laplace transform variable $s$. Hereafter the tilde denotes the Laplace transform of the function over which it appears. As discussed in Section 3-3, we frequently find it convenient to use the variable $s$ and operator $p$ interchangeably. The loop filter transfer function is related to $H_{\varphi}(s)$ through

$$F(s) = \frac{sH_{\varphi}(s)}{AK[1 - H_{\varphi}(s)]} \quad (4-7)$$

Figure 4-3 shows the nature of the loop response $H_{\varphi}(s)$, with $s = i\omega$, of interest.
in practice. For example, the tracked or accepted frequency components of the signal imply the range of modulating frequencies over which the loop accepts changes in frequency in its normal dynamic mode. The rejected frequency components of the signal by the loop refer to those changes in frequency that are ignored by the loop, namely the modulation sidebands.

Making use of (4-6) in (4-5), we can write

$$\varphi = [1 - H_\varphi(p)] \left[ d + M + \Delta \psi - \frac{K_r}{p} e \right] - H_\varphi(p) \frac{N_i}{A} \quad (4-8)$$

when $A$ is constant. From this expression it is clear that the phase error process is obtained by passing the noise process $\{N_i(t)\}$ through the linear filter $H_\varphi(s)$ with gain $-1/A$ and the "signal" $d + M + \Delta \psi - K_r e / s$ through the linear filter $1 - H_\varphi(s)$.

From (4-8) we see that, within the linear approximation, the loop error is due to several factors. The first two terms in the large brackets of (4-8) is that portion of the phase error due to the incoming input signal $d(t) + M(t)$; this is composed of the Doppler phase shift $d(t)$ and the phase-modulation process $M(t)$. Usually $d(t)$ is nonstationary, for its form can frequently be specified a priori. As pointed out in Chapter 3, $M(t)$ may be composed of one or more bi-phase modulated digital-data signals (e.g., timing or digital telemetry), or it may be analog modulation that is frequently assumed to be a stationary random process. The third term in the large brackets of (4-8) is that portion of the total phase error due to oscillator instabilities at the transmitter and at the receiver. We shall refer to these as transreceiver instabilities. The fourth term in the large brackets of (4-8) is that portion of the total phase error that can actually be controlled, for the engineer has $e(t)$ at his disposal. Finally, the last term in (4-8) is that portion of the total phase error due to the presence of additive noise process $\{n_i(t)\}$. When $e = 0$, the total mean-square phase error, say $\sigma^2$, is thus composed of four terms, namely,
Linear PLL Theory when $\theta(t)$ is Tracked by the Loop

$$\sigma^2 = \sigma^2_\Delta + \sigma^2_{\Delta \psi} + \sigma^2_M + \sigma^2_\nu$$  \hspace{1cm} (4-9)

and we have assumed that $M, d, \Delta \psi, \text{and } N_c$ are statistically independent random processes and that they have zero means. The first term in (4-9) represents the mean-square tracking error defined by

$$\sigma^2_\Delta \triangleq \frac{1}{2\pi i} \int_{-\infty}^{\infty} |1 - H_\psi(s)|^2 E[|\tilde{d}(s)|^2] \, ds$$  \hspace{1cm} (4-10)

where $E[\cdot]$ denotes that the expected value is taken, since, in general, $d(t)$ may include an initial phase offset that is random. The second and third terms in (4-9) are defined by

$$\sigma^2_{\Delta \psi} \triangleq \frac{1}{2\pi i} \int_{-\infty}^{\infty} |1 - H_\psi(s)|^2 S_{\Delta \psi}(s) \, ds$$

$$\sigma^2_M \triangleq \frac{1}{2\pi i} \int_{-\infty}^{\infty} |1 - H_\psi(s)|^2 S_M(s) \, ds$$  \hspace{1cm} (4-11)

Finally,

$$\sigma^2_\nu \triangleq \frac{1}{2\pi i} \int_{-\infty}^{\infty} |H_\psi(s)|^2 \frac{S_N(s)}{A^2} \, ds$$  \hspace{1cm} (4-12)

is that portion of $\sigma^2$ due to the additive noise $\{\eta(t)\}$. The amount of work involved in evaluating the integrals in (4-10), (4-11), and (4-12) is greatly simplified by using the results tabulated in Table 4-1.

<table>
<thead>
<tr>
<th>Table 4-1 Useful Integral Forms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_n = \frac{1}{2\pi i} \int_{-\infty}^{\infty} A_n(s)A_n(-s) , ds$</td>
</tr>
<tr>
<td>$A_n(s) = \frac{c_0 + c_1 s + c_2 s^2 + \cdots + c_{n-1} s^{n-1}}{d_0 + d_1 s + d_2 s^2 + \cdots + d_n s^n}$</td>
</tr>
<tr>
<td>$I_1 = \frac{c_1^2}{2d_0 d_1}$, $I_2 = \frac{c_1^3 d_2 + c_1^3 d_0}{2d_0 d_1 d_2}$</td>
</tr>
<tr>
<td>$I_3 = \frac{c_1^3 d_0 d_1}{2d_0 d_2 (d_1 d_2 - d_0 d_3)} + \frac{c_1^3 d_0 d_2}{2d_0 d_3 (d_1 d_2 - d_0 d_3)}$</td>
</tr>
</tbody>
</table>

The expression for $\sigma^2_\nu$ is interesting when the phase noise is white. With the two-sided noise spectral density given in (4-4), we have from (4-12) that
\[ \sigma_e^2 = \frac{N_0 W_L}{2A^2} = \frac{N_0 B_L}{A^2} \]  \hspace{1cm} (4-13)

where

\[ W_L \triangleq 2B_L \triangleq \frac{1}{2\pi i} \int_{-\infty}^{\infty} |H_\varphi(s)|^2 \, ds \]  \hspace{1cm} (4-14)

is defined to be the loop bandwidth, \( W_L \) a two-sided bandwidth, and \( B_L \) a single-sided bandwidth. If we define

\[ \rho \triangleq \frac{2A^2}{N_0 W_L} = \frac{A^2}{N_0 B_L} \]  \hspace{1cm} (4-15)

as the signal-to-noise ratio existing in the loop bandwidth, then the linear PLL theory states that

\[ \sigma_e^2 = \frac{1}{\rho} \]  \hspace{1cm} (4-16)

which says that the variance of the phase error due to the additive channel noise is inversely proportional to the signal-to-noise ratio in the loop bandwidth.

One point worth mentioning here is that any attempt to filter out (reject) more of the additive noise by narrowing the loop bandwidth must be paid for by an increase in \( \sigma_e^2 \), that is, accepting more of the transceiver instabilities. Obviously, when the loop tracks \( \theta(t) \), the linear PLL theory specifies a best loop bandwidth for a given statistical characterization of the processes \{\Delta \psi(t)\} and \{n_i(t)\}. We shall consider this in Sections 4-5.4 and 4-5.5.

### 4-3.1 Loop Bandwidth and the Closed-Loop Transfer Function

For future reference we shall compute \( W_L \) for three important situations. If we consider a first-order loop with \( F(s) = 1 \), then the use of (4-6) in (4-14) gives

\[ W_L = \frac{AK}{2} \quad \text{or} \quad B_L = \frac{AK}{4} \]  \hspace{1cm} (4-17)

and from (4-6) and (4-17) we can write

\[ H_\varphi(s) = \frac{2W_L}{s + 2W_L} \]  \hspace{1cm} (4-18)

When the loop filter is of the lag form [i.e., \( F(s) = 1/(1 + \tau s) \)], then \( W_L = AK/2 \).
In some system applications we are interested in storing information for a time period for subsequent comparison with the present state of the arriving signal. If we assume that the most recent information is of most importance, but that it must be tempered by what has gone before, we arrive at the concept of time-weighted information. Associated with the memory requirement is also a need to track sudden changes in phase which yields a step function out of the loop phase detector (multiplier). These apparently contradictory requirements are met by placing in the loop an RC filter of the integral-control proportional plus type. That is,

\[ F(s) = \frac{1 + \tau_2 s}{1 + \tau_1 s} = F_0 + \frac{1 - F_0}{1 + \tau_1 s} \quad (4-19) \]

See Fig. 4-4 for a typical mechanization. The memory requirement of phase information is achieved by the time constant \( \tau_1 = (R_1 + R_2)C \), while the quick response feature is provided by the resistor \( R_2 \), which is common to both the charging circuit and the output circuit and gives rise to a second time \( \tau_2 = R_2 C \). The current flowing through \( R_2 \) injects a quick reaction term in series with the voltage appearing across the capacitor \( C \). By proper choice of \( \tau_2 \) and \( \tau_1 \), we may control important characteristics of the PLL receiving system. In practice, \( F(s) \) is usually designed with \( F_0 = \tau_2 / \tau_1 \ll 1 \) so as to make \( F(s) \) appear to have a pole at the origin.

If we substitute (4-19) into (4-6), we find that

\[ H_\phi(s) = \frac{1 + \tau_2 s}{1 + (\tau_2 + 1/AK)s + (\tau_1/AK)s^2} \quad (4-20) \]

Defining

\[ r \triangleq \tau_2 AKF_0 \quad F_0 \triangleq \frac{\tau_2}{\tau_1} \quad (4-21) \]
then the loop bandwidth, as determined from (4-14), (4-20), and Table 4-1, is given by

$$W_L = \frac{r + 1}{2\tau_2(1 + \tau_2/r\tau_1)} \approx \frac{r + 1}{2\tau_2}$$  \hspace{1cm} (4-22)

the latter approximation being valid when $r\tau_1 \gg \tau_2$. The canonical form for $H_p(s)$ can be written, using (4-20), (4-21), and (4-22), as

$$H_p(s) = \frac{1 + \left(\frac{r + 1}{2W_L}\right)s}{1 + \left(\frac{r + 1}{2W_L}\right)s + \frac{1}{r}\left(\frac{r + 1}{2W_L}\right)^2s^2}$$  \hspace{1cm} (4-23)

Fig. 4-5(a). Closed-Loop Frequency Response for a High Gain Second-Order Loop.
when \( \tau_2 \gg 1/\text{AK} \). From (4-23) one may compute the system damping as a function of \( r \). For \( r = 4 \), critical damping occurs; for \( r > 4 \), overdamping occurs, for \( r < 4 \), the roots of the system are real and the loop is underdamped. In fact, the loop’s damping factor \( \zeta \) and loop natural frequency \( \omega_n \) are given by

\[
\zeta = \frac{1}{2} \left( 1 + \frac{\tau_2}{\tau_1} \right) \sqrt{r} \approx \frac{\sqrt{r}}{2}
\]

\[
\omega_n = \sqrt{\frac{\text{AK}}{\tau_1}} = \frac{\sqrt{r}}{\tau_2}
\]

(4-24)

With \( s = j\omega \), Fig. 4-5 illustrates a plot of \( |H_p(i\omega)| \) and \( |1 - H_p(i\omega)| \) vs. \( \omega/W_L \) for various values of \( r \).

Whenever an operational amplifier is permissible in the loop and \( F_0 \ll 1 \), the loop filter in (4-19) approximates the form of a perfect integrator.
\[ F(s) \approx \frac{1 + \frac{\tau_2 s}{\tau_1 s}}{F_0 \ll 1} \] (4-25)

As we shall see, if the term \( \tau_2 / \tau_1 \approx 0 \), the loop with the imperfect integrating filter of (4-19) performs very much like the perfect integrating filter with loop bandwidth given by (4-22).

### 4.3.2 Loop Stability as Determined by Root Locus Plots

So far we have assumed that all closed-loop systems were stable. A necessary and sufficient condition is that all the poles of the closed-loop transfer
function lie in the left half-plane. These poles change their position as the loop gain $AK$ is changed. The locus of the point which the poles trace out as the loop gain varies from zero to infinity is known as the root locus plot.

In the absence of noise and oscillator instabilities, the closed-loop transfer function (4-6) can have poles at those values of $s$ for which $AKF(s) = -s$. A considerable degree of insight into loop operation in the linear region can be obtained by determining the locations of the poles of the closed-loop transfer function. The technique for obtaining the root loci from the open-loop response can be summarized by a few rules that are given in most elementary texts on control theory (Refs. 1 and 2). Figure 4-6 illustrates the root locus plot for various loop filters. The plot in Fig. 4-6c is for an imperfect integrating loop filter. If the loop filter were a perfect integrator, both poles would originate at the origin and the diameter of the circle would be $1/\tau_2$.

For the third-order loop Fig. 4-6d, it should be pointed out that for low values of $AK$, the root locus enters the right half-plane. This is in direct contrast for first- and second-order loops, which were unconditionally stable for all values of $AK$. Obviously, when a third-order loop is used in practice, $AK$ must be large enough so that the poles do not fall in the unstable region—that is, the right half-plane.

### 4-3.3 Stability of Imperfect Second-Order PLLs with an Arbitrary Delay

From the linear PLL theory and the root locus plot of Fig. 4-6c, we have seen that an imperfect second-order PLL predicts unconditional stability. In practice, inadvertent delays are frequently introduced into the loop transfer function. Furthermore, the mechanization of the data-aided loop (Section 3-8.1) calls for the addition of a delay element in the loop. With the introduction of this delay, loop stability, as predicted by linear PLL theory, becomes more critical. Consequently, the purpose of this section is to address the general question of the stability of an imperfect second-order PLL when delay is present in the loop. The principal tool used in the linear stability analysis is the Nyquist diagram and the accompanying Nyquist criterion, Ref. 1. In Chapter 10 we shall consider the nonlinear stability performance based on an approximation to the phase-plane behavior.

The open-loop transfer $T_0(i\omega)$ can be seen from Fig. 4-1 to be given by

$$T_0(i\omega) = AK \frac{F(i\omega)}{i\omega} \exp(-i\omega\lambda)$$  \hspace{1cm} (4-26)

when a delay element of $\lambda$ seconds is placed in the loop. Substitution of (4-19) into (4-26) with the normalization $x = \omega \tau_1$ yields

$$T_0(ix) = \frac{r(\tau_1/\tau_2)^2[1 + ix(\tau_2/\tau_1)] \exp[-i(x\lambda/\tau_1)]}{ix(1 + ix)}$$  \hspace{1cm} (4-27)
It is desired to study the magnitude and phase characteristics of $T_0(i\chi)$ and from these, using the Nyquist criterion, determine the set of normalized frequencies, $\{x_k, k = 0, 1, \ldots\}$, at which the phase characteristic, arg $T_0(i\chi)$, equals $(2k - 1)\pi$ radians, $k = 0, 1, 2, \ldots$. Thus the set $\{x_k\}$ is the solution to

$$\tan^{-1}\left(\frac{x_k \tau_2}{\tau_1}\right) - \tan^{-1}(x_k) - \frac{\pi}{2} - \frac{x_k \lambda}{\tau_1} = (2k - 1)\pi$$  \hspace{1cm} (4-28)

In particular, the solution of (4-28) for $k = 0$—that is, the first crossing of the horizontal axis by the Nyquist plot—must have a corresponding magnitude of $T_0(i\chi)$ that is less than unity in order to ensure stability. Thus if $\lambda$ is such that

$$-\tan^{-1}\left(\frac{x_0 \tau_2}{\tau_1}\right) + \tan^{-1}(x_0) + \frac{x_0 \lambda}{\tau_1} = \frac{\pi}{2}$$  \hspace{1cm} (4-29)

and

$$|T_0(i\chi_0)|^2 = \frac{r^2(\tau_1/\tau_2)^4[1 + x_0^2(\tau_2/\tau_1)^2]}{\lambda_0(1 + \lambda_0^2)} < 1$$  \hspace{1cm} (4-30)

then the loop will be absolutely stable in the linear sense. Assuming $r \tau_1 \gg \tau_2$, the solution of (4-29) for $x_0$ is

$$x_0 > \frac{r\tau_1}{\tau_2} \sqrt{\frac{1 + \sqrt{1 + 4/r^2}}{2}}$$  \hspace{1cm} (4-31)

and hence

$$\tan^{-1}x_0 \approx \frac{\pi}{2}$$  \hspace{1cm} (4-32)

Substitution of (4-31) and (4-32) into (4-29) leads to

$$\frac{\lambda}{\tau_2} < \text{P.V.} \left\{ \tan^{-1}\left( r\sqrt{\frac{1 + \sqrt{1 + 4/r^2}}{2}} \right) \right\}$$  \hspace{1cm} (4-33)

where P.V. denotes the principal value. This equation represents a bound on normalized loop delay in order to guarantee stability. When $r = 2$, $\lambda/\tau_2$ must be less than 0.52. We note again that the preceding results rely heavily on the ability to make the small phase error assumption which allows for use of the linear PLL model and a linear stability criterion.
4-4 Carrier and Doppler Tracking in the Absence of Noise and Oscillator Instabilities

Assuming that \( \sin \varphi_c \approx \varphi_c \) and \( N(t, \varphi) = e(t) = 0 \) in (3-41), we can write

\[
\varphi_c = \frac{pd}{p + K \sqrt{P_c} F(p)} \triangleq [1 - H_{\varphi_c}(p)] d \tag{4-34}
\]

where \( P_c = [(A \cos M)]^2 \). Here \( P_c \) is the power appearing at the carrier frequency in the presence of the modulation \( M(t) \). Now the steady-state tracking error \( \varphi_{ss} \) can be determined from (4-34) by using the final value theorem of Laplace transform theory

\[
\varphi_{ss} \triangleq \lim_{t \to \infty} \varphi_c(t) = \lim_{s \to 0} \frac{s^2 d(s)}{s + K \sqrt{P_c} F(s)} \tag{4-35}
\]

where we have assumed that \( d(t) \) is deterministic. If we assume that the loop filter is expressible in the form

\[
F(s) = \frac{Q(s)}{s^l P(s)} \quad Q(0) \neq 0 \quad P(0) \neq 0 \tag{4-36}
\]

where \( l \) is an integer and \( Q(s) \) and \( P(s) \) are polynomials in \( s \), then

\[
\varphi_{ss} = \lim_{s \to 0} \left[ \frac{s^{l+2} P(s)}{s^{l+1} P(s) + K \sqrt{P_c} Q(s)} \right] \tilde{d}(s) \tag{4-37}
\]

Writing \( d(t) \), the Doppler signal to be tracked, in the form*

\[
d(t) = \theta_0 + \sum_{k=0}^{N-1} \frac{\Omega_k}{(k + 1)!} t^{k+1} \tag{4-38}
\]

then its Laplace transform becomes

\[
\tilde{d}(s) = \frac{\theta_0}{s} + \sum_{k=0}^{N-1} \frac{\Omega_k}{s^{k+2}} = \frac{D(s)}{s^{N+1}} \tag{4-39}
\]

when \( \theta_0 \) is constant and the \( \Omega_k \)'s are constant. In (4-39), \( D(s) \) is a polynomial of degree less than \( N + 1 \). Substituting this into (4-37) and taking the limit

*For convenience, we are using the term "Doppler" in a more general sense than implied by the Doppler effect named after the Austrian physicist C.J. Doppler, (1803-53). See the footnote on page 84 for a further description.
yields the steady-state phase error

$$\varphi_{ss} = \begin{cases} 
\infty & N > l + 1 \\
\frac{\Omega_0 P(0)}{K \sqrt{P_c} Q(0)} & N = l + 1 \\
0 & N < l + 1
\end{cases}$$

(4-40)

which says that for a PLL system to track an $N$th degree polynomial phase function with $\varphi_{ss} = 0$, $F(s)$ must have $l > N - 1$ poles at the origin.

### 4.4.1 Tracking a Constant Phase Offset

As a first example of our results, consider the steady-state phase error resulting from a step change in phase—$d(t) = \theta_0$. Since the Laplace transform of $d(t)$ is $\tilde{d}(s) = \theta_0/s$, we obtain from (4-35)

$$\varphi_{ss} = \lim_{s \to 0} \frac{\theta_0 s}{s + K \sqrt{P_c} F(s)} = 0$$

(4-41)

Thus the linear theory says that the loop will eventually track out any change of input phase in the absence of noise, provided that $F(0) \neq 0$.

### 4.4.2 Tracking a Constant Frequency Offset

A second case of practical interest occurs when $N = 1$—that is, $d(t) = \theta_0 + \Omega_0 t$. For this case the Doppler signal consists of the frequency offset $\Omega_0$, plus an unknown phase $\theta_0$. From (4-40) the linear PLL theory yields

$$\varphi_{ss} = \frac{\Omega_0}{K \sqrt{P_c}} = \frac{\Omega_0}{2W_L}$$

(4-42)

when $F(s) = 1$ and $W_L = K \sqrt{P_c}/2$. On the other hand, for the imperfect integrating loop filter defined in (4-19), we find from (4-35) or (4-40) that

$$\varphi_{ss} = \frac{\Omega_0}{K \sqrt{P_c}} \approx \frac{\Omega_0}{W_L} \left( \frac{r + 1}{2r} \right) \frac{\tau_2}{\tau_1}$$

(4-43)

when we replace the amplitude $A$ in the formulas for $r$ and $W_L$ by $\sqrt{P_c}$. For the perfect integrating filter (4-25), the steady-state phase error $\varphi_{ss} = 0$. We also note that the closed-loop transfer functions $H_{\varphi}(s)$ for the case where $F(s) = 1$ and $F(s)$ given in (4-19) are obtained by replacing $A$ by $\sqrt{P_c}$ in (4-17), (4-18), (4-22), and (4-23) respectively.
4-4.3 Tracking Doppler Rates with Second- and Third-Order Loops

If we assume that \( N = 2 \) in (4-38), then

\[
d(t) = \theta_0 + \Omega_0 t + \frac{\Omega_1 t^2}{2}
\]  

(4-44)

which corresponds to a Doppler signal produced by an accelerating target. For the imperfect integrating filter given in (4-19)—that is, a second-order loop—we find from (4-35) that the steady-state tracking error is given by

\[
\phi_{ss} = \frac{\Omega_0 + \Omega_1 t}{K \sqrt{P_e}} + \frac{\Omega_1 \tau_1}{K \sqrt{P_e}} \]  

reduction in frequency offset  
reduction in frequency rate  

(4-45)

Unfortunately, the steady-state phase error increases linearly with increasing time. Control of the VCO tuning can eliminate the \( \Omega_0 \) term; however, the term with \( \Omega_1 \) can only be eliminated by sweeping the VCO at the proper rate.

On the other hand, for the perfect integrating filter (4-25), there is a finite value for the steady-state phase error.

\[
\phi_{ss} = \frac{\Omega_1 \tau_1}{K \sqrt{P_e}}
\]  

(4-46)

Obviously, if an imperfect second-order PLL is used to track a signal possessing a Doppler rate, it is necessary to compensate for the linear time increase in the phase error by tuning the VCO. This amounts to “tuning out” the phase error due to the Doppler rate \( \Omega_1 \) by adjustment of the loop acquisition voltage \( e(t) \).

To reduce the steady-state phase error to zero requires a loop filter of the form

\[
F(s) = \frac{(1 + \tau_2 s)(1 + \tau_4 s)}{\tau_1 \tau_3 s^2} = \frac{\tau_3 \tau_4}{\tau_1 \tau_3} \left[ 1 + \frac{\tau_2 + \tau_4}{\tau_2 \tau_4 s} + \frac{1}{\tau_2 \tau_4 s^2} \right]
\]  

(4-47)

As we see from (4-47), this filter has two poles at the origin and gives rise to a third-order loop. The corresponding closed-loop transfer function is given by

\[
H_{pc}(s) = \frac{(1 + \tau_2 s)(1 + \tau_4 s)}{1 + (\tau_2 + \tau_4) s + \tau_2 \tau_4 s^2 + \tau_1 \tau_3 s^3 / K \sqrt{P_e}}
\]  

(4-48)

The loop filter given in (4-47) can be approximated by synthesizing the transfer function
\[ F(s) = \frac{1 + \tau_2 s}{1 + \tau_1 s} + \frac{1}{(1 + \tau_1 s)(\delta + \tau_3 s)} \] (4-49)

and selecting the time constants such that \( \tau_2/\tau_1 \ll 1 \), and \( \delta \ll 1 \). Setting \( k = \tau_2/\tau_3 \) the closed-loop transfer function becomes

\[ H_{\psi_1}(s) = \frac{rk(1 + \delta) + r(1 + \delta k)\tau_2 s + r(\tau_2 s)^2}{rk(1 + \delta) + (r + r\delta k + F_0 \delta k)\tau_2 s + (r + F_0 + \delta k)(\tau_2 s)^2 + (\tau_2 s)^3} \] (4-50)

while the loop bandwidth is given by

\[ W_L = \frac{r}{2\tau_2} \left[ \frac{r - k + 1 + kF_0/r + \delta k[(r + F_0 + k\delta)(1 + 1/r) + k/r]}{r - k + F_0 + \delta k(1 + F_0/r)(r + F_0 + \delta k)} \right] \approx \frac{r}{2\tau_2} \left( \frac{r - k + 1}{r - k} \right) \]

Comparing this with the bandwidth of the imperfect second-order loop given in (4-22) we see that their ratio is given by the terms in the parentheses.

In the absence of noise the steady-state phase error is

\[ \varphi_{ss} \approx \delta \left( \frac{\Omega_0 + \Omega_1 t}{K\sqrt{P_e}} \right) + \frac{\Omega_1 \tau_1}{K\sqrt{P_e}} \left( \frac{F_0}{k} + \delta \right) \]

when \( F_0 \) and \( \delta \) are small. Compared to an imperfect second-order loop, which produces (4-45), we see that the instantaneous frequency offset, \( \dot{\psi}(t) \), is reduced by the factor \( \delta \) while the error caused by the frequency rate is diminished by \((F_0/k + \delta)\). Such a comparison reflects the importance of making \( k \) and \( 1/\delta \) large as other design factors will allow while making \( F_0 \) small. (See Sections 4-5.3, 4-5.4, and Section 11-15 for further design and loop performance considerations.)

4-5 Optimum Design of Carrier and Doppler Tracking Loops in the Presence of Noise

As discussed in Chapter 3, a properly designed PLL can be used to track the carrier component in the incoming signal while simultaneously performing the function of phase-coherent demodulation. In space communications, a measure of the Doppler shift provides information about the vehicle’s velocity. Consequently, in most phase-coherent communication systems, a carrier tracking loop is usually present.

For the moment let us consider the problem of loop design when \( \psi_1 = \)}
\( \psi_2 = e = 0 \). Assuming that \( \varphi_c \approx \varphi_v \), we have from (3-45) that

\[
\varphi_c = [1 - H_{c}(p)] d - \frac{[H_{c}(p)N_x]}{\sqrt{P_c}} \quad (4-51)
\]

where the closed-loop transfer function is defined by

\[
H_{c}(s) \triangleq \frac{K\sqrt{P_c}F(s)}{s + K\sqrt{P_c}F(s)} \quad (4-52)
\]

and \( \sqrt{P_c} \triangleq A \cos M \). Due to the form of (4-51), the closed-loop transfer function that minimizes the mean-square carrier phase error \( \varphi_c \) is easily obtained from the Yovits-Jackson formula (2-49) derived in Chapter 2. Replacing \( i\omega \) by \( s/N_0/2 \) by \( N_0/2P_c \), and \( D_y(\omega) \) by \( d(s) = D(s)/s^{N+1} \) in (2-49) of Chapter 2, the optimum closed-loop transfer function is specified by

\[
H_{c}(s) = 1 - \frac{s^{N+1}}{(-1)^{N+1}s^{2N+2} + \frac{2\lambda^2P_c}{N_0}E[D(s)D(-s)]} \quad (4-53)
\]

The \( + \) subscript now implies taking those terms in the partial fraction expansion of the quantity in braces that have poles and zeros in the left half-plane. We also note that polynomial \( D(s) \) in (4-53) is defined in (4-39). We now present examples of this rather general result that are of interest in practice.

### 4-5.1 Optimum Tracking of a Random Phase Offset (First-Order Loop)

As a first example of the result given in (4-53), let us consider the case where \( d(t) = \theta_0 \) is a uniformly distributed random variable between \((-\pi, \pi)\). To find \( H_{c}(s) \) we set \( N = 0 \) in (4-53). Since the mean-squared value of \( \theta_0 \) is \( \pi^2/3 \), we find from (4-53) that

\[
H_{c}(s) = \frac{a}{s + a} \quad (4-54)
\]

since \( [a^2 - s^2]_+ = s + a \) and \( a = \sqrt{2\pi^2\Lambda^2P_c/3N_0} \). From (4-17) we can find the loop bandwidth.

\[
W_L = \frac{a}{2} = \sqrt{\frac{\pi^2\Lambda^2P_c}{6N_0}} \quad (4-55)
\]

The Lagrange multiplier can now be eliminated from the loop design by writing \( H_{c}(s) \) in terms of \( W_L \).
\[ H_p(s) = \frac{2W_L}{s + 2W_L} \]  

(4-56)

From (4-52) the corresponding optimum loop filter becomes \( F(s) = 1 \).

**4-5.2 Optimum Tracking of a Frequency Offset and a Random Phase Offset (Second-Order Loop)**

This section points out why so much effort has been devoted to the problem of analysis and design of second-order PLLs. We consider the problem of optimally tracking a frequency offset \( \Omega_0 \) with a uniformly distributed random phase. As we shall see, the theory specifies a second-order loop with a perfect integrating loop filter. This is particularly important since most tracking receivers in use today have been mechanized so as to approximate the behavior of a second-order loop with a perfect integrating loop filter by synthesizing (4-19) in the loop.

With \( N = 1 \) in (4-39) and \( D(s)/s^2 = \theta_o/s + \Omega_0/s^2 \) inserted into (4-53), the optimum closed-loop transfer function reduces to (see Example 2, Section 2-6)

\[ H_p(s) = \frac{1 + \sqrt{2c + \pi^2/3\Omega_0^2}s}{1 + (\sqrt{2c + \pi^2/3\Omega_0^2})s + s^2/c} \]  

(4-57)

where

\[ c = \sqrt{\frac{2\Lambda^2\Omega_0^2P_c}{N_0}} \]  

(4-58)

This result is due to Tausworthe (Ref. 3). When \( \Omega_0 \) is a zero mean r.v., then \( c = \sqrt{2\Lambda^2\sigma_0^2P_c/N_0} \), where \( \sigma_0^2 \) is the variance of \( \Omega_0 \). If we compare (4-57) with (4-20) and (4-21), we recognize that the filter constants for a high-gain loop are given by

\[ \tau_1 = \frac{K\sqrt{P_c}}{c} \quad \tau_2 = \sqrt{\frac{2}{c} + \frac{\pi^2}{3\Omega_0^2}} \]  

(4-59)

\[ r = c(\tau_2)^2 = 2 + \frac{\pi^2K\sqrt{P_c}}{3\tau_1\Omega_0^2} \]

The optimum loop filter corresponding to (4-57) is the perfect integrator of (4-25); however, rather than synthesize this, the engineer has found it easier to synthesize (4-19). In effect, (4-59) determines \( R_1, R_2, \) and \( C \) in the filter of Fig. 4-4 in terms of \( \Omega_0, K\sqrt{P_c}, N_0, \) and \( \Lambda \). From the canonical form (4-23), it is clear that we are interested in specifying our design in terms of the loop parameters \( r \) and \( W_L \). Using (4-22) and (4-59), we find that
\[ W_L = \frac{r + 1}{2\tau_2} = \frac{(r + 1) \Omega_0}{2\pi r} \left[ \sqrt{3(r - 2)} \right] \]  

(4-60)

which eliminates \( \Lambda \) from our design. Thus for a specified \( r > 2 \) our loop design depends on the frequency offset \( \Omega_0 \) or \( \sigma_0 \) when \( \Omega_0 \) is a r.v. The larger \( \Omega_0 \), the larger the loop bandwidth needs to be to accommodate the offset. For this special case, the steady-state phase error is easily determined. In fact, when (4-19) is mechanized in the loop, the steady-state phase error is given by (4-43). On the other hand, if (4-25) is mechanized in the loop, then \( \phi_{ss} = 0 \). In practice, high-gain loops are usually designed with \( F_0 = \tau_2 / \tau_1 \ll 1 \) and (4-43) exhibits the effects of making \( F_0 \) as small as possible; that is, the loop filter is well approximated by (4-25).

In most practical cases \( \Omega_0 \) is not known prior to phase locking of the loop. However, the design engineer can usually place bounds on \( |\Omega_0| \) by specifying an initial frequency uncertainty region. Usually, because of the noise, it is desirable to design a tracking loop such that its bandwidth is much narrower than this frequency uncertainty region. In such instances the loop acquisition voltage \( e(t) \) can be used to slowly sweep the VCO frequency through the uncertainty interval for purposes of achieving phase lock. Consequently, it is reasonable to design the loop such that when the incoming carrier lies inside the loop passband (i.e., when \( \Omega_0 < 2\pi B_L \)), the loop acquires the signal in a small amount of time. Inserting the condition \( \Omega_0 = 2\pi B_L \) into the loop bandwidth formula (4-60) yields the condition \( r = 2.28 \) and from (4-24) we find that \( \zeta = 0.760 \). This result, due to Tausworthe (Ref. 3), is interesting to compare with the now classical result due to Jaffe and Rechtin (Ref. 4).

Jaffe and Rechtin (Ref. 4) optimized the loop for the case in which \( d(t) = \Omega_0 t \), thereby neglecting any a priori random phase offset. The optimum closed-loop transfer function is found from (2-53) with \( b = 0 \). Comparing (2-53) with (4-20) and (4-23), we have

\[ \tau_2 = \frac{\sqrt{2}}{\beta}, \quad \beta^2 = \frac{K_\beta P_\varepsilon}{\tau_1}, \quad r = 2 \]

\[ W_L = \frac{3}{2} \sqrt{\frac{c}{2}} = \frac{3}{2\tau_2}, \quad \zeta = 0.707 \]

(4-61)

when \( AK\tau_2 \gg 1 \). The corresponding optimum loop transfer functions, as determined from (4-23), are given by

\[
H_{p_1}(s) = \frac{1 + (1.643/W_L)s}{1 + (1.643/W_L)s + (1.18/W_L^2)s^2} \begin{cases} \text{Tausworthe} \\ d(t) = \Omega_0 t + \theta_0 \end{cases}
\]

\[
H_{p_1}(s) = \frac{1 + (1.5/W_L)s}{1 + (1.5/W_L)s + (9s^2/8W_L^2)} \begin{cases} \text{Jaffe–Rechtin} \\ d(t) = \Omega_0 t \end{cases}
\]

(4-62)
For either case given in (4-62), the optimum loop filter is given by (4-25) with \( \tau_1 \) and \( \tau_2 \) found, respectively, from (4-59) and (4-61). It is interesting to note that Jaffe and Rechtin were the first to apply the Wiener theory discussed in Chapter 2 to the problem of optimum design of a deep-space tracking system. Had this theory not been available, system “optimization” would probably have proceeded by experimenting in the laboratory.

The mean-square tracking error, as determined from (4-10), (4-23), and Table 4-1, is given by

\[
\sigma_2^2 = \frac{\tau_1 \tau_2 \Omega_0^2}{2rK\sqrt{P_c}} \left( 1 + \frac{\pi^2 K\sqrt{P_c}}{3\Omega_0^2 \tau_1} \right)
\]  

(4-63)

The total mean square is found from (4-9), (4-15) with \( A \) replaced by \( \sqrt{P_c} \), and (4-63). For the optimum loop defined by (4-59), (4-63) reduces to

\[
\sigma_2^2 = \frac{\pi^4(r - 1)}{6\Omega_0^2 \sqrt{3r(r - 2)}}
\]  

(4-64)

With \( r = 2.28 \) and \( \Omega_0 = \pi W_L \), then \( \sigma_2^2 = 5.37/W_L \), whereas the parameters for the Jaffe-Rechtin model produce \( \sigma_2^2 = 5.4/W_L \). The results compare favorably.

### 4-5.3 Optimum Design for Tracking an Accelerating Target (Third-Order Loop)

If, for simplicity, we assume that \( N = 2, \theta_0 = 0, \Omega_0 = 0 \) in (4-38), then

\[ d(t) = \Omega_0 t^2/2. \]

The optimum closed-loop, rate-tracking transfer function (4-53), as determined by the Wiener theory, reduces to

\[
H_{\nu_c}(s) = \frac{2\beta_0 s^2 + 2\beta_0^3 s + \beta_0^3}{s^3 + 2\beta_0 s^2 + 2\beta_0^3 s + \beta_0^3}
\]  

(4-65)

where \( \beta_0 = \sqrt{2\Omega_0^2 A^2 P_c/N_0} \). Notice that this gives rise to a third-order loop. The optimum loop filter corresponding to (4-65) becomes

\[
F(s) = \frac{2\beta_0 s^2 + 2\beta_0^3 s + \beta_0^3}{K\sqrt{P_c} s^2} = \frac{2\beta_0^3}{K\sqrt{P_c}} \left( \frac{1 + s/\beta_0 + \beta_0}{2s^2} \right)
\]  

(4-66)

Notice that the first term on the right-hand side of (4-66) resembles (4-25). Equations (4-65) and (4-66) compare favorably with (4-50) and (4-47).

Fourth-order loops have been implemented by Electrac, Inc., Anaheim, Calif., for applications where rapidly moving vehicles must track one another. To the author’s knowledge, fifth- or higher-order loops have never been
constructed, for they have never been needed in practice. Furthermore, the closed-loop parameters of higher-order active networks tend to be overly sensitive to changes in the circuit components and loop gain; thus it is more difficult to stabilize a third or higher-order loop. Therefore, when third-order loops are used in practice, one would probably acquire the signal by sweeping the VCO in an imperfect second-order loop and switching to a third-order loop when the signal is acquired.

4-5-4 The Effects of Oscillator Instabilities In Second- and Third-Order Loops

One of the limiting factors usually neglected in the design of narrowband PLLs has to do with the effects that the oscillator instabilities produce on system performance. There are various factors that contribute to their spectral makeup; however, the most significant are thermal noise and noise with a 1/f spectrum. If we model the spectrum of the transceiver frequency instabilities as

\[ S_{\Delta \phi}(s) = N_{0v} + \frac{2\pi N_{1v}}{\omega} \]

where \( N_{0v} \) characterizes the thermal noise and \( N_{1v} \) characterizes the 1/f noise; then, if we use this spectrum and (4-23) in (4-11) we find that

\[ \sigma_{\Delta \phi}^2 = \left( \frac{r + 1}{4r} \right) \frac{N_{0v}}{W_L} + G(r) \frac{N_{1v}}{W_L^2} \]

where

\[
G(r) \triangleq \begin{cases} 
\frac{(r + 1)^2}{4\sqrt{r^3}(r - 4)} \ln \left[ \frac{r - 2 + \sqrt{r(r - 4)}}{r - 2 - \sqrt{r(r - 4)}} \right] & r > 4 \\
\frac{25}{16} & r = 4 \\
\frac{(r + 1)^2}{2\sqrt{r^3}(4 - r)} \left[ \frac{\pi}{2} - \tan^{-1} \left( \frac{r - 2}{\sqrt{r(4 - r)}} \right) \right] & r < 4
\end{cases}
\]

To arrive at these results, the loop filter is of the form given in (4-25). Neglecting that component of the total phase error variance due to Doppler, then there exists an optimum value of \( W_L \) and \( r \) for given values of \( N_0/P_c \), \( N_{0v} \), and \( N_{1v} \). This is due to the fact that the variance of the phase error due to the additive noise is proportional to \( W_L \), while the component due to \( \Delta \phi \) is inversely proportional to \( W_L \). In fact, Tausworthe (Ref. 3) argues that the optimum value for \( r \) is around \( r_{opt} \approx 7 \) corresponding to the loop damping \( \zeta \approx 1.32 \). Using this value of \( r \) then
\[ \sigma^2 = \frac{N_0 W_L}{2 P_e} + \sigma^2_{3}\nu \]

can be differentiated with respect to \( W_L \) to find the optimum loop bandwidth. This bandwidth solves the equation (Ref. 3, p. 32)

\[ W^2_L - \left(\frac{r + 1}{4r}\right) \frac{P_e N_{1v}}{N_0} W_L - 2G(r) \left(\frac{P_e N_{1v}}{N_0}\right) = 0 \]

For \( P_e/N_0 = 6 \times 10^4, N_{0v} = 0, N_{1v} = .08 \), the optimum \( W_L \) is 26 Hz for \( r = 7 \). These parameters lead to a phase error of \( \sigma = 1.43 \) degrees rms.

On the other hand, for the third-order loop defined by (4-49) and (4-50) we obtain

\[ \sigma^2_{3}\nu = \left(\frac{r(r + 1 - k)}{4(r - k)^2}\right) \frac{N_{0v}}{W^2_L} + G(r) \frac{N_{1v}}{W^2_L} \]

when we set \( F_0 = \delta = 0 \). This form resembles the above equation for the perfect second-order loop. At \( k = 0.25, r = 3.375 \), \( \sigma^2_{3}\nu \) is 10 to 15 per cent higher than it is for a second-order loop so that the use of a third-order loop does not allow one to relax the requirement for spectral purity in the VCO to be used. This result is due to Tausworthe and Crow (see Section 11-15 of Chapter 11 for further details.)

### 4-5.5 Comparison of Second- and Third-Order Loops

Owing to the fact that third-order loops are potentially unstable they have not found wide application in the past. However, as we shall see in Section 11-15, a third-order loop can outperform a second-order loop, not only in its ability to track a frequency offset or frequency ramp with practically zero phase error, but also in its ability to acquire lock more quickly and from greater offsets. Even when synthesized with imperfect integrators, as in (4-49), within the loop filter the third-order system will outperform a perfect second-order system by orders of magnitude improvement in steady-state phase error, acquisition time, and range. One further advantage to the third-order system is that there is less of a requirement for high loop gains and long time constants than needed by the second-order loop to maintain small tracking errors. Moreover, as noted in (4-66) the loop filter is an extension of presently mechanized loops; thus modification of existing equipment is minor.

Even though the previous section showed that the \( \sigma^2_{3}\nu \) was greater in third-order systems than second-order systems, there are other instabilities that were not modeled by \( S_{3}\nu(s) \). To the PD these appear as small "drifts" in the
frequency offset or rate of the incoming signal. Because the third-order loop minimized such effects, the drift requirement on VCOs may be relaxed.

4-6 Carrier and Doppler Tracking with a Second-Order PLL Preceded by a Bandpass Limiter (BPL)*

In the practical operation of a coherent receiving system employing PLLs for carrier tracking, the loop is preceded by a BPL. Any fluctuation in $P_e$ (owing to a change in range, for instance) causes the loop bandwidth to fluctuate similarly. Bandpass limiters are also used to protect various loop components—for example, the phase detector, where signal and noise levels can vary over several orders of magnitude and exceed the dynamic range of these components. The mechanizations of a BPL is illustrated in Fig. 4-7; the limiter incorporated in the PLL system is shown in Fig. 4-8. The signals $x(t)$, $y(t)$, and $y_i(t)$ appearing in these two figures are defined in Appendix I, Fig. AI-1.

![Fig. 4-7. Mechanization of a BPL.](image)

![Fig. 4-8. PLL Receiver Preceded by a BPL.](image)

Again the variance of the phase error is an important parameter in specifying the loop response to a sine wave plus noise. Assuming that the bandwidth of the spectrum at the phase detector input is large in comparison with the bandwidth of the loop, we can derive a result similar to (4-13).

*This section can be omitted upon first reading.
\[
\sigma_{\varphi^2} = \frac{N_0 P_L}{2\alpha_1^2 P_1}
\]  
(4-67)

where \(\alpha_1^2 P_1\) is the output signal power of the limiter in the first zone, \(N_0/2\) is the spectral density at the origin of the noise at the output of the phase detector, and \(\alpha_1\) is the signal amplitude suppression factor defined in Appendix I. Here \(W_L\) is the equivalent loop noise bandwidth when a limiter precedes the loop. In writing (4-67), we have made the approximation that the noise seen by the loop is white with a single-sided spectral density of \(N_0\) watts/Hz. In this context we can rewrite (4-67) as

\[
\sigma_{\varphi^2} = \frac{N_0 W_L}{2\alpha_1^2 P_1} = \frac{N_0 W_L}{2P_c} \Gamma_p = \frac{\Gamma_p}{\rho}
\]  
(4-68)

The quantity

\[
\Gamma_p \triangleq \frac{N_0/2\alpha_1^2 P_1}{N_0/2(\cos MA)^2}
\]  
(4-69)

is the limiter performance factor defined in Appendix I (Eq. I-60). For slide-rule calculations this factor is well approximated by

\[
\Gamma_p \approx \frac{1 + \rho_i}{\Gamma_H^{-1} + \rho_i}
\]  
(4-70)

where \(\Gamma_H\) (see Appendix I, Section I-3) depends on the intermediate frequency (IF) filter transfer characteristic and

\[
\rho_i = \frac{2(A\cos M)^2}{N_0 W_i} = \frac{2P_c}{N_0 W_i}
\]  
(4-71)

and \(W_i = 2B_i\) represents the equivalent bandwidth of the noise at the input to the limiter (see Fig. 4-2). In the limiting case of large \(\rho_i\), the value \(\Gamma_p\) approaches unity. For the rectangular predetection filter characteristic and small \(\rho_i\), its value approaches 1.16. The conclusion reached is that the effect of a limiter on loop performance is less than 16 per cent (0.66 dB) in effective input signal-to-noise ratio when the IF filter characteristic is rectangular.

When a limiter precedes the loop, the signal amplitude suppression factor, as determined from (I-28) of Appendix I,

\[
\alpha_1 = \sqrt{\frac{\pi}{2}} \sqrt{\frac{\rho_i}{2}} \exp \left( -\frac{\rho_i}{2} \right) \left[ I_0 \left( \frac{\rho_i}{2} \right) + I_1 \left( \frac{\rho_i}{2} \right) \right]
\]  
(4-72)
affects the loop bandwidth as we shall presently see. In the region of interest, in practice, (4-72) is closely approximated by (Ref. 3)

\[ \alpha_1 = \left( \frac{0.7854 \rho_t + 0.4768 \rho'_t}{1 + 1.024 \rho_t + 0.4768 \rho'_t} \right)^{\frac{1}{2}} \]

(4-73)

Thus the closed-loop transfer function, given by (4-20), for a PLL preceded by a BPL becomes

\[ H_\phi(s) = \frac{1 + \tau_s s}{1 + [(\tau_2 + 1)/\alpha_1 \sqrt{P_1 K})]s + (\tau_1/\alpha_1 \sqrt{P_1 K})s^2} \]

(4-74)

since \( \alpha_1 \sqrt{P_1} \) plays the role of the signal amplitude \( A \cos M = \sqrt{P_c} \) when a limiter does not precede the loop. The corresponding loop bandwidth, as determined from (4-14) and (4-74) with \( A \) replaced by \( \alpha_1 \sqrt{P_1} \), becomes

\[ W_L = \frac{1 + \alpha_1 \sqrt{P_1 K \tau_2^2}/\tau_1}{2\tau_2(1 + 1/\alpha_1 \sqrt{P_1 K \tau_2})} = \frac{1 + r}{2\tau_2(1 + \tau_2/r\tau_1)} \]

(4-75)

where

\[ r = \frac{\alpha_1 \sqrt{P_1 K \tau_2^2}}{\tau_1} \]

(4-76)

when a BPL precedes the loop. Notice that \( W_L \) is now a function of the limiter suppression factor \( \alpha_1 \), since \( \alpha_1^2 P_1 \) plays the same role as the carrier power \( P_c \) when the loop is not preceded by a BPL.

As a rule, PLL parameters are usually specified at what shall be referred to as the loop “design point.” The “design point” is frequently called the loop threshold.* In the past, this point has been taken to imply that condition where

\[ 2P_0 = 2(\cos M)^2 A_0^2 = N_0 W_{LO} \]

(4-77)

with

\[ W_{LO} \triangleq \frac{1 + r_0}{2\tau_2(1 + \tau_2/r_0 \tau_1)} \]

\[ r_0 = \frac{\alpha_{10} \sqrt{P_{10} K \tau_2^2}}{\tau_1} \]

(4-78)

*The term “design point” is preferred here over the term “loop threshold.” As we shall see, specifically in Chapters 9, 10, and 11, threshold in a PLL system appears as a result of the cycle slipping phenomenon. Since cycle slipping is in no way accounted for by the linear theory, the author prefers not to confuse the reader by referring to this condition as loop threshold. Loop threshold will be accounted for by specifying the cycle slipping rate, that is, the average number of times per second the system fails, for a given signal-to-noise ratio in \( W_L \) or \( W_{LO} \).
The subscript zero on $A$, $P$, $W_L$, and $r$ refers to the respective values of these parameters at the “design point.” Later on it will be shown that $P_0$ represents a signal power at which the linear PLL theory does not apply. At the “design point,” the loop bandwidth and signal suppression factor are denoted by $W_{LO}$ and $\alpha_{10}$, respectively. For values that differ from the “design-point” value, $W_L$ and $\alpha_1$ are used. Thus the canonical form of closed-loop transfer function of a high-gain loop preceded by a BPL becomes

$$H_\phi(s) = \frac{1 + \left(\frac{r_0 + 1}{2W_{LO}}\right) s}{1 + \left(\frac{r_0 + 1}{2W_{LO}}\right) s + \frac{\mu}{r_0} \left(\frac{r_0 + 1}{2W_{LO}}\right)^2 s^2}$$

(4-79)

when the time constants $\tau_2$ and $\tau_1$ are evaluated at the design point. In (4-79),

$$\mu = \frac{\alpha_{10}}{\alpha_1}$$

(4-80)

is defined as the limiter suppression factor. When a limiter precedes the loop, the loop bandwidth as determined from (4-14) and (4-79) becomes

$$W_L = W_{LO} \left(\frac{1 + r_0/\mu}{1 + r_0}\right)$$

(4-81)

and the variance of the phase error becomes

$$\sigma^2_\phi = \frac{1}{\alpha_1} = \frac{N_0 W_L}{2P_e}$$

(4-82)

where

$$\alpha_1 \triangleq \frac{2P_e}{N_0 W_{LO}} \frac{1}{\Gamma_p} \left(1 + \frac{r_0}{1 + r_0/\mu}\right)$$

(4-83)

and $\Gamma_p$, $r_0$, and $\mu$ are defined by (4-70), (4-78), and (4-80) respectively. Figure 4-9 illustrates a plot of (4-81) for various values of $\rho_i$ for $r_0 = 2$.

Figure 4-10 is a plot of (4-81) for various values of $\eta = W_{LO}/W_i$. This figure plots the variance of the phase error $\sigma^2_\phi$ vs. the signal-to-noise ratio $x = 2P_e/N_0 W_{LO}$ existing in the “design-point” loop bandwidth as determined from the linear PLL theory. The variance is essentially independent of the parameter $\eta$; the parameter $\eta = W_{LO}/W_i$ is the ratio of the “design-point” loop bandwidth to the bandwidth of the filter (IF amplifier) preceding the BPL of Fig. 4-8. Note that the variance of the phase error is relatively insensitive to $\eta$. Later on we shall make a comparison between the linear and nonlinear PLL theory.
Fig. 4-9. Loop Bandwidth Variation versus $\rho_1$ for Various Loop Design Points with $r_0 = 2$. 
Fig. 4-10. Variance of the Phase-Error $\sigma_{\phi}^2$, versus the Loop Signal-to-Noise Ratio $x$; Linear PLL Theory is Assumed with a BPL Preceding the Loop.

4-7 Linear PLL Angle Demodulation Theory

In the previous six sections we have considered the problem of carrier tracking based on the linear PLL theory. This section begins our study of the linear PLL theory of angle demodulation. We will be concerned with the demodulation of both phase and frequency modulation. At various times in the development we shall allude to the optimum filtering theory presented in Chapter 2. Throughout this section we assume that $e = \psi_1 = \psi_2 = 0$.

To be more specific, the problem we wish to consider is illustrated in Fig. 4-11. At the transmitter a random signal $m(t)$ is used to control the instantaneous frequency of and phase of a VCO with a gain of $K_c$ rad/sec-volt. The random process $\{m(t)\}$ is assumed to be wide-sense stationary and statistically independent from channel noise process $\{n_i(t)\}$. For this case the phase modulation $\theta(t) = d(t) + M(t)$, consisting of the transmitter modulation

$$M(t) = \begin{cases} K_c \int m(\lambda) \, d\lambda & \text{FM} \\ K_c m(t) & \text{PM} \end{cases}$$

(4-84)
and the channel Doppler signal, is related to the phase error $\varphi_M$ through [see (3-39)]

$$\varphi_M = \theta - \frac{KF(p)}{p} [A \sin \varphi_M + N(t, \varphi)] \quad (4-85)$$

Thus, to study the receiving system of Fig. 4-11 from the input $x(t)$ to the output $y(t)$, (4-85) evidences the pertinent quantities, and the receiver may be conveniently represented by the baseband model of Fig. 4-12. Even though

the system of Fig. 4-11 has been replaced by that of Fig. 4-12, it is still impossible to apply the theory presented thus far to the problem of optimizing loop design. Consequently, in order to proceed, we must introduce the linearity assumption $\sin \varphi_M \sim \varphi_M$ so that (4-85) reduces to (4-1) with $\varphi = \varphi_M$ and $\psi_2 = \epsilon = 0$. The linearized baseband model is illustrated in Fig. 4-13. Now the variance of the phase error of this receiver, as determined from (4-11) and (4-12), is given by

$$\sigma^2 = \frac{1}{2\pi i} \lim_{n \to \infty} \left[ 1 - H_\varphi(s) \right]^2 S_\theta(s) + |H_\varphi(s)|^2 \frac{S_{\varphi_2}(s)}{A^2} \right] ds \quad (4-86)$$

where
Since the noise process \( \{N_c(t)\} \) is white, the Yovits-Jackson formula (2-41) can be used to specify \( H_a(s) \), given in (4-6). The loop filter is then determined by (4-7).

For reasons that will subsequently become obvious, we define the following transfer function

\[
H_f(s) \triangleq \frac{\bar{\theta}(s)}{\tilde{\theta}(s)}
\]

and referring to Fig. 4-13, it is obvious that

\[
F_0(s) = \frac{KH_f(s)}{sH_a(s)}
\]

where \( p \) is replaced by \( s \). From Chapter 2 we note that variance of the demodulation error, \( m(t) - y(t) \), is given by

\[
\sigma_f^2 \triangleq E[(m - y)^2] = \frac{1}{2\pi i} \int_{-\infty}^{\infty} [L(s) - H_f(s)]^2 S_y(s) + |H_f(s)|^2 S_n(s) \, ds
\]

where \( L(s) = 1/K_r \) for PM and \( L(s) = s/K_r \) for FM. Notice that in writing (4-90) we have moved the noise component \( N_c(t) \) to the input and designated it as \( n'(t) \). Its spectral density is
\[ S_n(s) \triangleq \frac{S_{N_n}(s)}{A^2} = \frac{N_0}{2A^2} \tag{4-91} \]

Notice that the first part of the error in (4-90) is obtained because the transfer function \( H_f(s) \) differs from \( L(s) \). We shall define this portion of the error as the signal distortion

\[ S_d \triangleq \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} S_\phi(s)[L(s) - H_f(s)]^2 \, ds \tag{4-92} \]

and the second part is due to the additive noise.

The causal Wiener filter \( H_f(s) \) that minimizes the mean square error \( \sigma_r^2 = E(m - y)^2 \) is determined from (2-34). In terms of the Laplace transform variable \( s = i\omega \), this filter is given by

\[ H_f(s) = \frac{1}{S_x(s)} \left[ \frac{L(s)S_\phi(s)}{S_x(s)} \right] \tag{4-93} \]

where we use the fact that \( S_{dd}(s) = |L(s)|^2 S_\phi(s) \) from (2-25). The spectral density \( S_x(s) \) is given by

\[ S_x(s) = S_\phi(s) + S_n(s) = S^*_x(s)S_x(s) \tag{4-94} \]

The use of the theory of angle demodulation by means of a PLL is demonstrated in the following sections.

4-8 Phase Demodulation Using the Linear PLL Model

In this section we apply the theory given in the previous section and portions of the linear filtering theory developed in Chapter 2 to the problem of phase demodulation. In order to apply the theory, however, we must specify the message spectral density \( S_m(\omega) \). For purposes here, we assume that \( \{m(t)\} \) is a unit-variance stationary random process whose spectrum is taken to be a member of the Butterworth family discussed in Chapter 2, Section 2-7. That is,

\[ S_m(\omega; k) = \frac{K_i(k)}{1 + (\omega/a)^{2k}} \tag{4-95} \]

for \( k = 1, 2, \ldots, \infty \). Here, however, we select \( K_i(k) \) such that \( S_m(\omega, k) \) has unit variance.

\[ K_i(k) = \frac{\pi}{a} \sin \left( \frac{\pi}{2k} \right) \triangleq \frac{\pi}{a} \text{sinc} \left( \frac{\pi}{2k} \right) \tag{4-96} \]
When \( d(t) = 0 \), the spectrum of the phase modulation is due to \( \theta(t) = M(t) \) and

\[
S_d(\omega) = K_i^2 S_m(\omega; k) \quad (4-97)
\]

whereas that of the equivalent phase noise is given in \( (4-91) \). Substituting \( (4-91), (4-95), \) and \( (4-97) \) into \( (2-22) \) and \( (2-41) \) of Chapter 2, we find that the optimum filter functions for noncausal and causal filtering are given respectively by

\[
H_d(i\omega) = \frac{C(k)}{1 + C(k) + (\omega/\alpha)^{2k}} \quad \text{noncausal} \quad (4-98)
\]

\[
H_d(i\omega) = 1 - \left[ \frac{\omega^{2k} + a^{2k}}{\omega^{2k} + a^{2k}(1 + C(k))} \right]_+ \quad \text{causal}
\]

where

\[
C(k) = 2\pi K_i^2 R \sin \left( \frac{\pi}{2k} \right) \quad (4-99)
\]

and \( R \triangleq A^2/aN_0 \). In using \( (2-22) \) and \( (2-41) \), we note that \( S_d(\omega) = S_s(\omega) = S_\delta(\omega) \), that \( N_0 \) is replaced by \( N_0/A^2 \) in \( (2-41) \), and that \( S_s(\omega) = S_n(\omega) \) is defined in \( (4-91) \).

For \( k = 1 \) we note that the optimum causal filter in \( (4-98) \) reduces to

\[
H_d(i\omega) = \frac{a(\gamma' - 1)}{i\omega + a\gamma'} \quad (4-100)
\]

where

\[
\gamma' = \sqrt{1 + 4K_i^2 R} \quad (4-101)
\]

Using \( (4-93) \) and noting that \( L(i\omega) = K_i^{-1} \), we find that

\[
H_f(i\omega) = \frac{a(\gamma' - 1)}{K_i(i\omega + a\gamma')} \quad (4-102)
\]

when \( k = 1 \). The output filter \( F_0(s) \), as determined from \( (4-89), (4-100), \) and \( (4-102) \) becomes

\[
F_0(i\omega) = \frac{K}{i\omega K_i} \quad (4-103)
\]
Before the performance characteristics of the receiver are computed analytically, it remains to define the region of validity of our results. In fact, it has been the practice to design the loop at a point where \( \sigma^2 = 1 \) (as determined from the linear theory) and operate the loop in the region where \( \sigma^2 < 1 \). The region for which \( \sigma^2 \gtrsim 1 \) is that region in the \((\sigma^2, R)\) plane for which the linear theory does not apply. Our later treatment of the nonlinear theory will resolve this uncertainty.* From (4-7) and (4-100) the corresponding optimum loop filter becomes

\[
F(i\omega) = \frac{i\alpha(\gamma - 1)\omega}{AK(i\omega + a)}
\]

(4-104)

The minimum mean-squared value of the phase error for noncausal filters can be determined from (2-23), while for causal filters its value can be determined using the Yovits-Jackson formula (2-42) with \( N_0 \) replaced by \( N_0/A^2 \) and \( S_v(\omega) \) by \( S_\phi(\omega) \) of (4-95). Thus

\[
\sigma^2(k) = K_1^2[1 + C(k)]^{1/2k-1} \text{ noncausal}
\]

\[
\sigma^2(k) = \frac{2K_1^2k}{C(k)}[[1 + C(k)]^{1/2k} - 1] \text{ causal}
\]

(4-105)

We use as the basis of comparison of all systems the output signal-to-noise ratio (SNR)—that is, the SNR appearing at the output of \( F_\phi(s) \). This quantity is defined as the ratio of the desired output signal power to the Wiener error \( \sigma^2(k) \), or

\[
\rho_\phi(k) \triangleq \frac{\int_{-\infty}^{\infty} S_\phi(\omega) d\omega}{2\pi\sigma^2(k)} = \frac{K_3^2}{\sigma^2(k)}
\]

(4-106)

and from (4-105) we find

\[
\rho_\phi(k) = [1 + C(k)]^{-1/2k} \text{ noncausal}
\]

\[
\rho_\phi(k) = \frac{C(k)}{2k}[[1 + C(k)]^{1/2k} - 1]^{-1} \text{ causal}
\]

(4-107)

For \( k = 1 \), (4-107) reduces to

\[
\rho_\phi(1) = \sqrt{1 + 4K_1^2R} \text{ noncausal}
\]

\[
\rho_\phi(1) = 2K_3^2R[(1 + 4K_1^2R)^{1/2} - 1]^{-1} \text{ causal}
\]

(4-108)

*The reader is referred to the footnote on p. 155.
when one uses (4-109). For \( k = \infty \), (4-107) reduces to

\[
\rho_0(\infty) = 1 + 2\pi K_i^2 R \quad \text{noncausal}
\]

\[
\rho_0(\infty) = \frac{2\pi K_i^2 R}{\ln(1 + 2\pi K_i^2 R)} \quad \text{causal}
\]

(4-109)

The factor \( K_i \) plays the role of the modulation index for PM.

The loop design point is defined as the locus of a point that moves such that \( \sigma^2(k) = 1 \). From (4-105) this requires that

\[
K_i^2 = [1 + C(k)]^{1-1/2k} \quad \text{noncausal}
\]

\[
\frac{C(k)}{2kK_i^2} = [1 + C(k)]^{1/2k} - 1 \quad \text{causal}
\]

(4-110)

for arbitrary \( k \). For \( k = 1 \), (4-110) can be rearranged to give

\[
R_D = \frac{K_i^2 - 1}{4K_i^2} \quad \text{noncausal}
\]

\[
R_D = K_i^2 - 1 \quad \text{causal}
\]

(4-111)

---

**Fig. 4-14.** Output SNR versus \( R \) for \( k = 1 \) and Noncausal Filters.
where $R_D = A_D^2/aN_0$ is the design point signal-to-noise ratio for a given modulation index. For $k = \infty$, the corresponding locus of the design point in (4-110) moves such that

$$R_D = \frac{K_i^2 - 1}{4K_i^2} \text{ noncausal}$$

$$2\pi R_D = \ln(1 + 2\pi K_i^2 R_D) \text{ causal} \quad (4-112)$$

The output SNR given in (4-108) is plotted as a function of $R$ in Figs. 4-14 and 4-15 for various values of the modulation index. The threshold locus or design point locus (shown dotted) is given by (4-111). It is clear from these figures that PM performance can be improved relative to amplitude modulation (AM) by increasing the modulation index. However, the maximum value of $K_i$ is limited by the system threshold $\sigma^2(1) = 1$. If the modulation index is set at its maximum as determined by (4-111), the resulting output SNR is maximized. For large $R$, $k = 1$, and causal filtering, it follows from (4-111) that this maximum index becomes $K_i \approx \sqrt{R_D}$ and from (4-108)

$$\rho_0(1) \approx R \quad (4-113)$$

Fig. 4-15. Output SNR versus $R$ for $k = 1$ and Causal Filters.
Thus the maximum output SNR for PM with \( k = 1 \) is asymptotically proportional to the \( R \), while for AM (\( K_r = 1 \)) it is asymptotically proportional to \( \sqrt{R} \). The receiver performance for other values of \( k \) can be evaluated by means of a digital computer using (4-99), (4-107) and (4-110).

4-9 Frequency Demodulation Using the Linear PLL Model

It is clear from Fig. 4-13 that two types of errors are made at the receiver when used as a demodulator of frequency modulation. One is the total phase error \( \sigma^2 \), which is the error due to the inability of the receiver VCO to follow the incoming phase \( \theta(t) \) exactly. To ensure proper operation of the PLL—that is, to make the loop operate as linearly as possible—it is necessary to minimize \( \sigma^2 \) by properly selecting the loop filter \( F(s) \). The other error is the mean-squared frequency tracking error \( \sigma_f^2 = E[(m - y)^2] \). In order that the receiver reproduce the modulation waveform \( m(t) \) as faithfully as possible and that the PLL follow the observed data as closely as possible, it is necessary that we simultaneously minimize \( \sigma_f^2 \) and also \( \sigma^2 \).

It may seem impossible at first to minimize the phase and frequency errors simultaneously; however, since the receiving system has been linearized, the filter \( H_p(s) \), and hence the loop filter \( F(s) \), may be determined so as to minimize \( \sigma^2 \). Having determined the loop filter \( F(s) \) that minimizes \( \sigma^2 \), we may adjust the output filter \( F_0(s) \) so as to minimize \( \sigma_f^2 \) when \( H_p(s) \) is fixed. The ability to carry out the optimization procedure in this manner is a result of the linear receiver model.

For the case of FM we have from (4-84) and (4-95) that

\[
S_d(\omega) = \frac{K_i^2 S_m(\omega; k)}{\omega^2} = \frac{K_i^2 K_i(k)}{\omega^2[1 + (\omega/a)^{2k}]} \tag{4-114}
\]

when \( d(t) = 0 \). Direct substitution of (4-114) into the Yovits-Jackson formula (2-41), with \( N_0 \) replaced by \( N_0/a^2 \) and \( S(\omega) \) by \( S_d(\omega) \), produces the optimum realizable linear filter that minimizes \( \sigma^2 \), namely

\[
H_p(i\omega) = 1 - \frac{\omega^2(\omega^{2k} + a^{2k})}{\omega^2(\omega^{2k} + a^{2k}) + 2a^{2k}K_iA^2K_i(k)/N_0} \tag{4-115}
\]

For the case \( k = 1 \) this can be simplified to give

\[
H_p(i\omega) = \frac{a[i(\gamma - 1)\omega + \delta a]}{a^2\delta + i\gamma\omega - \omega^2} \tag{4-116}
\]

where
\[ \gamma \triangleq \sqrt{1 + 2\delta} \quad \delta \triangleq 2m_r\sqrt{R} \]

\[ m_r \triangleq \frac{K_i}{a} \quad R \triangleq \frac{A^2}{aN_0} \]  

(4-117)

It seems appropriate to refer to \( m_r \), the ratio of the modulation index \( K_i \) to the 3 dB radian frequency \( a \), as the deviation ratio.

Since \( \dot{\theta}(t) \) is the phase variation of the signal portion of the received waveform \( x(t) \) and since the desired output is \( m(t) \)—that is, the time derivative of \( K_i \dot{\theta}(t) \)—the linear operation that must be performed on the input signal \( \theta(t) \) is one of differentiation; hence \( L(s) = s/K_i \). Direct substitution of this fact and (4-114) into (4-93) with \( s = i\omega \) produces the optimum linear filter that minimizes \( \sigma_r^2 \),

\[ H_f(i\omega) = \frac{ia\gamma (\gamma - 1)\omega}{2K_i(a^2\delta + ia\gamma \omega - \omega^2)} \]  

(4-118)

for \( k = 1 \).

The loop filter \( F(i\omega) \) may be determined by using (4-116) in (4-7). Thus

\[ F(i\omega) = \frac{1}{AK} \left[ \frac{ia\gamma (\gamma - 1)\omega + a^2\delta}{i\omega + a} \right] \]  

(4-119)

whereas the output filter is determined by using (4-116) and (4-118) in (4-89). With \( s = i\omega \) in the result, this filter is given by

\[ F_\phi(i\omega) = \frac{a(\gamma - 1)^2K}{2K_i[\gamma(\gamma - 1)\omega + a\delta]} \]  

(4-120)

Both filters \( F_\phi(i\omega) \) may be realized with RC networks.

Consider first the realization of the loop filter \( F(i\omega) \) by means of a lag-lead network. Rewriting \( F(i\omega) \), we have

\[ F(i\omega) = \frac{a\delta}{AK} \left[ \frac{1 + (\frac{\gamma - 1}{a\delta})i\omega}{1 + i\omega/a} \right] \]

\[ = K_0 \left( 1 + \frac{i\tau_z\omega}{1 + i\tau_z\omega} \right) \]  

(4-121)

where \( K_0 = a\delta/AK \) and \( \tau_1 = 1/a \); \( \tau_2 = (\gamma - 1)/a\delta = 2\tau_1/(\gamma + 1) \). Thus the time constants \( \tau_1 \) and \( \tau_2 \) are directly related to the basic communication parameters \( A, N_0, m_r, \) and \( a \). The output filter \( F_\phi(i\omega) \) may be accomplished with
a simple $RC$ network. Rewriting $F_0(i\omega)$ and relating this to the pertinent time constants, we have

\[ F_0(i\omega) = K_0' \left( \frac{1}{1 + i\tau \omega} \right) \quad (4-122) \]

where

\[ K_0' = K \left( \frac{\gamma - 1}{\gamma + 1} \right) \quad (4-123) \]

and $\tau = R_1C_2$.

An important conclusion is that the design of the filter function $H_f(i\omega)$ does not depend on $K$; however, design of the output filter $F_0(i\omega)$ and the loop filter $F(i\omega)$ does. Furthermore, we see that the optimum filter’s pole-zero configuration depends, unfortunately, on both the input signal and the noise powers. This means that in order to maintain optimum demodulation at the receiver in the presence of variations in the received signal and noise mean-squared strengths, the filter must be adaptive. Designing such a filter would be extremely difficult, if not impossible. Therefore, in a later section, we shall fix the filter structure that is optimum only for a certain signal power $A^2$ and noise power $N_0$, and compute the filter performance when the filter is subjected to new signal and noise levels.

**4.9.1 The Receiver Design Point Criterion and the Output Signal-to-Noise Ratio, $k = 1$, Causal Filtering**

Having determined the various filters that comprise the receiver structure, we shall now determine its performance. The problem here is to “compare” a fixed filter, which is optimum for only one value of input signal-to-noise ratio, say $R_0 = A_0^2/aN_0$, with the optimum receiver whose pole-zero configuration varies with $R$. Briefly, we shall fix the filter designed to be optimum at a value of $R = R_0$ and then subject it to a new signal power $A \neq A_0$ and a new noise power density $N_1$. Of course, even though the loop filter is fixed, the loop bandwidth changes as a function of $A^2$. Practically speaking, the loop would normally be preceded by an automatic gain control (AGC) amplifier that holds $A = A_0$ for the range of interest. In this section we therefore assume that $A = A_0$. A later section treats the case where $A$ may vary.

First we must define what we mean by receiver performance. One performance criterion that has been used successfully for design purposes is that of output signal-to-noise ratio. For our purposes here, we define the “output” signal-to-noise ratio for FM as the ratio of the mean-squared value of the desired signal $m(t)$ to the mean-squared frequency tracking error $\sigma_f^2$; that is, the
noise power in the system is taken to be equal to $\sigma_f^2$. Thus for FM*

$$\rho_0(k) = \frac{\bar{m}^2(t)}{E[y(t) - m(t)]^2} = \frac{1}{\sigma_f^2}$$  \hspace{1cm} (4-124)

is our measure of signal fidelity, and we see that this definition is nothing more than the reciprocal of the Wiener error. [Note that $\bar{m}^2(t) = 1$.]

The "output" signal-to-noise ratio may be easily computed using (4-124) and the appropriate filter and signal spectral density equations. First we compute the signal distortion from (4-92). This term is easily obtained by direct substitution of (4-114) and (4-118) into (4-92) with $L(s) = K^{-1}r_s$ and $s = i\omega$. Carrying out the integration in the complex plane using Table 4-1 gives

$$S_d = \frac{3\gamma - 1}{\gamma(\gamma + 1)}$$  \hspace{1cm} (4-125)

Similarly, that portion of the total error due to the additive noise is easily found from the second term of (4-90). This turns out to be

$$\sigma_n^2 = \frac{(\gamma - 1)^2}{\gamma(\gamma + 1)^2}$$  \hspace{1cm} (4-126)

while for a new noise spectral density of $S_n(s) = N_f/2A^2$ we get

$$\sigma_{n_1}^2 = \frac{N_f}{N_0} \frac{(\gamma - 1)^2}{\gamma(\gamma + 1)^2}$$  \hspace{1cm} (4-127)

If we substitute (4-125) and (4-127) into (4-124), we find the "output" signal-to-noise ratio for $k = 1$, namely,

$$\rho_0(1) = \frac{1}{S_d + [((\gamma - 1)^2/\gamma(\gamma + 1)^2)/\kappa}$$  \hspace{1cm} (4-128)

where

$$\kappa \triangleq \frac{A^2}{aN_1} \frac{aN_1}{A^2} \triangleq \frac{R_D}{R}$$  \hspace{1cm} (4-129)

and $R_D$ is the design point input signal-to-noise ratio while $R$ is the actual

*Although we treat here the case $k = 1$, in general, $\sigma_f^2$ depends upon $k$ in (4-95). For convenience, we do not show the dependence on $k$ in (4-124).
input signal-to-noise ratio. As a novel by-product of the analysis, the output signal-to-noise ratio for the optimum receiver with \( k = 1 \) is obtained when \( R = R_{D} \). That is,

\[
\rho_{o}(1) = \frac{(\gamma + 1)^{2}}{4\gamma} \tag{4-130}
\]

for all \( R \geq R_{D} \).

One final computation is that of determining the threshold locus of the PLL. This locus is found from (4-86), (4-97), and (4-116) and the use of Table 4-1. The noise spectral density \( S_{n}(s) \) given by \( N_{r}/2A^{2} \) is used, however. Substituting these quantities into (4-86) and performing the integration yields

\[
\sigma^{2} = \frac{2m_{f}^{2}}{\gamma(\gamma^{2} - 1)} \left[ 1 + \frac{\kappa(3\gamma - 1)}{\gamma + 1} \right] \tag{4-131}
\]

for the suboptimum or fixed-receiver structure. Letting \( \kappa = 1 \), we obtain the desired "threshold" locus

\[
\sigma^{2} = \frac{8m_{f}^{2}}{(\gamma - 1)(\gamma + 1)^{2}} \tag{4-132}
\]

### 4-9.2 Receiver Performance for Reception of a Signal whose Power Is Different from the Design Level, \( k = 1 \), Causal Filtering

It was pointed out earlier that the closed-loop transfer function of the PLL depends on the received signal power \( A^{2} \). In this section we wish to determine the effects on receiver performance when the transmitter power, say \( A^{2} \), differs from the design level \( A_{0}^{2} \). We assume further that the noise spectral density \( N_{r} \) watts/Hz remains fixed.

Since the closed-loop transfer function of the PLL varies with the received signal power, it is necessary to determine the new transfer function when the filter is subjected to the new signal power \( A^{2} \). From (4-6) we can write the closed-loop transfer function for the design point power

\[
H_{p}(s) = \frac{A_{p}KF(s)}{s + A_{p}KF(s)} \tag{4-133}
\]

Substituting the fixed-loop filter given in (4-119) evaluated at the design levels into (4-133) and rearranging yields

\[
H_{p}(i\omega) = \frac{\beta[i\alpha(\gamma - 1)\omega + a^{2}\delta]}{a^{2}\delta^{2} + i\alpha\omega[1 + \beta(\gamma - 1)] - \omega^{2}} \tag{4-134}
\]
where

\[ \beta \triangleq \frac{A}{A_0} \]  \hspace{1cm} (4-135)

At \( \beta = 1 \), \( H'_\phi(s) \) reduces to \( H_\phi(s) \) in (4-116) as it should. The overall fixed FM receiver is obtained from \( H'_r(s) = F(s)H'_\phi(s) \).

\[ H'_r(i\omega) = \frac{i\beta\alpha^2(\gamma - 1)\omega}{2K_0[\alpha^2\delta\beta + i\omega(1 + \beta(\gamma - 1))]} \]  \hspace{1cm} (4-136)

The new frequency tracking error \( \sigma^2 \) is easily obtained from (4-90). First, we compute the new signal distortion term, using (4-114) with \( k = 1 \) and (4-136) in (4-92). Carrying out the necessary integration, using Table 4-1, yields

\[ S'_d = \frac{4\delta\beta(\delta\beta + \alpha) + 4\delta\beta(\alpha^2 - \lambda) + \lambda(\alpha + 1)}{4\delta\beta[(\alpha + 1)(\delta\beta + \alpha) - \delta\beta]} \]  \hspace{1cm} (4-137)

where

\[ \alpha \triangleq 1 + \beta(\gamma - 1) \]
\[ \lambda \triangleq \beta(\gamma^2 - 1) - (\gamma - 1)^2 \]  \hspace{1cm} (4-138)

and \( \beta \) is defined in (4-135).

On the other hand, the variance of the new noise is easily found by substituting (4-136) into the second term of (4-90) and integrating with \( S'_n(s) = N_0/2A^2 \). The result is

\[ \sigma^2 = \frac{(\gamma - 1)^2}{\alpha(\gamma - 1)^2} \]  \hspace{1cm} (4-139)

Hence the new "output" signal-to-noise ratio \( p'_\phi(1) \) is obtained from (4-124), (4-137), and (4-139). That is,

\[ p'_\phi(1) = \frac{1}{S'_d + [(\gamma - 1)^2]/\alpha(\gamma + 1)^2} \]  \hspace{1cm} (4-140)

which is the required result.

The new total mean-squared phase error \( \sigma^2 \) that results from a change in received signal power becomes

\[ \sigma^2 = \frac{2m^2}{\alpha^2(\gamma^2 - 1)} \left[ 1 + \frac{2\beta(\gamma - 1)^2 + \gamma^2 - 1}{\gamma^2 - 1} \right] \]  \hspace{1cm} (4-141)

when (4-134) is substituted into (4-86) and the integration is performed. Thus
(4-140) and (4-141) are the required results needed for specifying the receiver performance based on the linear PLL theory.

The loop-noise bandwidth $W_L$ at the loop design point can be found by substituting (4-116) into (4-14) and integrating. This yields

$$\frac{W_L}{2a} = \frac{(\gamma - 1)(3\gamma - 1)}{8\gamma} \quad (4-142)$$

Solving (4-142) for $\gamma$, we find

$$\gamma = \frac{2}{3} \left(1 + \frac{W_L}{a}\right) + \frac{1}{2} \sqrt{\left[\frac{4}{3} \left(1 + \frac{W_L}{a}\right)\right]^2 - \frac{4}{3}} \quad (4-143)$$

which is approximately equal to

$$\gamma \approx \frac{4}{3} \left(1 + \frac{W_L}{a}\right) \quad (4-144)$$

when $W_L/a > 1$. For large $\gamma$ or a large normalized 3-dB loop bandwidth $W_L/a$, we have

$$\gamma \approx \frac{4W_L}{3a} \quad (4-145)$$

Substituting (4-145) into (4-116), we obtain, for reasonable demodulator input conditions,

$$H_p(i\omega) \approx \frac{1 + (3/2W_L)i\omega}{1 + (3/2W_L)i\omega - (9/8W_L)^2\omega^2} \quad (4-146)$$

which has a damping factor of $1/\sqrt{2}$ or $r = 2$. This agrees with the Jaffe-Rechtin model (4-62) found for Doppler tracking and we again have a second-order loop.

4-9.3 Graphical Results for Linear PLL Frequency Demodulators, Causal Filtering

In Fig. 4-16 we have graphically illustrated the performance of the linear FM demodulator for two different receiver terminations: the optimum PLL demodulator and the PLL demodulator preceded by an ideal AGC amplifier. From Fig. 4-16 it is evident that the optimum demodulator outperforms the other realization when $R > R_p$. In the suboptimum system, the signal-to-noise ratio $\rho_s(1)$ becomes asymptotic (for large $R$) to the reciprocal of the signal distortion $S_d$ initially designed into the system. For comparison purposes we
Fig. 4-16. Performance Characteristics for a PLL Type Frequency Demodulator.

indicate the performance (derived in Chapter 2) of an amplitude-modulated double-sideband or single-sideband suppressed carrier system that demodulates the noisy received data by coherent frequency translation and by smoothing the resulting waveform with a Wiener filter. The improvement obtained by using FM is clearly manifested. According to the theory, it is evident that, for any \( R > R_p \), the output SNR can be increased by increasing the deviation ratio \( m_f \) until \( \sigma^2 > 1 \).

4-9.4 Receiver Performance for Other Values of the Parameter \( k \)

It is interesting to study the total mean-squared phase error for the class of modulation processes defined in (4-114). Using the Yovits-Jackson formula (2-42), we have

\[
\sigma^2(k) = \frac{N_0}{4\pi A^2} \int_{-\infty}^{\infty} \ln \left[ 1 + \frac{2K_c^2 A^2 K_i(k)}{N_0 \omega^2 [1 + (\omega/a)^{2k}]} \right] d\omega \quad (4-147)
\]

Evaluating this integral for general \( k \) is a formidable task; however, it may be numerically integrated on a general-purpose computer with no difficulty.

Of particular interest are the cases where \( k = 1 \) and \( k = \infty \). For \( k = 1 \), \( \sigma^2(1) \) is given by (4-132). For \( k = \infty \), (4-147) reduces to
\[ \sigma^2(\infty) = \frac{1}{\pi R} \left[ \ln \left( 1 + 2\pi m_f^2 R \right) + \sqrt{2\pi m_f^2 R} \tan^{-1} \left( \frac{1}{\sqrt{2\pi m_f^2 R}} \right) \right] \] (4-148)

The total mean-squared phase error that arises in filtering the asymptotically Gaussian process defined in (2-62) is found from (2-42) with \( N_0 \) replaced by \( N_0/2A^2 \). This turns out to be

\[ \sigma^2(k) = \frac{N_0}{4\pi A^2} \int_{-\infty}^{\infty} \ln \left[ 1 + \frac{2A^2 K_2^2(k)/N_0}{\omega^2 \left( 1 + (\omega/a\sqrt{k})^2 \right)^{k/2}} \right] d\omega \] (4-149)

where \( K_2(k) \) is defined in (2-60) with \( P = 1 \). Again the integral may be evaluated in terms of well-known functions only for special cases, and numerical integration must be used in general. For \( k = 1 \), (4-149) reduces to (4-132), while for \( k = \infty \), we find that

\[ \sigma^2(\infty) = \frac{1}{2\pi R} \int_{-\infty}^{\infty} \ln \left[ \frac{1 + 4\sqrt{\pi R} m_f^2 \exp \left( -x^2 \right)}{x^2} \right] dx \] (4-150)

The loop design point (so-called threshold) loci corresponding to the case \( k = 1 \) and causal filtering is given by

\[ 8m_f^2 = (\gamma - 1)(\gamma + 1)^2 \] (4-151)

while for \( k = \infty \) and the maximally flat spectra we have from (4-148) that

\[ \pi R = \ln \left( \sqrt{1 + 2\pi m_f^2 R} \right) + \sqrt{2\pi m_f^2 R} \tan^{-1} \left( \frac{1}{\sqrt{2\pi m_f^2 R}} \right) \] (4-152)

and for \( k = \infty \) and the asymptotically Gaussian spectra we have from (4-150) that

\[ 2\pi R = \int_{-\infty}^{\infty} \ln \left[ \frac{1 + 4\sqrt{\pi R} m_f^2 \exp \left( -x^2 \right)}{x^2} \right] dx \] (4-153)

The plot in Fig. 4-17 indicates how large \( m_f \) may be made at the transmitter, for a given signal-to-noise ratio \( A^2/aN_0 \) in the channel, before the phase-locked loop frequency demodulator reaches "threshold". From this figure it is clear that the maximally flat (MF) spectra yield a better "threshold" characteristic than do the asymptotically Gaussian spectra.* Hence the output signal-to-noise ratio will be larger (for a given \( k \), \( m_f \), and \( A^2/aN_0 \)) for the MF processes than for the asymptotically Gaussian processes. The important practical con-

*By "maximally flat spectra" is meant the case \( k = \infty \) in (4-95).
Fig. 4-17. Threshold Characteristics as Specified by Linear PLL Theory.

The conclusion that may be reached is that the telemetry spectra to be conveyed to the receiver should be shaped prior to transmission by means of a Butterworth filter rather than by isolated-cascaded RC networks. This shaping serves to reduce the “threshold” of the PLL receiver provided the loop filter $F(s)$ is selected appropriately.

Notice that for small signal-to-noise ratios in the channel, the asymptotically Gaussian process for $k = 1$ and $\infty$ are relatively close together and that the receiver will “threshold” rather easily. Under the same channel conditions, the converse is true for the MF process with $k = \infty$. For $1 < k < \infty$, other “threshold” characteristics can lie between the respective $k = 1$ and $k = \infty$ characteristics shown in Fig. 4-17.

4-10 Bandpass PLL Design Using Multiple Filters in the Loop

To this point we have considered the problem of PLL design using linear theory at baseband. For certain types of modulation of interest, for example, in the Apollo missions, it has been pointed out by Develet (Ref. 5) that bandpass design can be used to obtain improved performance; in particular, for
those in which large amounts of energy are contained in the modulation sidebands—for example, in tone-ranging systems or systems where multiple subcarriers are sparsely scattered in the baseband.

Consider the situation where

$$\theta(t) = M(t) + \sum_{i=1}^{N} x_i(t) \sin[\omega_i t + a_i(t)]$$  \hspace{1cm} (4-154)

and the purpose of the loop is to maintain phase-lock to the oscillations.

$$x(t) = \sqrt{2} A \sin[\omega_d t + \theta(t)] + n_i(t)$$  \hspace{1cm} (4-155)

Here we assume that $M(t)$ is a low-pass modulation and we elect to demodulate $x(t)$ (i.e., track $\theta$) by a PLL whose loop filter consists of the set of filters—that is, \{\(F_i(s), \ldots, F_N(s)\)\}. Based on the methods of Chapter 3 and those of this chapter, it is easy to show that the equivalent loop model is illustrated in Fig. 4-18.

![Fig. 4-18. The Multifilter Linear PLL Equivalent Loop Model.](image)

The closed-loop transfer function can be written as

$$H_\varphi(s) = \frac{\sum_{k=1}^{N} AKF_k(s)}{s + AK \sum_{j=1}^{N} F_j(s)}$$  \hspace{1cm} (4-156)

which evidences an interesting phenomenon that we presently note. Under the condition that the individual closed-loop responses $AKF_k(s)/[s + AKF_k(s)]$ develop and go to zero at frequencies remote from that of any of the others, then to a good approximation
\[ H_k(s) \approx \sum_{k=1}^{N} \frac{AKF_k(s)}{s + AKF_k(s)} \quad (4-157) \]

The cases under which (4-157) hold are those in which the multiple subcarriers are separable at baseband. On the basis of (4-157) it is important to note that in order to specify loop behavior, we need only study loop response of a single element. The loop behavior is then simply the sum of the individual behaviors; that is, assuming uncorrelated baseband signals and noise, one need only root sum square the errors due to signal distortion and the noise errors due to each element of the bank. The noise bandwidths of each element of the bank simply add.

4-11 Phase-Locked Loop Mechanizations for Carrier Tracking

In this section we discuss various PLL mechanizations of practical interest. The main feature of these receivers is that they employ heterodyning of the incoming signal at various stages. Thus the engineer is able to combine the advantages of intermediate frequency (IF) amplification to produce high-loop gains with those of the PLL—coherent communications capability at narrow loop bandwidths and predictable stability.

A superheterodyne phase-locked loop receiver is illustrated in Fig. 4-19.

![Fig. 4-19. The Superheterodyne PLL Receiver.](image)

The incoming signal at frequency \( f_r \) is RF amplified and then mixed with a heterodyning local reference at frequency \( f_0 \), which is produced by a frequency multiplier at frequency \( f_0/M \). The intermediate frequency is \( f_1 = f_r - f_0 \) or \( f_1 = f_0 - f_r \), depending on whether low-side or high-side injection is used. The IF amplifier consists of filtering and amplifying properties. A fixed-reference oscillator at frequency \( f_3 \) is compared with the IF amplifier output in the phase detector (multiplier). The loop is then closed through the loop filter, VCO, frequency multiplier and mixer.
Fig. 4.20. The Double Superheterodyne PLL Receiver Preceded by a BPL.
Fig. 4-21. The Coherent Transponder of the PLL Type.
A double-superheterodyne PLL receiver preceded by a BPL is illustrated in Fig. 4-20. This receiver differs from the one illustrated in Fig. 4-19 in that there are two separate intermediate frequency amplification stages and a band-pass limiter is inserted in the loop. Such receivers can be designed to operate with very narrow bandwidths and low-signal levels, and they can achieve a great deal of stability, precision, and reliability. The receiver’s loop gain is provided by the two IF mixer amplifiers. The loop equation of operation can be shown to reduce to the equation of loop operation derived in Chapter 3.

A coherent transponder of the PLL type is illustrated in Fig. 4-21. A transponder is said to be coherent if its transmitted frequency \( f_t \) is a rational multiple of the received frequency \( f_r \)—that is, \( f_t = (m/n)f_r \), where \( m \) and \( n \) are integers. Thus there are exactly \( n \) cycles out for every \( m \) cycles that enter the transponder. The frequency received at the reference system can be phase locked to and then multiplied by \( n/m \). The result is then compared to produce a measure of the frequency difference. This principle can be used to measure so-called two-way Doppler shifts, sense the Earth’s atmosphere with occultation satellites, and so on. Because the retransmitted frequency \( f_t = M_4f_r \), the ratio of the retransmitted frequency to the received frequency \( f_r \) can be shown to be given by

\[
\frac{f_t}{f_r} = \frac{M_4}{M_1 \pm M_2 \pm M_3} \tag{4-158}
\]

where \( M_1, M_2, M_3, \) and \( M_4 \) are frequency multiplication ratios.

4-12 Sample Design of a Superheterodyne PLL Receiver Preceded by a BPL

This section presents the design of a PLL receiver based on the theory given thus far*. The loop is to be used for carrier tracking purposes, and it is specified that the loop filter be of the form given in Fig. 4-4. The steps of the design progress in a prescribed sequence, using system data, design equations, circuit data, and values that have been obtained as the result of calculations already performed. Without loss in generality, we set \( A = \sqrt{P_c} \).

A. Definitions

A-1 VCO Output Frequency Multiplication Factor

\[
M = \frac{\text{frequency into Mixer}}{\text{frequency of VCO}}
\]

*This section can be omitted upon first reading.
This is limited by the signal frequency, the type of oscillator desired, and the maximum tunable frequency range needed to accommodate the Doppler plus transmitter frequency excursions.

A-2 Predetection Bandwidth, $W_i$ Hz (two-sided)
This is limited by the maximum data channel bandwidth and by the maximum signal suppression ($\alpha_{10}$) that can be tolerated.

A-3 Open-Loop Phase Gain at Loop Design Point

$$G_0 = \alpha_{10}K_mK_vM \text{ sec}^{-1}$$

This is a measure of the tightness of lock; typical phase error values lie between $\pm 4$ and $\pm 20$ deg, depending on the application. This is analogous to $\Delta \theta_0$ in (4-6). The gain must be made high enough to ensure a satisfactorily small phase error between the incoming frequency and the frequency of the local VCO oscillator at all times.

A-4 Threshold or Design Point

$$2P_0 = N_6W_{LO} \quad N_6 = kT^o$$

This corresponds to unity phase error variance as specified by linear PLL theory.

A-5 Loop Noise Bandwidth, $W_L$ Hz
This is limited by system dynamics and by the time constants in the loop filter.

A-6 dc Loop Gain
$K_1$ is the gain provided by the loop amplifier; it is frequently unnecessary to provide this gain.

A-7 Phase-Detector Gain
$K_m$ volts/rad; this is a function of the peak voltages handled in the phase-detector circuitry.

A-8 VCO Gain Constant
$K_v$ rad/sec-volt; this is dependent on the type of oscillator, the type of controlling circuitry, and the frequency of operation. It can be varied over a wide range of values.

A-9 Tracking Filter Output Time Constant, $\tau_2$

$$\tau_2 = R_2C = \left(\frac{r_0 + 1}{2W_{LO}}\right)$$

This time constant is obtained from (4-78). In practice it is limited by the output impedance of the phase detector, the input impedance of the VCO, and the values of capacitance that can be obtained.

A-10 Second-Order Loop-Parameter Ratio
\[ r_0 = \frac{G_0 \tau_2^2}{\tau_1} \]

**B. Design Equations**

B-1 Noise Power
\[ N_0 = kT^0 = -174 \text{ dBm Hz when } T^0 = 290 \]
\( k = \text{Boltzmann's constant} \)
\( T^0 = \text{system noise temperature in degrees Kelvin} \)

B-2 Signal Amplitude Suppression Factor
This is determined from (4-72) or (4-73).

B-3 Open-Loop Gain at Any Signal Level (analogous to AK)
\[ G = K_i K_m K_f M \alpha_1 \sec^{-1} \]

B-4 Loop-Noise Bandwidth \((r_0 = 2)\)
\[ W_L = \frac{W_{L0}}{3} \left(1 + \frac{2\alpha_1}{\alpha_{10}}\right) \]

At the design point, \(W_L = W_{L0}\).

B-5 Loop Threshold Sensitivity (in dBm)
\[ 2P_0 = -174 \text{ dBm} + 10 \log W_{L0} \]

**C. Given System Parameters**

C-1 Nominal Transmitter Frequency
Design-center frequency \(400 \text{ MHz}\).

C-2 Transmitter Frequency Tolerance
Maximum difference between actual transmitter frequency and design-center frequency. Tolerance = 0.002%

C-3 Maximum Doppler Frequency Shift
Maximum difference between actual transmitter frequency and design-center frequency. \(\pm 10 \text{ kHz}\)

C-4 Minimum Expected Signal-to-Noise Ratio at Limiter Input
This is usually determined by the worst conditions that the receiver must operate. \(\rho_{10} = 1/200\)

C-5 Design-Point Loop Bandwidth
This is determined by the quality of the local VCO and the bandwidth that will accommodate the fast rise time expected in the signal phase characteristic. \(W_{L0} = 20 \text{ Hz}\)

C-6 Loop Damping at the Design Point
The value of \(r_0\) is chosen by the ingenuity of the design engineer.
\[ \zeta_0 = \frac{\sqrt{\tau_0}}{2} = 0.707 \]

**D. Design Circuit Values**

D-1 Maximum VCO Frequency Shift
Measured: Based on a medium \( Q \)-crystal oscillator. \( \pm 3 \) kHz

D-2 VCO Center Frequency
Experimentally determined in conjunction with D-1
51 MHz

D-3 Phase Detector Gain, \( K_m \)
Selected by design engineer.
0.35 volt/rad

D-4 VCO Gain, \( K_v \)
This quantity is measured.
400 rad/volt-sec

D-5 Loop Filter Capacitor
Measured; largest practical and very low loss.
100 \( \mu \)F

D-6 Tracking-Filter Output Time Constant, \( \tau_1 \)
Selected by design engineer.
920 sec

D-7 Reference Oscillator Frequency
Equals second IF Amplifier Frequency; 4 MHz

D-8 Reference Oscillator Multiplication Factor: 90/4

**E. Derived Receiving System Parameters**

E-1 Second IF Amplifier
Selected by design engineer.
4 MHz

E-2 Maximum Transmitter Frequency Tolerance
Product: C-1 times C-2.
\( \pm 8 \) kHz

E-3 Maximum Received Frequency Excursion
Sum of C-3 and E-2.
\( \pm 18 \) kHz

E-4 Phase Multiplication Ratio, \( M \)
E-3 divided by D-1. This ensures that the VCO can deviate between the greatest expected range of input frequencies.
\( M = 6 \)

E-5 Second Mixer Injection Frequency
Product of D-7 times reference oscillator multiplication factor D-8.
90 MHz

E-6 First Mixer Frequency
Product of D-2 times E-4.
306 MHz

E-7 First IF Amplifier Frequency
Difference of C-1 and E-6.
94 MHz

E-8 Threshold Signal Suppression Factor
From (4-73) and C-4.
\( \alpha_{10} \approx 0.063 \)

E-9 Strong Signal Suppression Factor
Assume \( \rho_r = 0.1 \) in predetection bandwidth.
\( \alpha_r \approx 0.96 \)

E-10 Strong Signal-Tracking Loop Bandwidth
Assuming \( r_0 = 2 \), one computes \( W_L \) from B-4, C-5, E-8 and E-9.

E-11 Filter Output Time Constant, \( \tau_2 \)
Computed from A-9, C-5, and the fact that \( r_0 = 2 \).
\[ \tau_2 = 0.075 \text{ sec} \]

E-12 Open-Loop Phase Gain, \( G_0 \), at the Design Point
Determined from A-10, D-6, and E-11 and the fact that \( r_0 = 2 \).
\[ G_0 = 3.25 \times 10^5 \]

E-13 Minimum dc Loop Gain
Determined from E-3, D-4, E-4, and E-8.
\[ K_1 = 388 \]

E-14 Shunt Filter Resistor, \( R_2 \)
Determined from A-9, D-5, and E-11.
\[ R_2 = 750 \Omega \]

E-15 Series Filter Resistor, \( R_1 \)
Determined from D-5, D-6, E-14, and \( \tau_1 = (R_1 + R_2)C \).
\[ R_1 = 9.2 \text{ M\Omega} \]

E-16 Tracking Threshold Sensitivity
Determined from B-5 and C-5.
\[ -161 \text{ dBm} \]

dBm implies dB below one milliwatt

4-13 Related Studies

The basis of the linear PLL theory was predicated on writing a series expansion for the loop nonlinearity \( \sin \varphi \) and then neglecting higher-order terms. Various approximation techniques have also been developed for the purpose of retaining the higher-order terms. Historically the first was Booton's equivalent gain technique (Ref. 6). The technique was applied by Develet (Ref. 7) to the PLL, and the theory developed became known as the quasi-linear theory. The basis behind this theory is one of attempting to extend the region of validity of the approximation \( \sin \varphi \approx \varphi \) by defining an equivalent loop gain that is a function of the mean-squared value of the phase error process.

Tausworthe (Ref. 3) has developed an extension of the quasi-linear theory, which he has called the linear-spectral theory. In this theory one selects two system gains, both of which are analogous with the equivalent gain defined by Booton. The development of the linear spectral theory proceeds on the basis of writing an equation for the spectral density of the phase error process in the steady state. It turns out that this equation is an approximation, for the phase error process \( \{\varphi(t)\} \) is only stationary in the linear region of loop operation. In order to proceed with this equation, one must further assume that the loop phase error obeys a Barrett-Lampard expansion (Ref. 8), which, as we shall later see, is not the case.

Another approach that has been explored in attempting to explain the nonlinear behavior of a PLL is the so-called Volterra function expansion tech-
nique. This technique was first applied to the loop analysis by Van Trees (Ref. 9). Unfortunately, the procedure is extremely tedious and does not give any insight to what is actually happening in the loop. Another approach based on the perturbation technique was introduced by Margolis (Ref. 10); it suffers from the same disadvantages as does the Volterra method. Further details pertaining to the development of these approximation theories will be given in Chapters 9 and 10.

Various efforts to apply estimation theory to the demodulation problem have been made. Lehan and Parks (Ref. 11) apparently were the first, while Youla (Ref. 12) extended their work and introduced the maximum a posteriori probability (MAP) estimate. Viterbi (Ref. 13) discusses the problem of optimum phase demodulation using the MAP criteria and shows that the PLL demodulator is optimum for high signal-to-noise ratios. Gilchriest (Ref. 14) was among the first to consider the problem of optimum design of a PLL as an FM discriminator or tracking filter.

Van Trees (Refs. 15, 16, and 17) and Thomas and Wong (Ref. 18), among many others, have used maximum a posteriori probability (MAP) or Bayes’ criteria to study various analog communication system problems. The MAP approach leads to an integral equation for the message estimate, and the solution to this equation corresponds to a physically unrealizable demodulator. Van Trees (Ref. 16) suggests making an approximation to the unrealizable demodulator for the purpose of implementation. Snyder (Ref. 19), on the other hand, takes the state variable approach to the optimum demodulation problem based on the theory of Markov processes. It turns out that the state variable approach leads directly to a physically realizable demodulator that is “equivalent” to the physically realizable portion of Van Trees (Ref. 16) MAP estimator. In fact, Snyder (Ref. 19) establishes quasi-optimum (optimum for large signal-to-noise ratios) phase and frequency demodulators of a stationary Gaussian message corrupted by additive white channel noise. These simplified demodulators reduce to a PLL mechanization. Since the appearance of the works of Tikhonov (Refs. 21, 22) there has been considerable interest (Refs. 19 and 23) in applying the theory of continuous Markov processes to determine the performance of nonlinear demodulators of the PLL type. We will cover this material in later chapters.

Finally, we note that the book by Gardner (Ref. 24) gives an elementary treatment of phase-lock techniques. This book also contains an extensive bibliography of the work contributed by American authors prior to 1966.
APPENDIX I

I-1 Passage of a Narrowband Process Through a Bandpass Limiter

In this appendix we outline and develop the theory required for understanding certain effects of passing a narrowband process through a bandpass limiter. The intent, in practice, is not simply to limit the power delivery to the receiver but to have the receiver adapt itself to varying signal-to-noise input conditions. This appendix develops and describes the properties of a bandpass limiter and shows how these properties lead to a useful adaptive behavior. For the moment, however, we shall not restrict ourselves to a particular zero-memory transformation law, for a few general results can be obtained.

Suppose that the input to the device in question is

\[ x(t) = s(t) + n_s(t) \]  \hspace{1cm} (I-1)

where the signal \( s(t) \) and noise \( n_s(t) \) are sample functions of stationary processes. Denoting the output of the device by \( y(t) \), the fundamental problem for the moment is to determine what is meant by the "signal" and "noise" portions of \( y(t) \). We write

\[ y(t) = s_y(t) + n_y(t) \]  \hspace{1cm} (I-2)

where \( s_y(t) \) is the output "signal" portion and \( n_y(t) \) is the output "noise" portion. The sample function \( n_y(t) = y(t) - s_y(t) \) is assumed to be decomposed
from \( y(t) \) in such a way that
\[
E[n_y(t)f[s(t)]] = E[n_y(t)]E[f[s(t)]]
\]  
(I-3)

where \( f \) is any function of the input signal process. This says that the output noise \( n_y(t) \) must be uncorrelated with any function of the input signal process, which implies that
\[
E[y(t_1)y(t_2)] = E[s_y(t_1)s_y(t_2)] + E[n_y(t_1)n_y(t_2)] + E[n_y(t_1)]E[s_y(t_2)]
+ E[n_y(t_2)]E[s_y(t_1)]
\]  
(I-4)

and hence the spectral density of \( y(t) \) is, except possibly for a dc term, the sum of the spectral densities of \( s_y \) and \( n_y \). If \( y(t) \) is to be decomposed such that the process \( \{n_y(t)\} \) must be orthogonal to the process \( \{s(t)\} \), then we must have
\[
E[(y - s_y)s] = 0
\]  
(I-5)

which says that
\[
E(sy) - E(ss_y) = 0
\]  
(I-6)

Equation (I-6) can be written as
\[
E[s(E(y)s) - E(s_s)s)] = 0
\]  
(I-7)

Since the processes \( \{s_y(t)\} \) and \( \{n_y(t)\} \) are orthogonal and \( \{n_y(t)\} \) is orthogonal to \( \{s(t)\} \), we note that \( s_y = E(s_y|s) \). If (I-7) is to hold for all \( s(t) \), then
\[
s_y(t) = E[y(t)|s(t)]
\]  
(I-8)

defines the output signal process. This is the key result—namely, the signal portion of the output at time \( t \) is obtained by fixing the input signal and averaging the output at time \( t \) over all possible noise inputs belonging to the noise process \( \{n_y(t)\} \).

We now assume that the input to the nonlinear device is
\[
x(t) = V(t) \sin [\omega_0t + \theta(t)] + n_y(t)
\]  
(I-9)

where \( n_y(t) \) is a narrowband Gaussian process and \( s(t) = V(t) \sin [\omega_0t + \theta(t)] \) is also a narrowband process. Letting \( \Phi(t) = \omega_0t + \theta(t) \) and \( V = V(t) \), the output signal \( s_y = E(y|V \sin \Phi) \) has the Fourier series expansion
\[
s_y = \sum_{k=-\infty}^{\infty} c_k \exp (ik\Phi)
\]  
(I-10)

where
\[ c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} E(y|V \sin \Phi) \exp (-ik\Phi) \, d\Phi \]  
(I-11)

Since \( \Phi(t) = \omega_0 + \theta(t) \), the terms in (I-10) are the signal components in the narrow frequency zones about each frequency component \( \pm k\omega_0 \). If we follow the nonlinear device by a bandpass zonal filter that passes the \( k \)th harmonic unchanged and annihilates all the others, the \( k \)th signal component in the output is

\[ s_k(t) = 2 \text{Re} \left( c_k \exp \{ ik[\omega_0 t + \theta(t)] \} \right) \]  
(I-12)

where \( \text{Re} \) denotes the real part of the quantity in parenthesis. Notice that in the first zone, \( k = 1 \), the phase modulation \( \theta(t) \) passes through unchanged! That is a very important result, and is exploited in the analysis of phase-locked loops.

As an example of our results, we assume that the zero-memory device has the ideal limiter characteristic defined by

\[ G[x(t)] = \text{sgn} [x(t)] = \begin{cases} +1 & x(t) \geq 0 \\ -1 & x(t) < 0 \end{cases} \]  
(I-13)

Then the conditional expectation in (I-8) can be evaluated directly.

\[ s_y(t) = \int_{-\infty}^{\infty} G[s + n_i] p(n_i) \, dn_i \]  
(I-14)

where \( p(n_i) \) is the p.d.f. of the noise. Here we have made use of the fact that \( p(y|s) = p(n_i - s) \). Making use of (I-13) in (I-14), we find that

\[ s_y(t) = \text{erf} \left( \frac{s(t)}{\sigma} \right) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \exp \left( -x^2 \right) \, dx \]  
(I-15)

For small signal-to-noise ratios,

\[ s_y(t) \approx \frac{s(t)}{\sigma} \]  
(I-16)

and the transformation is almost linear, whereas for large signal-to-noise ratios, the characteristic \( \text{erf} \, x \) behaves like the ideal limiter characteristic defined in (I-13).

If we now assume that \( V(t) = \sqrt{2} A \) and note that \( \sin \Phi \) is an odd function, then the conditional expectation \( E(y|V \sin \Phi) \) has the Fourier series expansion
Appendix I

\[ s_r(t) = \sum_{k=1}^{\infty} b_k \sin [k\omega_d t + k\theta(t)] = E(y|\sqrt{2}A \sin \Phi) \]  
(I-17)

where

\[ b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} E(y|\sqrt{2}A \sin \Phi) \sin k\Phi \, d\Phi \]  
(I-18)

and each term in the series is referred to as the \( k \)th zonal term due to the signal. Making use of (I-15) in (I-18), we have

\[ b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \text{erf}(v \sin \Phi) \sin k\Phi \, d\Phi \]  
(I-19)

where \( v = \sqrt{2A^2/\sigma^2} \). For \( k \) even, \( b_k = 0 \); while for \( k \) odd, (I-19) reduces to

\[ b_k = \sqrt{\frac{2}{\pi}} \frac{v}{k} \exp \left( \frac{-v^2}{4} \right) \left[ I_{\frac{k-1}{2}}(v^2/4) + I_{\frac{k+1}{2}}(v^2/4) \right] \]  
(I-20)

and the functions \( I_v(z) \) are modified Bessel functions of the first kind and order \( v \).

Consider now the bandpass-limiting-filter model illustrated in Fig. AI-1.

Fig. AI-1. Bandpass Limiter-Filter Model.

Here \( H(i\omega) \) is the frequency response of the predetection (IF) filter preceding the limiter. This filter is assumed to have a center frequency of \( \omega_0 \) rad/sec and an equivalent noise bandwidth

\[ W_i = 2B_i = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|H(i\omega)|^2}{|H(i\omega_0)|^2} \, d\omega \]  
(I-21)

The input signal \( x(t) \) is defined in (I-9). A particular sample function representation for the narrowband process \( \{n(t)\} \) can be written as

\[ n(t) = \sqrt{2} [n_i(t) \cos \Phi(t) - n_i(t) \sin \Phi(t)] \]  
(I-22)
where \( \sigma^2 = N_0 B_0 \). If \( V(t) = \sqrt{2} A \) in (I-9), a sample function of the input process assumes the form

\[
x(t) = E(t) \cos [\Phi(t) - \gamma(t)]
\]

where

\[
E(t) = \sqrt{2} [A - n_r(t)]^2 + 2n^2_r(t)
\]

\[
\gamma(t) = \tan^{-1} \left[ \frac{n_r(t)}{A - n_r(t)} \right]^{-1}
\]

Using the complex Fourier transform approach, we can relate the output \( y(t) \) to the input through

\[
y(t) = \frac{1}{\pi i} \int_C \exp \left[ i\lambda E(t) \cos [\Phi(t) - \gamma(t)] \right] d\lambda
\]

where \( C \) is the dynamic path of integration taken along the real axis with an indentation around the pole at \( \lambda = 0 \) and closed by a semicircle in the upper half-plane. If we expand the exponential function by using the Jacobi-Anger expansion, we get

\[
y(t) = \frac{1}{\pi} \sum_{k=0}^{\infty} \left\{ \epsilon_k i^{k-1} \int_C \frac{J_k(\lambda E(t))}{\lambda} \cos \{k[\Phi(t) - \gamma(t)]\} d\lambda \right\}
\]

\[
= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k + 1} \cos \{(2k + 1)[\Phi(t) - \gamma(t)]\}
\]

where \( \epsilon_k = 1 \) for \( k = 0 \), \( \epsilon_k = 2 \) for \( k \neq 0 \), and \( J_k(x) \) is the \( k \)th-order Bessel function. We note here that the total output power in the \( k \)th zonal term is zero for \( k \) even and becomes

\[
P_k = \frac{8}{\pi^2 k^2}
\]

for \( k \) odd. We also observe the remarkable fact that the value of \( P_k \) is independent of \( A \) and \( \sigma \)!

Define the limiter signal amplitude suppression factor in the \( k \)th zone as the ratio of the power in the \( k \)th harmonic of \( s_r(t) \) to the total power delivered to the output. Thus

\[
\alpha_k^2 = \frac{b_k^2}{2P_k} = \frac{S_k}{P_k}
\]

\[
= \frac{\pi}{4} \rho_i \exp (-\rho_i) \left[ \frac{I_{(k-1)/2} (\rho_i/2)}{2} + \frac{I_{(k+1)/2} (\rho_i/2)}{2} \right]^2
\]
and we have made use of (I-20) and (I-27). In (I-28) the parameter $\rho_i$ represents the signal-to-noise ratio at the output of $H(\omega)$; that is,

$$\rho_i = \frac{A_i^2}{\sigma^2} = \frac{2A_i^2}{N_0W_i} = \frac{A_i^2}{N_0B_i} \quad (I-29)$$

Since the total power $P_k$ in the $k$th zone is constant, it follows that the output power due to the $k$th component in $s_j(t)$ is

$$S_k = \alpha_k^2 P_k \quad (I-30)$$

and that due to the additive noise is

$$N_k = (1 - \alpha_k^2) P_k \quad (I-31)$$

so that the signal-to-noise ratio in the $k$th zone of the limiter output is

$$\rho_k = \frac{S_k}{N_k} = \frac{\alpha_k^2}{1 - \alpha_k^2} \quad (I-32)$$

for $k$ odd. We see from that the output signal-to-noise ratio $\rho_k$ in the $k$th zone is only a function of $\alpha_k$. In the limiting cases, it can be shown that

$$\rho_k \approx \frac{\pi}{\left(\frac{k - 1}{2}\right)!} \frac{2^{2k}}{\rho_i} \quad \text{for small } \rho_i$$

$$\approx \frac{2\rho_i}{k^2} \quad \text{for large } \rho_i \quad (I-33)$$

For $k = 1$, we note from (I-33) that

$$\frac{\rho_1}{\rho_i} \approx \frac{\pi}{4} (-1.05 \text{ dB}) \quad \text{for small } \rho_i \quad (I-34)$$

$$\frac{\rho_1}{\rho_i} \approx 2 (+3 \text{ dB}) \quad \text{for large } \rho_i$$

These results are due to Davenport (Ref. 25). An excellent engineering approximation for the ratio $\rho_1/\rho_i$ produced by Tausworthe (Ref. 3) is

$$\frac{\rho_1}{\rho_i} \approx \frac{0.7854 + 0.4768\rho_i}{1 + 0.2384\rho_i} \quad (I-35)$$

Illustrated in Fig. AI-2 is a plot of (I-35) which is also compared with an exact
plot produced from (I-28) and (I-32) with $k = 1$. Fig. AI-3 represents a plot of the signal power-suppression factors $\alpha_1^2$ and $\alpha_3^2$ for the first and third zones.

I-2 Signal-to-Noise Ratios Following Coherent Demodulation

In practice, the output $y_1(t)$ of the zonal filter is frequently multiplied by the reference signal $r(t, \varphi)$ to produce the signal $y_2(t, \varphi)$. The angle $\varphi$, now
assumed fixed, is dependent on the application. For example, when $\varphi = \pi/2$, the output represents the dynamic phase error in a phase-locked loop system, while $\varphi = 0$ produces coherent demodulation of amplitude modulation or phase-shift keying. The low-pass filter $H_d(iw)$ is used to remove the double-frequency term that appears in $y_d(t, \varphi)$. Assuming that the zonal filter passes only the first term in (I-26), then

$$y_1(t) = \frac{4}{\pi} \cos [\Phi(t) - \gamma(t)]$$  \hspace{1cm} (I-36)

If the reference signal takes the form

$$r(t, \varphi) = \sqrt{2} \sin [\Phi(t) + \varphi]$$  \hspace{1cm} (I-37)

then

$$z(t, \varphi) = \frac{2\sqrt{2}}{\pi} \sin [\gamma(t) + \varphi]$$  \hspace{1cm} (I-38)

since $H_d(iw)$ in Fig. A1-1 does not pass the double-frequency terms. Equation (I-38) can be expanded, using (I-24), to produce

$$z(t, \varphi) = \frac{2\sqrt{2}}{\pi} \left[ \left( \frac{A - n_e \cos \varphi}{\sqrt{(A - n_e)^2 + n_e^2}} \right) + \frac{n_e \sin \varphi}{\sqrt{(A - n_e)^2 + n_e^2}} \right]$$  \hspace{1cm} (I-39)

In evaluating the output signal-to-noise ratio, the first two moments of $z(t, \varphi)$ are needed. These can be shown (Ref. 26) to be given by

$$\mu_z = \frac{2\sqrt{2}}{\pi} \alpha_i \cos \varphi$$  \hspace{1cm} (I-40)

$$\sigma_z^2 = \frac{8}{\pi^2} \left( \cos^2 \varphi - \left[ \frac{1 - \exp(-\rho_i)}{2\rho_i} \right] \cos 2\varphi \right) - \mu_z^2$$

For large $\rho_i$, we have, using (I-28) for $k = 1$,

$$\mu_z \approx \frac{\sqrt{2}}{\pi} \left( 2 - \frac{1}{2\rho_i} - \frac{3}{16\rho_i^2} \right) \cos \varphi$$

$$\sigma_z^2 \approx \frac{4}{\pi^2\rho_i} \left( \frac{\cos^2 \varphi}{4\rho_i} + \sin^2 \varphi \right)$$  \hspace{1cm} (I-41)

while for small $\rho_i$, (I-40) is approximated by (using (I-28) for $k = 1$)

*The angle $\varphi$ is not to be confused here with the actual phase error in a PLL. It merely represents a convenient dummy variable.*
\[ \mu_z \approx \sqrt{\frac{2\rho_i}{\pi} \cos \varphi} \]  
\[ \sigma_z^2 \approx \frac{4}{\pi^2} \left[ 2 \cos^2 \varphi - \exp(-\rho_i) \cos 2\varphi \right] - \frac{2\rho_i \cos^2 \varphi}{\pi} \]  
(I-42)

If the output signal-to-noise ratio (SNR) is defined by

\[ \rho_o(\varphi) \triangleq \frac{\mu_z^2}{\sigma_z^2} \]  
(I-43)

then the ratio, \( \rho_o(\varphi)/2\rho_i \), of the output signal-to-noise ratio to the equivalent input signal-to-noise ratio is given by

\[ \frac{\rho_o(\varphi)}{2\rho_i} = \frac{\mu_z^2}{2\rho_i \sigma_z^2} \]  
(I-44)

The factor of \( \frac{1}{2} \) in (I-44) is required, for a bandwidth change of \( \frac{1}{2} \) accompanies the bandpass to low-pass transformation appearing at the multiplier output.

For \( \varphi = 0 \), (I-28), (I-40), (I-43), and (I-44) can be combined to give

\[ \frac{\rho_o(0)}{2\rho_i} = \frac{\pi \exp(-\rho_i) [I_o(\rho_i/2) + I_i(\rho_i/2)]^2}{8 \left[ 1 - \left[ 1 - \exp(-\rho_i/2\rho_i) \right] \right] - \rho_i \exp(-\rho_i) [I_o(\rho_i/2) + I_i(\rho_i/2)]^2} \]  
(I-45)

Asymptotically this reduces to

\[ \frac{\rho_o(0)}{2\rho_i} = \begin{cases} 
4\rho_i & \rho_i \gg 1 \\
\frac{\pi}{4 - 2\pi \rho_i} & \rho_i \ll 1
\end{cases} \]  
(I-46)

For a phase-locked loop mechanization, the average signal component into the loop is given by

\[ \mu' \left( \frac{\pi}{2} \right) = E \left[ \frac{dz(t, \varphi)}{d\varphi} \right]_{\varphi=\pi/2} \]  
(I-47)

so that the ratio of the input-to-output, signal-to-noise ratio becomes

\[ \frac{\rho_o(\pi/2)}{2\rho_i} = \frac{\pi \rho_i \exp(-\rho_i) [I_o(\rho_i/2) + I_i(\rho_i/2)]^2}{4 \left[ 1 - \exp(-\rho_i) \right]} \]  
(I-48)

The asymptotic values for this case are
Figure AI-4 represents a plot of (I-45) and (I-48), plus the ratio \( \rho_i/\rho_o \), as found from (I-32) with \( k = 1 \).

\[
\frac{\rho_o(\pi/2)}{2\rho_i} = \begin{cases} 
1 - \frac{1}{2\rho_i} & \rho_i \gg 1 \\
\frac{\pi}{4(1 - \rho_i)} & \rho_i \ll 1 
\end{cases}
\] (I-49)

Fig. AI-4. Ratio of the Input-to-Output SNRs for Various Demodulator Models (Courtesy of J. C. Springett).

I-3 Signal-to-Noise Spectral Density Ratios

In many applications of the limiter to communication system design, the output of the zonal filter is followed by a narrowband filter. In these cases, it is desirable to know the noise spectral density of the zonal filter output at \( \omega_0 \) in the incoherent case—that is, preceding the multiplier in Fig. AI-1. For the coherent case we will need to know the noise spectral density of the multiplier output around \( \omega = 0 \). We proceed in this section to develop these relationships.

Springett and Simon (Ref. 26) have carried out a numerical evaluation of the ratio of the output-to-input, signal-to-noise spectral density ratio, assuming that the filter characteristic \( H(i\omega) \) has an ideal rectangular frequency characteristic. Their results show that the ratio of incoherent (prior to multiplication) output-to-input, signal-to-noise spectral density ratios
\[(\Gamma_{\text{inc}})^{-1} = \frac{\alpha_i^2 P / N_0}{A^2 / N_0} = \frac{(S/N_0)_o}{(S/N_0)_i} = \frac{\rho_i B_i}{\rho_i B_i} = \frac{\alpha_i^2 B_i}{\rho_i(1 - \alpha_i^2) B_i} \] (I-50)

approaches 2 for large \(\rho_i\) and 0.86 for small \(\rho_i\). Here \(B_i\) is the first zone noise bandwidth, which we shall subsequently define. In (I-50) \(N_0\) is the value of spectral density of the zonal filter output noise at \(\omega = \omega_0\). The complete result for \(1/\Gamma_{\text{inc}}\) is produced in Fig. AI-5 and compared with

\[\frac{\rho_i}{\rho_i} = \frac{(S/N)_o}{(S/N)_i} = \frac{\alpha_i^2}{\rho_i(1 - \alpha_i^2)} \] (I-51)

![Graph showing the ratio of SNRs in the first zone](image)

**Fig. AI-5.** Ratio of SNRs Appearing in the First Zone (Courtesy of J. C. Springett).

In (I-51) we have introduced the notation \(\rho_i = (S/N)_o\) and \(\rho_i = (S/N)_i\). For small \(\rho_i\) we see from Fig. AI-5 that

\[\frac{(S/N)_o}{(S/N)_i} < 1 \] (I-52)

which says that the equivalent single-sided noise bandwidth of the first zonal output filter, say \(B_i\), is greater than that of the input noise. Tausworthe (Ref. 3) suggests the rational approximation

\[\Gamma_{\text{inc}} \approx \frac{1 + 0.345 \rho_i}{0.862 + 0.690 \rho_i} \] (I-53)

to (I-50) for the incoherent case when \(H(i\omega)\) has a rectangular filter characteristic.

It is also desirable to evaluate results analogous with (I-50) for the
coherent case—that is, post multiplication. Without presenting all the details, it is shown in Ref. 26 that the correlation function appearing at the output of the zonal filter in Fig. AI-1 is given by

\[ R_z(\tau) = \frac{2r(\tau)}{\sigma^2} + \sum_{k=1}^{\infty} \left[ \frac{1 \cdot 3 \cdot 5 \ldots (2k - 1))^2}{(2 \cdot 4 \cdot 6 \ldots 2k)^2} \right] \frac{2r(\tau)}{\sigma^2} \left( \frac{1}{k+1} \right) \int_{-\infty}^{\infty} \left[ \frac{2r(\tau)}{\sigma^2} \right]^{2k+1} d\tau \]  

(I-54)

where

\[ R_n(\tau) = 2r(\tau) \cos \omega_0 \tau \]

is the correlation function of the noise appearing at the output of \( H(i\omega) \) in Fig. AI-1. From the coefficient of \( \cos \omega_0 \tau \) in (I-54), we find that the single-sided spectral density around the origin is given by

\[ N_{0n} = \frac{2}{\pi} \left( \frac{1}{B_z} + \sum_{k=1}^{\infty} \left[ \frac{1 \cdot 3 \cdot 5 \ldots (2k - 1))^2}{(2 \cdot 4 \cdot 6 \ldots 2k)^2} \right] \left( \frac{1}{k+1} \right) \int_{-\infty}^{\infty} \left[ \frac{2r(\tau)}{\sigma^2} \right]^{2k+1} d\tau \]  

(I-55)

From this result and the fact that the output power delivered to the first zone is \( 8/\pi^2 \), one can deduce the bounds

\[ \frac{\pi}{4} < \frac{1}{\Gamma_H} < 1 \]  

(I-56)

where \( 1/\Gamma_H \) is the ratio of the output-to-input noise spectral density ratio when \( \rho_i = 0 \). This bound is also valid for the incoherent and coherent cases. For an arbitrary predetection filter characteristic, \( \Gamma_H \) is given exactly by

\[ \Gamma_H = 1 + B_z \sum_{k=1}^{\infty} \left[ \frac{1 \cdot 3 \cdot 5 \ldots (2k - 1))^2}{(2 \cdot 4 \cdot 6 \ldots 2k)^2} \right] \left( \frac{1}{k+1} \right) \int_{-\infty}^{\infty} \left[ \frac{2r(\tau)}{\sigma^2} \right]^{2k+1} d\tau \]  

(I-57)

Typical values of \( \Gamma_H \) for various predetection filters are given below:

<table>
<thead>
<tr>
<th>Filter</th>
<th>( 1/\Gamma_H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single pole RC</td>
<td>0.944</td>
</tr>
<tr>
<td>Gaussian</td>
<td>0.893</td>
</tr>
<tr>
<td>Ideal rectangular</td>
<td>0.864</td>
</tr>
</tbody>
</table>

The value of \( \Gamma_H \) for the Gaussian filter is probably typical of that value produced by a two-to-four pole RC filter. Moreover, Springett and Simon (Ref. 26) develop a relationship \( B_z(\rho_i, \phi) \) for the equivalent noise bandwidth of the zonal filter output as a function of \( \rho_i \) and \( \phi \). They show that, to a good approximation,
\[ B_i(\rho_i, \varphi) = B_i \left[ 1 + \left( \frac{4}{\pi \Gamma_H} - 1 \right) \exp \left( -\rho_i \left[ 1 - \frac{\varphi}{2} \right] \right) \right] \]  

(I-58)

for \( 0 \leq \varphi \leq \pi/2 \). To obtain \( B_i \), needed in (I-50) for the incoherent case, one averages (I-58) over \( \varphi \), assuming that \( \varphi \) is uniformly distributed.

For the phase-locked loop case, \( \varphi = \pi/2 \), and we have

\[ \Gamma_p = \frac{2\rho_i}{\rho_0(\pi/2)} \frac{B_i}{B_i} \]  

(I-59)

which reduces to

\[ \Gamma_p = \frac{4[1 - \exp(-\rho_i)][1 + (4/\pi \Gamma_H) - 1] \exp[-\rho_i(1 - \pi/4)]^{-1}}{\pi \rho_i \exp(-\rho_i)[I_0(\rho_i/2) + I_1(\rho_i/2)]} \]  

(I-60)

when we make use of (I-48) and (I-58) in (I-59). When \( \Gamma_H = 0.893 \), an excellent approximation to this result is given by

\[ \Gamma_p \approx \frac{1 + 0.542 \rho_i}{0.893 + 0.542 \rho_i} \]  

(I-61)

For \( \varphi = 0 \), the case of coherent demodulation, we have

\[ \Gamma_o \triangleq \frac{2\rho_i}{\rho_0(0)} \left[ \frac{1}{1 + (4/\pi \Gamma_H) - 1} \exp(-\rho_i) \right] = \frac{2\rho_i}{\rho_0(0)} \frac{B_i}{B_i} \]  

(I-62)

where \( \rho_0(0)/\rho_i \) is given in (I-45). When \( 1/\Gamma_H = 0.862 \), the resulting curves—produced from (I-50), (I-60), and (I-62)—are shown in Fig. AI-6. Shown also

\[ \frac{1}{\Gamma_H} = 0.86 \]

\[ \frac{1}{\Gamma_0} \]

\[ \frac{1}{\Gamma_{inc}} \]

\[ \phi = 0 \]

\[ \phi = \pi/2 \]

\[ \text{Upper Bounds} \]

\[ \text{Lower Bounds (Zonal Ratios)} \]

\[ \text{Limiter Input SNR, } \rho_i (\text{dB}) \]

Fig. AI-6. Various Limiter Performance Factors versus \( \rho_i \) (Courtesy of J. C. Springett).
are the lower and upper bounding curves obtained by letting \( \Gamma_L = \pi/4 \) and \( \Gamma_U = 1 \) respectively.

In summary, Davenport (Ref. 25) performed the classic analysis of limiters, whereas Springett and Simon (Refs. 26, 27) and Tausworthe (Ref. 28) reconsidered the important aspects of Davenport's earlier work from an engineering point of view. Blackman (Ref. 29) has considered the problem of determining the various signal and noise components appearing at the output of a nonlinearity. Aein (Ref. 30) gives a rather complete list of references pertaining to the limiter problem.

Problems

4-1 A 1-MHz clean-up loop for Mariner-Venus 67 precision ranging and occultation experiment was designed such that \( \tau_1 = 7600 \text{ sec}, \tau_2 = 124 \text{ sec}, \) and \( AK = 2 \text{ sec}^{-1} \). This clean-up loop was used to provide spectrally pure reference signals. Find (a) the loop damping, (b) natural frequency, and (c) the loop bandwidth when an imperfect integrating filter is placed in the loop. (d) Choose \( C = 250 \mu\text{F} \) and find \( R_1 \) and \( R_2 \).

4-2 A narrowband carrier tracking loop used in the deep-space instrumentation facility for tracking planetary spacecraft is designed such that \( W_L = 18 \text{ Hz} \) with \( \tau_1 = 2630 \text{ sec} \) and \( \tau_2 = 0.0834 \text{ sec} \). Find (a) the loop damping, (b) natural frequency, and (c) the loop gain. (d) Choose a value for \( C \) and determine \( R_1 \) and \( R_2 \). Assume no limiter is present and that an imperfect integrating filter is placed in the loop.

4-3 The transfer function of a second-order PLL is given by

\[
H_p(s) = \frac{2\zeta \omega_n s + \omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

where \( \omega_n \) is the loop natural frequency and \( \zeta \) is the loop damping. (a) Show that the loop bandwidth is given by

\[
W_L = \omega_n \left( \zeta + \frac{1}{4\zeta} \right)
\]

(b) Determine the damping factor that minimizes \( \sigma_{\phi}^2 \) in the presence of white Gaussian noise.

(c) What is the corresponding bandwidth to natural frequency ratio.

(d) Determine the loop filter that corresponds to \( H_p(s) \).

(e) If \( \zeta = 2 \) and \( \omega_n = 1.485 M \text{ rad/sec} \), what is the loop bandwidth?

4-4 A second-order PLL results if the loop filter is of the form

\[
F(s) = \frac{1}{1 + \tau s}
\]
(a) Find the closed-loop transfer function.
(b) Develop an expression for the loop damping, $\zeta$, in terms of $AK$ and $\tau$.
(c) Develop an expression for the natural frequency, $\omega_n$, of the loop in terms of $\tau$ and $AK$.
(d) What is the loop bandwidth?

4-5 An imperfect second-order PLL is to be used in the design of a frequency synthesizer. If $R_1 = 206.5 \, k\Omega$, $R_2 = 998 \, \Omega$, and $C = 250 \, \mu F$, find (a) $\tau_1$ and $\tau_2$, (b) the loop damping, (c) the loop natural frequency, and (d) the loop bandwidth if $AK = 3328$.

4-6 An imperfect second-order PLL is to be implemented with the active filter illustrated in Fig. P4-6. The gain of the amplifier is adjusted such that $AK = 5800$.

![Fig. P4-6. Loop Filter Mechanization.]

(a) Determine the transfer function of the loop filter.
(b) Find $\tau_1$, $\tau_2$, the loop natural frequency $\omega_n$, $r$, and the loop bandwidth.
(c) Write an expression for the closed-loop transfer function.
This filter was used by Lindsey and Charles (IEEE Proceedings, vol. 54, September 1966) to obtain laboratory measurements of the variance of the phase error.

4-7 With $H_p(s)$ as defined in Prob. 4-3, determine and sketch the phase error trajectories versus time for $\zeta > 1$, $\zeta = 1$, $\zeta < 1$ when (a) $\theta(t) = \theta_0$, (b) $\theta(t) = \Omega_0 t$, (c) $\theta(t) = \Omega_0 t^2/2$. Check your answer by evaluating the steady-state phase error.

4-8 The closed-loop transfer function of a third-order PLL is given by

$$H_p(s) = \frac{AK(s^2 + as + b)}{s^3 + AKs + aAKs + bAK}$$

(a) Find the loop bandwidth using the results in Table 4-1.
(b) Find the loop filter that gives rise to the above closed-loop transfer function.

4-9 Determine $H_p(s)$ when

$$F(s) = \frac{a_0 + a_1 s + a_2 s^2}{s^3}$$
\[ F(s) = \frac{a_0 + a_1s + a_2s^2 + a_3s^3}{s^3} \]

(c) If \( d(t) = \theta_0 + \Omega_{\theta}t + \Omega_2t^2/2 \), find the steady-state phase error when the filters in (a) and (b) are implemented in the loop.

4-10 The closed-loop transfer function of a second-order PLL is given in Prob. 4-3.
(a) If \( d(t) = \theta_0 + \Omega_{\theta}t \), find the steady-state phase error.
(b) Repeat (a) if \( d(t) = \theta_0 + \Omega_{\theta}t + \Omega_2t^2/2 \).

4-11 Using (4-53), show that the optimum closed-loop transfer function is given by (4-57) when \( d(t) = \theta_0 + \Omega_{\theta}t \) and \( \theta_0 \) is a uniformly distributed r.v.

4-12 Consider the problem of carrier tracking in the absence of angle modulation \( M(t) \).
(a) Using the closed-loop transfer function found in Prob. 4-11, show that the loop bandwidth is given by (4-60).
(b) When \( \Omega_0 = 2\pi B_{L} \), determine the value of \( r \) and \( \zeta \) that ensures that the heterodyned incoming carrier lies inside the loop bandwidth.

4-13 A perfect second-order PLL is implemented with the loop filter \( F(s) = [1 + \tau_2s]/\tau_1s \).
(a) Find the closed-loop transfer function.
(b) Determine the single and double-sided loop bandwidths.
(c) Determine the natural frequency \( \omega_n \) and damping factor \( \zeta \) as a function of \( \tau_1, \tau_2, \) and \( \lambda K \).

4-14 Based on the linear PLL assumption, develop the stability criteria for the second-order loop with loop filter \( F(s) = (1 + \tau_2s) \exp(-\lambda s)/\tau_1s \). Compare your results with the results given in Section 4-3.3 for the imperfect second-order PLL.

4-15 The arriving Doppler signal is characterized by the frequency step \( d(t) = \Omega_{\theta}t \).
(a) Determine the optimum closed-loop transfer function from (4-53).
(b) Determine the bandwidth of the loop in terms of the Lagrange multiplier.
(c) Specify the form of the loop filter.
(d) Compare the closed-loop design with (4-62) and that found in Prob. 4-11.

4-16 A PLL is to be used to track \( \theta(t) = d(t) = \Omega_2t^2/2 \) in the presence of white Gaussian noise.
(a) Using (4-53), show that the closed-loop transfer function is given by (4-65).
(b) From part (a), show that the corresponding loop filter is given by (4-66).

Notice that this gives rise to a third-order loop.

4-17 Mariner 69's command link was designed such that the total power \( P \) to noise spectral density was \( P/N_0 = 2060 \). The ratio of the carrier power to total power was \( P_c/P = 0.685 \). Assuming a carrier-tracking loop bandwidth of \( W_L = 10 \) Hz, determine the variance of the loop phase error based on the linear PLL theory.

4-18 Develop the linear theory of Costas and squaring loops developed in Sections 3-7.1 and 3-7.2 when \( \psi_1 = \psi_2 = 0 \). In particular, develop. (a) the equation
of loop operation when the phase error is small, (b) the closed-loop transfer function, and (c) the variance of the phase error due to the additive noise.

4-19 Show that the variance of the phase error in a Costas or squaring loop is given by

$$\sigma_\phi^2 = \frac{N_0 W_E}{2S} \left(1 + \frac{N_0 W_i}{4S}\right)$$

when an ideal bandpass filter precedes the loop. Using this result and assuming that $S/N_0 = 2060$ (typical of Mariner 69), evaluate the variance when $W_i = 4000$ Hz. Now assume that the bandwidth $W_i$ is reduced to $W_i = 200$ Hz and recompute the variance. Compare your answers with the results obtained in Prob. 4-18. (Hint: See Chapter 3 for the equations of loop operation.)

4-20 Consider the problem of tracking the received waveform $x(t) = s(t, \Phi) + n(t)$ with a reference signal, $r(t, \hat{\Phi})$, established by means of a PLL. If $s(t, \Phi)$ and $r(t, \Phi)$ are sinusoidal signals of average power $A^2$ and unity, respectively, then the mean-squared value of the phase error (as predicted by the linear PLL theory) is $\sigma_\phi^2 = 1/\rho = N_0 K/4A$ when the loop is first-order.

(a) If $s(t, \Phi)$ and $r(t, \Phi)$ are now both square waves of average power $A^2$ and unity, respectively, then find the increase in $\sigma_\phi^2$ over that of the sinusoidal case if the bandwidth is held fixed.

(b) Answer the same question as in (a) if $s(t, \Phi)$ again a sinusoid of average power $A^2$ but $r(t, \Phi)$ is still the unit power square wave. Again the comparison should be made on the basis of equal linear loop bandwidths.

(c) Compare and discuss the answers in (a) and (b) with that of the sinusoidal PLL where $\sigma_\phi^2 = 1/\rho$.

4-21 Since a bandpass limiter frequently precedes a narrowband PLL in many carrier tracking applications (e.g., the deep-space instrumentation facility), the communications design engineer needs to convert the linear carrier-tracking theory formulas (which do not incorporate IF limiting) into formulas that account for IF limiting. Show that $r$ is replaced by $r_0/\mu$, $\omega_n$ by $\omega_{n0}/\mu$, $\rho$ by $\alpha_i$ and that $\zeta = \frac{1}{2} \sqrt{r_0/\mu}$.

4-22 An imperfect second-order carrier tracking loop is to be preceded by a bandpass limiter and designed such that $\alpha_{10} \sqrt{P_{10}} = 0.07$, $r_0 = 2$, and $W_{I0} = 12$ Hz. A convenient mechanization of the loop filter is illustrated in Fig. P4-22.

---

**Fig. P4-22.** Loop Filter Mechanization.
(a) Write an expression for the loop filter transfer function in terms of $R_1$, $R_2$, $C_2$, and $R_3$ when the amplifier gain is very high. What is the significance of the negative sign that occurs?

(b) Assuming $\tau_2/r_0 \tau_1 \gg 1$ and defining $\tau_3 = R_3 C_3$ and $\tau_1 = (R_2 + R_3) C_2$, find $\tau_1$ if $K_m = 0.318$ volt/rad, $K_o = 600\pi$ rad/volt, and $K_1 = 200$.

(c) Choose $R_3 = 750$ k$\Omega$, $C_2 = 87 \mu$F, and find $R_2$ and $R_1$.

(d) An additional 6 dB/octave roll-off can be introduced for $\omega \gg 1/\tau_2$ without affecting the operation of the loop by simply adding a capacitor $C_3$ in parallel with $R_3$. A convenient choice for this third break point is $\tau_3 \leq \tau_2/10$. If $\tau_3 = R_3 C_3 = 1/80$, find $C_3$. This capacitor is very effective in removing the double-frequency terms and filtering out extraneous transients.

4-23 A coherent transponder is to be used to measure the radial velocity of a moving target. Show that the ratio of the output frequency to input frequency is given by (4-158).

4-24 Justify each step in the sample design of the superheterodyne PLL receiver given in Section 4-12.

4-25 Consider the problem of analog phase demodulation by means of a PLL.

(a) Derive the formulas for the Wiener filters given in (4-98).

(b) Evaluate the minimum mean-squared error and show that your results reduce to (4-105).

4-26 For the frequency demodulation problem considered in Section 4-9, verify formulas (4-115), (4-116), (4-118), (4-119), and (4-120).

4-27 For $k = 1$ in Prob. 4-25, develop the expression (4-128) for output signal-to-noise ratio. Also justify the threshold locus defined by (4-132) with $\sigma^2_b = 1$.

4-28 Using the Yovits-Jackson formula, derive the expressions (4-147) and (4-149) for the receiver threshold locus.

4-29 From Fig. 4-17 we see that for $k = 1$, $R = 4$, $m_f = 10$ the receiver reaches threshold. Given that $R = 4$, what is the corresponding value of $m_f$ for which the receiver reaches threshold when the spectrum is asymptotically Gaussian.

4-30 The spectral density of the oscillator instabilities in a PLL system are to be characterized by

$$S_{\Delta\omega}(\omega) = \frac{N_n}{2} + \frac{N_v}{2} \left[ \frac{\omega^2 + \omega^2_{\omega}}{\omega^2 + \omega^2_r} \right]$$

where $N_n$, $N_v$, $\omega_n$, and $\omega_r$ are constants. This spectral density could be obtained by adding white noise ($N_n$) to the output of an $RC$ filter with gain driven by another uncorrelated white noise ($N_v$) process. The parameter $\omega_n$ is the cutoff radian frequency of the network and $\omega_r$ is the reference radian frequency of the network. Notice that when $\omega_r = 0$, $S_{\Delta\omega}(\omega) = N_n \omega^2_n/2\omega^2$—that is, the shaped portion varies as $1/f^2$ noise.

(a) Find the optimum closed-loop transfer function that minimizes the mean-squared value of the total phase error when $\theta(t) = \Omega_o t + \theta_o$. 
(b) Assume $\Omega_0 = 0$ and see what the filter in (a) becomes.
(c) Assume $\omega_r = \Omega_0 = 0$ and see what the filter in (b) becomes.
(d) Assume $N_w = \theta_0 = \Omega_0 = 0$ and see what the filter in (c) becomes.

4-31 For the optimum filters determined in Prob. 4-25, determine the loop filters required when (a) $\Omega_0 = 0$, (b) $\Omega_0 = 0$, $\omega_r = 0$, and (c) $N_w = \theta_0 = \Omega_0 = 0$.

4-32 For the oscillator instability problem defined in Prob. 4-30, find the minimum mean-squared value of the total phase error when (a) $\omega_r = 0$, $N_w = 0$ and (b) $N_w = 0$, $\Omega_0 = 0$.

4-33 For the oscillator instability problem defined in Prob. 4-30, determine the optimum loop bandwidth when $\omega_r = 0$, $N_w = 0$.

4-34 A PLL located in the deep-space instrumentation facility is mechanized such that the loop bandwidth can be varied by switching relays using a computer. For the loop filter mechanization illustrated in Fig. P4-34, find the loop band-

![Fig. P4-34. Loop Filter Mechanization.](image)

width when all relays are unenergized, K3 is energized, and K2 and K3 are energized. Assume $r = 2$ in all cases.

4-35 A PLL tracking receiver operational in the deep-space network is preceded
by a bandpass limiter and designed to operate with three different loop bandwidths. The tracking filter characteristics are given in the table below.

(a) Find $r_0$ and the bandwidths at loop threshold.
(b) Find the received power, $P_0$, which is observed at design point.
(c) If the input signal-to-noise ratio $\rho_i = -10 \, \text{dB}$, find the ratio $W_L/W_{LO}$.

**Characteristic of the Tracking Filter**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase detector constant, $K_1 K_m$</td>
<td></td>
</tr>
<tr>
<td>VCO constant, $K_v$</td>
<td></td>
</tr>
<tr>
<td>Signal suppression at threshold, $a_{10}\sqrt{P_{10}}$</td>
<td></td>
</tr>
<tr>
<td>For $W_{L1}$</td>
<td>334 mV/deg</td>
</tr>
<tr>
<td>For $W_{L2}$</td>
<td>50 Hz/V</td>
</tr>
<tr>
<td>For $W_{L3}$</td>
<td>0.06848</td>
</tr>
<tr>
<td>Multiplication factor in receiver local oscillator</td>
<td>0.03430</td>
</tr>
<tr>
<td>Noise temperature of receiving system, $T^o$</td>
<td>0.01981</td>
</tr>
<tr>
<td>Loop bandwidth components:</td>
<td></td>
</tr>
<tr>
<td>$W_{L1} = ?$</td>
<td></td>
</tr>
<tr>
<td>$W_{L2} = ?$</td>
<td></td>
</tr>
<tr>
<td>$W_{L3} = ?$</td>
<td></td>
</tr>
<tr>
<td>$R_1 = 813.3583 , \text{k\Omega}$</td>
<td></td>
</tr>
<tr>
<td>$R_2 = 329.34 , \Omega$</td>
<td></td>
</tr>
<tr>
<td>$C = 380 , \mu\text{F}$</td>
<td></td>
</tr>
<tr>
<td>$R_1 = 6.5525 , \text{M\Omega}$</td>
<td></td>
</tr>
<tr>
<td>$R_2 = 1.3153 , \text{k\Omega}$</td>
<td></td>
</tr>
<tr>
<td>$C = 380 , \mu\text{F}$</td>
<td></td>
</tr>
<tr>
<td>$R_1 = 33 , \text{M\Omega}$</td>
<td></td>
</tr>
<tr>
<td>$R_2 = 3.9463 , \text{k\Omega}$</td>
<td></td>
</tr>
<tr>
<td>$C = 380 , \mu\text{F}$</td>
<td></td>
</tr>
</tbody>
</table>

4-36 Show that the Fourier series output of the bandpass limiter is given by (I-26).

4-37 (a) Show that (I-18) reduces to (I-19).
(b) Verify (I-20).

4-38 (a) Find the mean and variance of (I-39).
(b) From this result, justify (I-45) and (I-48).

4-39 Using (3-17), show that the noise process $\{N_c(t)\}$ is approximately white if $r(\tau)$ is given by (3-34), $B_i \gg B_0$, and $B_i$ is large.

4-40 The signal $m(t)$, generated from the left-channel signal $l(t)$ and right-channel signal $r(t)$ by the stereo-multiplex network in Fig. P4-40, is used to frequency modulate a transmitter.
(a) Write an expression for the stereo-multiplex signal $m(t)$ and the transmitted signal $s(t, \Phi)$. Assume that $M(t) = \int s(t) \, dt$.
(b) Draw a block diagram of a PLL receiving system which extracts the signal $m(t)$.
(c) Using bandpass filters, low-pass filters, multipliers, adders, and a carrier tracking loop which phase-locks to $f_s = \omega_s/2\pi = 19 \, \text{kHz}$, as network elements, construct a stereo-reproduction network which produces $r(t)$ and $l(i)$ from $m(t)$.
Fig. P4-40. Stereo-Multiplex Transmitter.

(d) Make the network in part (c) monophonic compatible, i.e., extract \( r(t) \), \( l(t) \), and \( r(t) + l(t) \).

4-41 (a) Show that the transfer function of the loop filter in Fig. P4-41 is given by
(4-49) with
\[
\tau_1 = (R_1 + R_2)C_1, \quad R_3C_2 = \frac{\tau_2}{k} \left[ \frac{1 - k\delta}{1 + k\delta} \right]
\]
\[R_2C_1 = R_3C_2 = (2\tau_2)/(1 + k\delta)\]

and \(\delta = R_3/R_4\).

(b) Verify (4-50) and evaluate the loop bandwidth in terms of loop parameters.

References


4-12 Youla, D. C., The Use of Maximum Likelihood in Estimating Continuous-


References


PART TWO
5

FUNDAMENTALS OF NONLINEAR OSCILLATIONS

"The periodicity . . . is to a certain extent inherited." C. F. Darwin, 1880

5-1 Introduction

It is not at all surprising that the engineer must grapple with nonlinear problems, for many of the phenomena that occur in the world around him are governed by nonlinear relationships. In the development of the mathematical sciences, the difficulties of nonlinear analysis have hindered the formulation of nonlinear concepts that would permit us to understand such phenomena.

If one looks back over the long history of accomplishments in the sciences, one is struck by the fact that most of our effort has been centered on linear systems and linear concepts. If, at the same time, one takes a look at the world around us, one is everywhere confronted by phenomena that are nonlinear in their behavior. Linear concepts give only a superficial understanding of much that we see in nature. If we are to make our analyses more realistic, we must acquire greater facility in comprehending and using nonlinear concepts.

In recent years we have developed large digital computers, and in many instances we have fed nonlinear problems into these machines in the hope that the resulting solutions would give us a better understanding of nonlinear behavior. In general, we have found that simply grinding out solutions gives
us little more understanding of nonlinear behavior than, for example, watching nature herself grind out solutions for such nonlinear problems as the weather. Understanding lies not in terms of equations or solutions but rather in terms of fundamental and well-understood concepts. For example, two engineers sit in an airplane during takeoff. The ash tray starts to buzz. One looks at the other and says “Resonance!” The other nods and both believe they understand the phenomenon. We understand nature only when we can piece together its behavior in terms of concepts so simple they may be well understood and so broad they may be stated abstractly without reference to a particular situation. The list of such concepts is long and includes such terms as resonance, hysteresis, waves, feedback, boundary layers, turbulence, shock waves, buckling, weatherfronts, contagion, immunity, inflation, and depression. Many of our most useful concepts are nonlinear in character, and a part of our inability to express in the exact language of mathematics such everyday phenomena as the flow of water in the gutter or the curling of smoke from a cigarette lies in the fact that we have been reluctant to plunge in and understand nonlinear mathematics.

The phenomenon of resonance is known to be frequently represented in living matter. According to Wiener (Ref. 1), Szent-Gyorgyi has suggested its importance in the construction of muscles. It turns out that substances with high resonance generally have an abnormal capacity for storing both energy and information, and such a storage certainly occurs in muscle contraction.

Nonlinear oscillations, random nonlinear oscillations, and coupled (phase-locked) nonlinear oscillations are at the heart of the matter in a great many scientific fields—for example, telecommunications and power system engineering, rhytmical phenomena occurring in biological and physiological systems (see Section 3-2.2, Chapter 3). The biophysicist, meteorologist, geophysicist, atomic physicist, seismologist, etc. are all concerned with nonlinear oscillations, frequently phase locked, in one form or another. For example, the power system engineer is concerned with the problem of hunting, the communication engineer with timing or synchronization jitter, the physiologist with clonus and clonic vibration, the neurologist with ataxia, the meteorologist with the frequency of air-pressure oscillations, the cardiologist with oscillations produced by the heartbeat, and the biologist with oscillations produced by the physiological clock.

The main purpose of this chapter is the treatment of a variety of nonlinear oscillation problems that are pertinent to the fundamental understanding of the notions of entrainment or synchronization, tracking, demodulation, and phase-coherent communications. An attempt is made to give a review of nonlinear problems of practical interest and whose solutions have been worked out in a form that appeals to the electrical engineer. The review is not exhaustive, but it has attempted to include examples of problems that serve to illustrate elementary concepts pertinent to understanding the nonlinear behavior of phase-
locked loops. The question of existence and uniqueness of solutions is only superficially touched; emphasis is placed on methods for obtaining solutions. Many physicists and engineers have the feeling that so-called mathematical proofs of existence and uniqueness of solutions often merely verify facts that are more or less evident from physical considerations. Unfortunately, in most nonlinear problems, physical reasoning is not sufficient or not fully convincing, so that in these cases the questions of existence and uniqueness represent a real challenge to the mathematician. For this reason, the writer considers this aspect of the subject beyond the scope of this book.

The material covered can be grouped under essentially three topics. Topic one consists of a short, although fairly complete, summary of the theory of linear oscillations for a system of one degree of freedom having constant characteristics. This material serves both for reference and for comparison with the results to be given relative to the nonlinear oscillation theory. Topic two treats easily integrable nonlinear systems in which no external forces depending on time occur. Here free nonlinear oscillation problems are studied in detail by working in the phase plane. The theory of Poincaré of singularities of first-order differential equations is abbreviated. The usefulness of the idea of singularity is illustrated by solving a number of physical problems. This theory will be of use in Part III of this book. Finally, topic three concerns nonlinear oscillations of the forced, self-sustained, and relaxation type. In particular, we discuss the Van der Pol theory as applied to the synchronization and tracking problem and close the chapter by discussing Hill's equation.

5.2 Linear-Free Oscillations

It is of value and interest to summarize the essential features of linear oscillation problems. There are a number of reasons for doing so here. One of our principal objectives is to compare and contrast linear with nonlinear oscillation problems. Also, it has been the practice to carry over, as much as possible, the terminology used in linear problems to nonlinear problems. Finally, it is useful to have a summary of the main ideas and formulas of the linear theory available for reference.

Perhaps the simplest example of a linear oscillation problem is furnished by the simple electric system consisting of an inductance $L$ in series with a capacitance $C$ and resistance $R$ (see Fig. 5-1). The mechanical analog, depicted in Fig. 5-1, consists of a mass $m$ attached to a spring that exerts a force (called the restoring force) proportional to the displacement $x$ of the mass. For the electrical system, we have, using Kirchhoff's law, that

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{i}{C} = \frac{de}{dt}$$

(5-1)
If the mass in the mechanical system is considered to move in a medium that exerts a resistance, \( c \), proportional to the velocity (a viscous damping force), the equation of motion for the oscillations of the mechanical systems is given by

\[ m\ddot{x} + c\dot{x} + kx = F(t) \]  

(5-2)

In the analogy we have that \( m = L, \ R = c, \ k = 1/C, \) and \( F(t) = \frac{de}{dt} \) with the current \( i(t) \) analogous to the displacement \( x(t) \). Assuming for the moment that the external force \( F(t) = 0 \) and introducing the quantities

\[ r \triangleq \frac{c}{2m}, \quad \omega_0^2 \triangleq \frac{k}{m}, \quad \omega \triangleq \sqrt{\omega_0^2 - r^2} \]  

(5-3)

then (5-2) becomes

\[ \ddot{x} + 2r\dot{x} + \omega_0^2 x = 0 \]  

(5-4)

Since \( F = 0 \), the oscillations governed by this linear, homogeneous, differential equation are called free linear oscillations. The general solution of this linear equation with constant coefficients is a linear combination of two exponential functions.

\[ x(t) = C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t) \]  

(5-5)

where \( C_1 \) and \( C_2 \) are arbitrary constants that are determined by initial conditions and \( \lambda_1 \) and \( \lambda_2 \) are roots of

\[ \lambda^2 + 2r\lambda + \omega_0^2 = 0 \]  

(5-6)
Thus $\lambda_1$ and $\lambda_2$ are given by

$$\lambda_1 = -r + i\omega \quad \lambda_2 = -r - i\omega \quad i = \sqrt{-1}$$

(5-7)

If we wish to express the solution (5-5) in real form, we consider three cases in which $\omega$ is (a) real, (b) zero, (c) imaginary. The solutions are readily found to be

$$x = \exp(-rt)(c_1 \cos\omega t + c_2 \sin\omega t)$$

(5-8a)

$$x = \exp(-rt)(c_1 t + c_2)$$

(5-8b)

$$x = c_1 \exp(\lambda_1 t) + c_2 \exp(\lambda_2 t), \quad \lambda_1, \lambda_2 \text{ real}$$

(5-8c)

where $c_1$ and $c_2$ are arbitrary constants that are determined by prescribing the values of the displacement $x$ (current $i$) and the velocity $\dot{x}$ at some initial time $t = t_0$.

Equation (5-8a) arises most frequently in practice. As is easily seen from (5-3), this case occurs if $r$ (the damping constant) is small compared with $\omega^2_0$. Equation (5-8a) then represents an oscillatory motion such that any two successive maxima $x_1$ and $x_2$ of the displacement $x$ satisfy the relation

$$x_2 = x_1 \exp\left(-\frac{2\pi r}{\omega}\right)$$

(5-9)

Consequently, if $r > 0$, the oscillations die out exponentially with the passage of time ($t > 0$). However, if $r < 0$ (corresponding to a negative damping or coefficient-of-friction), the oscillations increase exponentially. The cases where $r > 0$ are most common in practice.

If $r = 0$, the system is without damping and the motion is frequently referred to as self-sustained oscillations. For this case (i.e., $\omega = \omega_0$)

$$x = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$$

(5-10)

which is indicative of simple harmonic motion in which the angular or radian frequency $\omega_0 = \sqrt{k/m} = \sqrt{1/LC}$. For $\omega$ real, the oscillations given by (5-4) and (5-10) are called natural oscillations. The quantity $f_0 = \omega_0/2\pi$ is called the natural or resonant frequency.

Finally, the solution (5-8b) for $\omega = 0$ corresponds to the transition from oscillatory to nonoscillatory motion; this motion is that corresponding to critical damping.
5-3 Linear Oscillations in the Presence of a Deterministic External Force

We now study the motion that results when the external force $F(t)$, depending only on time, is present. We treat here the case where $F(t)$ is deterministic; however, of great concern in later chapters will be that case where $F(t)$ is nondeterministic—that is, the case of random nonlinear oscillations. The most important case for our purpose is that in which $F(t)$ is periodic. For example, $F(t)$ is sinusoidal,

$$F(t) = A \cos (\omega_c t + \theta)$$  \hspace{1cm} (5-11)

in which $A$ is the amplitude, $\omega_c$ the angular frequency, and $\theta$ a constant called the phase of $F(t)$. For this case the solutions of (5-2) consist of the sum of the homogeneous equation (i.e., free oscillations just discussed) and of any solution of the nonhomogeneous equation. Assuming that the free oscillation is of the type given in (5-8a), the solutions of (5-1) with $F(t)$ given by (5-11) are readily obtained in the form

$$x = \exp (-rt) \left[ c_1 \cos \omega t + c_2 \sin \omega t \right] + \frac{A \cos (\omega_c t + \theta - \delta)}{m \sqrt{4r^2 \omega_c^2 - (\omega_0^2 - \omega_c^2)^2}}$$  \hspace{1cm} (5-12)

The square root of the denominator is zero only if $\omega_c = \omega_0$ and $r = 0$—that is, the case of resonance. In other words, the resulting motion is a superposition of the free oscillation and an oscillation called the forced oscillation due to the external force $F(t)$. Note that with the passage of time the frequency of the forced oscillation is the same as that of the external force. The amplitude $H$ of the forced oscillation (see 5-12) is given by

$$H \triangleq \frac{A}{m \sqrt{(\omega_0^2 - \omega_c^2)^2 + 4r^2 \omega_c^2}}$$  \hspace{1cm} (5-13)

while the phase shift $\delta$ of $x$ relative to $F(t)$ is given by

$$\delta = \tan^{-1} \left( \frac{2r\omega_c}{\omega_0^2 - \omega_c^2} \right)$$  \hspace{1cm} (5-14)

When $r > 0$, it is clear from (5-12) that the free oscillation is damped out and only the forced oscillation would remain with the passage of time.

In the case where $r = 0$, the relative phase shift $\delta_p$ is seen from (5-14) to be zero for $\omega_c < \omega_0$ and $\pi$ for $\omega_c > \omega_0$; in other words, the forced oscillation is in phase with the external force if the free or resonant frequency $\omega_0$ is
greater than the frequency of the external force and is 180° out of phase with it when \( \omega_c > \omega_0 \). For \( r = 0 \), we obtain from (5-12) the solution

\[
x = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{A}{m[\omega_0^2 - \omega_c^2]} \cos (\omega_c t + \theta - \delta)
\]

(5-15)

provided \( \omega_c \neq \omega_0 \).

For \( r = 0 \), \( \omega_c = \omega_0 \) (the case of resonance) the solution to (5-2) becomes

\[
x = c_1 \cos \omega_c t + c_2 \sin \omega_c t - \frac{At}{2\omega_c m} \sin (\omega_c t + \theta)
\]

(5-16)

Note here that the motion due to the external force is no longer periodic but is oscillatory with an amplitude \( At/2m\omega_c \) that increases linearly with time. In both electrical as well as mechanical systems—for example, rotating machinery or other engineering structures such as missiles—it is often vital to design machine parts in such a way as to avoid resonance with periodic forces that may be impressed on the system.

When damping is present, it is clear from (5-13) that the response amplitude \( H \) is always finite. Nevertheless, the engineer must frequently investigate the amplitude of the forced oscillation, for a rupture of the spring could occur if \( H \) became too large. From (5-3) and (5-13) it is easy to write the normalized response as

\[
H_n = \frac{k|H|}{A} = \frac{1}{\sqrt{[1 - (\omega_c/\omega_0)^2]^2 + 4(r/\omega_0)^2(\omega_c/\omega_0)^2}}
\]

The extreme values for \( H_n \) are attained for \( \omega_c = 0 \) and \( (\omega_c/\omega_0)^2 = 1 - 2(r/\omega_0)^2 \). If \( 1 - 2(r/\omega_0)^2 < 0 \), a maximum occurs when \( \omega_c = 0 \); if \( 1 - 2(r/\omega_0)^2 > 0 \) and \( \omega_c > 0 \), there is a maximum where \( \omega_c/\omega_0 = \sqrt{1 - 2(r/\omega_0)^2} \) and a minimum for \( \omega_c = 0 \). For small values of the damping constant, the maximum amplitude is approximately the natural frequency. Figure 5-2 shows the response curves for \( H_n \) as a function of \( \omega_c/\omega_0 \) for various values of \( r/\omega_0 \). Obviously the curves yield the amplitude, or response, of the system for an external force of any given frequency.

In closing this section we acknowledge the principle of superposition, which states: A linear system forced by \( F_1(t) \) yields the response \( x_1(t) \); when forced by \( F_2(t) \), it yields the response \( x_2(t) \); and when forced by \( F_1 + F_2 \), the response is \( x_1 + x_2 \). This fundamental fact, a direct consequence of the linearity of the differential equation (5-1), is called the superposition principle. We note explicitly that the principle does not hold for motion of systems described by nonlinear differential equations. The theory of linear differential equations has
been thoroughly studied and developed, particularly for linear systems with constant coefficients. On the other hand, there is almost nothing of a general character known about obtaining solutions to nonlinear differential equations (forced or nonforced); consequently, the communications engineer faced with a problem involving nonlinearity has, for the most part, had to go off and grapple with it all alone. In the next section we discuss methods and techniques for dealing with certain types of nonlinear oscillation problems and apply the principles to a variety of nonlinear problems of special interest in elementary radio engineering. We continue to treat the case of deterministic forces (signals) and reserve a later chapter for the case of nondeterministic forces (signals).
5-4 Free Nonlinear Oscillations of Undamped Systems with Nonlinear Restoring Forces

The main purpose of this section is the presentation of a concise treatment of mechanical or electrical systems governed by the differential equation

$$m\ddot{x} + f(\dot{x}) + h(x) = F(t)$$  \hspace{1cm} (5-17)

when $f(\dot{x}) = 0$ and $F(t) = 0$. In the analogy with the linear system just discussed, we shall find it convenient here and in future chapters to refer to the term $m\ddot{x}$ as the inertia force, to $f(\dot{x})$ as the damping force or frictional force, to $h(x)$ as the restoring force, and to the term $F(t)$ as the external force, excitation, or applied signal. With $x = \varphi, f(\dot{x}) = \dot{\varphi}$, and $F(t) = N(t, \varphi)$, (5-17) arises in a PLL tracking a constant phase offset and a loop filter of the lag type—that is, $F(p) = 1/(1 + mp)$. The problem of hunting (Ref. 2) in synchronous electrical machinery due to randomly time-varying load is another example of a physical problem that leads to the same equation.

We will not study the solutions to (5-17) in all generality, for such knowledge that we presently have about the nonlinear oscillation phenomena is largely confined to certain special cases. We will, however, consider a build-up of theory that is essential to a fundamental understanding of the processes of synchronization, tracking, and coherent demodulation.

Consider now the simplest case of (5-17),

$$m\ddot{x} + h(x) = 0$$  \hspace{1cm} (5-18)

This is the case of undamped ($f(\dot{x}) = 0$) and free oscillations ($F(t) = 0$) with a nonlinear restoring force $h(x)$. The best-known example of an oscillatory motion governed by (5-18) is that of the simple pendulum where $h(x) = mg \sin x/l$, $l$ is the length of the pendulum, $g$ is the acceleration of gravity, and $m$ is the attached mass. The $x$ variable is analogous to the phase error $\varphi$ in a PLL. If it is assumed that $x$ is small so that (5-18) can be linearized (i.e., $\sin x \approx x$), then (5-18) can be replaced by

$$\ddot{x} + \frac{gx}{l} = 0, \quad x \ll 1$$  \hspace{1cm} (5-19)

to produce the linear theory. This equation, from our preceding discussion when $r = 0$, implies that the motion is periodic with period $T = 2\pi\sqrt{l/g}$; that is, the period is independent of the initial velocity and displacement. For large displacements, (5-19) does not hold and one must resort to other methods of analysis.

A first integral of $m\ddot{x} + h(x) = 0$ can be easily obtained, for the substitution
\[ v = \frac{dx}{dt}, \quad \frac{d^2 x}{dt^2} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx} \]  
(5-20)

reduces (5-18) to the time-suppressed, first-order differential equation

\[ m \frac{dv}{dx} + h(x) = 0 \]  
(5-21)

in which the variables are separable. Thus

\[ mv \frac{dv}{dx} = -h(x) \, dx \]  
(5-22)

and if \( v = v_0, x = x_0 \) when \( t = t_0 \), then integration of both sides of (4-22) yields

\[ \frac{m}{2} (v^2 - v_0^2) = -\int_{x_0}^{x} h(y) \, dy \triangleq -[U(x) - U(x_0)] \]  
(5-23)

which expresses the law of conservation of energy. The left side of this equation represents the change in kinetic energy; the right side represents work done by the restoring force, or the change in potential energy. Solving for the velocity, we have

\[ v = \frac{dx}{dt} = \pm \sqrt{\frac{2}{m} [U(x) - U(x_0)]} \]  
(5-24)

Similarly, separating variables and integrating, time can be found as a function of displacement.

\[ t - t_0 = \int_{x_0}^{x} \frac{dy}{\sqrt{v_0^2 - (2/m)[U(y) - U(x_0)]}} \]  
(5-25)

It must be understood that generally it is necessary to pass from one branch of the square root to the other whenever \( v = dx/dt \) passes through zero. The curves given by (5-23) in the \((x, v)\) plane are curves of constant energy; we frequently refer to them as energy curves in the phase plane. As we shall subsequently see, important information about the motion of a fundamental and qualitative character can be obtained rather easily. Since \( x \) and \( v = dx/dt \) are functions of \( t \), the energy curves in the \((x, v)\) plane may be regarded as given in parametric form with \( t \) as a parameter. From \( v = dx/dt \) it follows, then, that \( x \) increases with \( t \) when \( v \) is positive and decreases with \( t \) when \( v \) is negative.
When the energy curves are closed, it is important to note that this corresponds to periodic oscillations in \( x(t) \) [i.e., \( x(t) = x(t + T) \)], where \( T \) is the period or the amount of time required to reach the displacement and velocity \( t - T \) seconds earlier. The period \( T \) can be calculated by the line integral

\[
T = \oint \frac{dx}{v}
\]  

(5-26)

taken along the closed energy curves in the positive direction of \( t \).

For example, consider first the case in which \( h(x) \) is linear—that is, \( h(x) = kx \). The differential equation for the energy curves is

\[
v \frac{dv}{dx} = -kx
\]  

(5-27)

from which we obtain

\[
v^2 + kx^2 = v_0^2 + k x_0^2 = E
\]  

(5-28)

where \( x_0 \) and \( v_0 \) are the initial conditions at \( t = t_0 \). All energy curves are ellipses for \( k > 0, k \neq 1 \), and hence every motion is periodic (see Fig. 5-3).

![Phase-Plane Plot for Simple Harmonic Motion](image)

(a) Circular paths \((k=1)\)  
(b) Elliptical paths \((k > 0, k \neq 1)\)

**Fig. 5-3.** Phase-Plane Plot for Simple Harmonic Motion.

From the previous section we know that this corresponds to simple harmonic motion with \( r = 0 \) and with \( x(t) \) and \( v(t) \) given by

\[
x(t) = a \cos \omega_0 t \quad v(t) = -a \omega_0 \sin \omega_0 t
\]  

(5-29)
where $\omega_0 = \sqrt{k}$ and $a = \sqrt{(v_0^2 + kx_0^2)/k}$. This assumes the initial conditions $x_0 = a, v_0 = 0$ (at $t = 0$). The period of the oscillations, as obtained from (5-26), is

$$T = \frac{4}{\omega_0} \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \frac{2\pi}{\omega_0} \quad (5-30)$$

In this linear case we note that the period of the oscillations is independent of the amplitude $a$; that is, all of the closed solution energy curves in the phase plane are transversed in the same amount of time. If $k < 0$, the energy curves become hyperbolas and no periodic oscillations exist.

Secondly, consider the case of the pendulum, since its behavior is analogous to that of a phase-locked loop with loop filter $F(p) = 1/p$ operating in the absence of noise [see (5-28)]. The differential equation (5-18) for the $(x, v)$ curves becomes

$$v \frac{dv}{dx} = -a \sin x, \quad a = \frac{l}{g} \quad (5-31)$$

and the energy curves are given by

$$v^2 = 2a \cos x + 2E \quad E = \frac{v_0^2}{2} - a \cos x_0 \quad (5-32)$$

where $E$ represents the total energy in the system. These energy curves are illustrated in Fig. 5-4. We observe that $E > -a$ must be required or $v^2$ will

![Fig. 5-4. Phase-Plane Plot for the Simple Pendulum.](image) always be negative. When $|E| < a$, one sees from (5-32) that the curves are closed curves encircling the points $v = 0, x = 2n\pi$ ($n$ any integer). In these cases the amplitude $a$ satisfies the relationship $0 = a \cos x + E$, and the period of the oscillations that they represent is given by
\[ T = 4 \int_0^\theta \frac{dx}{\sqrt{2E + 2a \cos x}} \]  
\hspace{1cm} (5-33)

If the new integration variable \( \sin \left( \frac{x}{2} \right) = \sin \theta \sin \left( \frac{a}{2} \right) \) is introduced and the relationship between \( a \) in terms of \( E \) is used, we find from (5-33) that

\[ T = \frac{4}{\sqrt{a}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \sin^2 \theta \sin^2 \left( \frac{a}{2} \right)}} \]  
\hspace{1cm} (5-34)

We note that \( T \) increases with the amplitude \( a \) (a consequence of nonlinearity and independent of \( a \) in the linear range) and is given in terms of the complete elliptic integral of the first kind.

When \( E > a \), we note from (5-32) that \( v \) never becomes zero; the curves are open as indicated in Fig. 5-4. In the upper half-plane, motion of the point \((x, v)\) is from left to right; while in the lower half-plane, motion of the point \((x, v)\) is from right to left. The transition boundary (drawn more heavily)—that is, the transition from open to closed curves—obtained when \( E = a \) is defined from (5-32) by \( v^2 = 4a \cos^2 \left( \frac{x}{2} \right) \). Sometimes this trajectory is referred to as the separatrix.

The physical interpretation of these facts is quite clear and is analogous with the behavior of a phase-locked loop. The pendulum either oscillates (phase error oscillates) about its lowest position \((x = 2n\pi)\) and energy curves are closed or it has been given so high an initial velocity that it rotates always in the same direction (clockwise—upper half-plane or counterclockwise—lower half-plane) about the point of suspension. In the latter case the angular displacement increases or decreases without limit, but the angular velocity fluctuates periodically about a certain mean value. It can easily be shown that the time required to reach its highest point \( x = n\pi, n \) odd, with velocity zero is infinite. In the analogy, the oscillations or motion about the points \( x = 2n\pi \) for \( E < a \) corresponds in the phase-locked loop to the case where the loop phase error is periodic. The rotation of the pendulum about the suspension point \( E > a \) is analogous to an unlocked phase-locked loop that is slipping cycles. In fact, motion from left to right (\( E > a \)) in the phase plane corresponds to the case where the phase of the synchronized oscillator in a PLL retards by \( 2\pi \) multiples relative to the phase of the synchronizing signal. Of course, motion from right to left accounts for an advancement by \( 2\pi \) multiples of the VCO relative to the incoming oscillations. An analogous skipping phenomenon occurs in the heart (Ref. 3) and in overloaded rotating machinery (Ref. 4). In later chapters we shall explore how this phenomenon occurs in systems employing the phase-lock principle.

The equilibrium positions correspond to what we shall later define as singular points in the \((x, v)\) plane. It turns out that the character of these singular points pretty well establishes the general character of the energy curves.
In the next section we study the detailed nature of the various singularities with the object of applying the knowledge so gained to the applications of interest in this book.

5-5 Free Nonlinear Oscillations of Damped Systems with Nonlinear Restoring Force

In this section we shall be concerned with the differential equation

$$\ddot{x} + f(\dot{x}) + h(x) = F(t) \quad (5-35)$$

when \( f(\dot{x}) \neq 0 \) and \( F(t) = 0 \). The presence of \( f(\dot{x}) \) does not allow for explicit integration, as was accomplished in the previous section. The differential equation (5-35) arises in the case of a pendulum when damping forces are present. For the moment we make use of (5-35) to introduce the notion of singularities.

Since time does not occur explicitly in (5-35), it is possible to reduce the equation to one of first order by letting \( \dot{x} = v \), the velocity. Thus (5-35) becomes

$$\frac{dv}{dx} = -\frac{h(x) - f(v)}{v} \quad (5-36)$$

which is the slope \( m = \dot{v}/\dot{x} \). It is useful to note that in the half-plane \( v > 0 \), \( x \) increases with increasing \( t \), and hence the representative point moves from left to right with increasing \( t \). Conversely, for \( v < 0 \), the representative point moves from right to left. Because of the presence of \( f(v) \) in (5-36), it is not possible to separate the variables in general and produce the energy curves. There is some advantage in replacing the single equation (5-36) by the first-order differential equations

$$\frac{dv}{dx} = -h(x) - f(v) \quad \frac{dx}{dt} = v \quad (5-37)$$

which yield a vector field with the components \((\dot{x}, \dot{v})\). The field vector is always tangent to an energy curve and points along it in the direction of motion of the representative point \([x(t), v(t)]\) in the \((x, v)\) plane with increasing \( t \). However, the field has no direction at points where the numerator and denominator of (5-36) vanish simultaneously—that is, \( \dot{x} = \dot{v} = 0 \). From (5-37) we see that the components of the vector field are zero at such points. Such a point is called a singular point of (5-35). For our purposes, a singular point corresponds to a definite physical situation—to a position of equilibrium with zero velocity. Such singularities were encountered in the case of the undamped pendulum, namely, at \( x = 2n\pi \). The nature of the singularities is a decisive factor in deter-
mining the qualitative nature of the solutions, as well as the existence of periodic solutions.

It turns out, due to the work of Poincaré (Refs. 4, 5, 6), that a complete discussion of the types of singularities required here can be generated by a discussion of the character of the energy curves in the neighborhood of an isolated singular point of the differential equation

\[ m = \frac{dv}{dx} = \frac{P(x, v)}{Q(x, v)} \]  \hspace{1cm} (5-38)

We repeat, by a singularity is meant a singular point \((x_0, v_0)\) for which \(P(x_0, v_0) = Q(x_0, v_0) = 0\).

Poincaré showed by neglecting higher-order terms in a Taylor series expansion of (5-38), that the differential equation

\[ m = \frac{dv}{dx} = \frac{ax + bv + P_2(x, v)}{cx + vd + Q_2(x, v)} \]  \hspace{1cm} (5-39)

in which the constants \(a, b, c,\) and \(d\) are such that the determinant \(ad - bc \neq 0\) and in which \(P_2\) and \(Q_2\) vanish like \(v^2 + x^2\) as \((x, v)\) approaches \((x_0, v_0)\), has as its only singularities those of the simple equation

\[ \frac{dv}{dx} = \frac{ax + bv}{cx + dv} \]  \hspace{1cm} (5-40)

In addition, Poincaré showed that criteria for distinguishing the types of singularities of (5-39) can be derived in terms of \(a, b, c,\) and \(d\). In certain cases \(D = ad - bc = 0\) occur; that is, singularities of higher order occur; however, for the case of interest here we confine our attention to those situations for which \(D \neq 0\). We shall assume without proof the results of Poincaré relating the singularities of (5-39) to (5-40). The interested reader is referred to the original work due to Poincaré (Refs. 4, 5, 6), the book of Bieberbach (Ref. 7), and Stoker (Ref. 8).

### 5-6 The Various Types of Singularities

We now wish to study four special cases of (5-40) in which the differential equation can be solved explicitly. For our purposes, the four cases exhibit the four types of singularities of interest in the work to follow.

**Case 1.** \(dv/dx = bv/x\). Integration of the equation gives the energy curves \(v = v_0(x/x_0)^k\). If \(b = 1\), it is clear that all the curves are straight lines through the origin (see Fig. 5-5a). If \(b > 1\), all curves pass through the origin and all
are tangent to the $x$ axis except the curve $x = 0$. If $0 < b < 1$, all energy curves pass through the origin and all are tangent to the $x$ axis except the curve $v = 0$ (see Fig. 5-5). For all these cases, the origin is called a *nodal point* or *node*.

![Fig. 5-5. Phase-Plane Plot Illustrating Nodal Point Type Singularities.](image)

This situation is quite different if $b < 0$. The energy curves $vx^{-b} = v_0 x_0^{-b}$ are asymptotic to $x = v$ and $x = -v$. Only the curves $x = 0, v = 0$ pass through the origin; all others avoid it. This type of singularity is called a *saddle point* (see Fig. 5-6). A saddle point is a singularity to which there tends only paths that are asymptotes of the energy curves. Each such asymptote is referred to as a *separatrix*. In the case of the pendulum with no damping (see Fig. 5-4), the unstable equilibrium positions $x = n\pi$, $n$ odd, correspond to singularities of this kind.

![Fig. 5-6. Phase-Plane Plot in the Vicinity of a Saddle Point Type Singularity.](image)
**Case 2.** $dv/dx = -\mu^2 x / v$. For this case, $v^2 + \mu^2 x^2 = v_0^2 + \mu^2 x_0^2$ and the curves are ellipses ($\mu \neq 1$) with the origin as the center and circles when $\mu = 1$. This singularity is called a center (see Fig. 5-3), and such a phase-plane plot represents a periodic motion. The stable equilibrium points $x = 2n\pi$, $n$ any integer, of the undamped pendulum (Fig. 5-4) are of this type.

**Case 3.** $dv/dx = (x + av) / (ax - v), a \neq 0$. This equation can be solved by introducing polar coordinates $x = R \cos \theta, v = R \sin \theta$. In these variables the equation reduces to $dR/d\theta = aR$, and the energy curves are given by $R = c \exp (a\theta)$; they are logarithmic spirals (see Fig. 5-7). The singularity is called

![Fig. 5-7. Phase-Plane Plot in the Vicinity of a Spiral Point.](image)

![Fig. 5-8. Phase-Plane Plot in the Vicinity of a Nodal Point.](image)
a spiral point and arises in the nonlinear analysis of a phase-locked loop in the absence of noise. If the point is stable, the motion converges toward the origin; in the unstable case, the motion diverges from the origin.

**Case 4.** \( dv/dx = (x + v)/x \). The energy curves are given by \( v = xv_0/x_0 + x \log |x/x_0| \). All curves (see Fig. 5-8) pass through the origin. The origin is again called a nodal point.

Finally, according to (5-36), the slope of the path in the phase plane is some function of \( x \) and \( v \), say \( m = G(x, v) \). The locus of a point that moves such that \( G(x, v) = c \) (\( c \) is constant) was defined by Van der Pol (Ref. 9, 10, 11, 12) to be an isocline. Frequently one can readily determine by the method of isoclines the type of singularity and whether the point corresponds to a stable or unstable point.

**5.7 The Pendulum with Damping Force Proportional to the Absolute Velocity \(|v|\)**

For this case, \( f(\dot{x}) = c|\dot{x}|\dot{x} \) and \( h(x) = k \sin x \) in (5-36). Instead of (5-35), we consider the usual first-order equation

\[
\frac{dv}{dx} = \frac{-k \sin x - c|v|v}{v} \quad (5-41)
\]

which was studied by Stoker (Ref. 8). Obviously the singularities, corresponding to the equilibrium positions, occur at \( x = n\pi, n \) any integer and \( v = 0 \). At \( x = 0 \), the right-hand side of (5-41) takes the form \((-kx + \cdots)/v\) and the singularity is a center (motion as indicated by Fig. 5-3) or a spiral; however, our criteria for the singularity fail to distinguish between the two cases. For \( c = 0, x = 0, v = 0 \), we know that the singularity is a center; that is, the case of the pendulum with no damping (see Fig. 5-3). When damping is present, however, the singularity is a stable spiral point. At \( x = n\pi \) (\( n \) odd), \( v = 0 \), we know from earlier discussions that the singularity is a saddlepoint. Therefore the singularities at \( x = n\pi, v = 0 \), are stable spiral points if \( n \) is even and saddlepoints if \( n \) is odd. The phase-plane trajectories are sketched in Fig. 5-9 for the case where \( v = 0 \) at \( x = n\pi \) (\( n \) odd) and \( v_0^2 = 2[1 + \exp (2\pi)]/[1 + 4c^2] \).

Every motion tends to a stable equilibrium position. The points \( v = 0 \) at \( x = 2n\pi \) (\( n \) even) correspond to phase-lock points in a PLL.

The phase-plane trajectories of Fig. 5-9 can be obtained explicitly in this case by introducing into (5-41) the variable \( y = v/\sqrt{k} \) to obtain

\[
\frac{d(y^2)}{dx} + 2cy^2 = -2 \sin x, \quad y > 0
\]

\[
\frac{d(y^2)}{dx} - 2cy^2 = -2 \sin x, \quad y < 0 \quad (5-42)
\]
These equations are first-order linear differential equations with constant coefficients for the function $y^2 = v^2/k$. Thus the solution to (5-42) is easily written as

$$v^2 = kc_1 \exp(-2cx) + \frac{1}{1 + 4c^2}(2k \cos x - 4ck \sin x) \quad (5-43)$$

for $y > 0$ and

$$v^2 = kc_2 \exp(2cx) + \frac{1}{1 + 4c^2}(2k \cos x + 4ck \sin x) \quad (5-44)$$

for $y < 0$. In (5-43) and (5-44), $c_1$ and $c_2$ are arbitrary constants. In passing, we point out that Levinson and Smith (Ref. 13) have considered the more general equation $dv/dx = -[f(x, v)v + g(x)]/v$.

5-8 Self-Sustained and Relaxation Oscillations

In this section we proceed to consider the nonlinear problem in which the damping or frictional force is nonlinear while the restoring force is assumed linear. The nonlinear damping force will be such that as the amplitude of the
oscillations increases, the velocities decrease, whereas the amplitude decreases as the velocity increases. Therefore the state of rest will not be stable and an oscillation will build up from rest even in the absence of external forces or signals. This phenomenon accounts for these oscillations being called self-excited or self-sustained oscillations.

For the electrical engineer the most important systems, in practice, that lead to self-sustained oscillations are electrical circuits involving vacuum tubes or transistors.* These circuits are used in communication engineering as voltage-controlled oscillators, modulators, and so on. An electrical circuit depicting such a device is illustrated in Fig. 5-10. This circuit, referred to as a feedback oscillator circuit, is chosen because it is important in practice and leads easily to the differential equation that we wish to study. We proceed to establish the differential equation for the current \( i \) flowing through the inductance coil of the oscillator.

![Fig. 5-10. A Feedback Oscillator Circuit.](image)

Assuming that the current in the grid circuit may be neglected, we note that the total current flowing in the plate circuit is given by

\[
i_p = i + i_R + i_c
\]  \hspace{1cm} (5-45)

Using elementary current voltage relationships, we can easily write

\[
LC\frac{d^2i}{dt^2} + \frac{L}{R} \frac{di}{dt} + i = i_p
\]  \hspace{1cm} (5-46)

The grid potential \( e_g \) is provided by the mutual inductance \( M \) so that

*For didactic and historical sake, we shall use the vacuum tube circuit originally studied by Van der Pol (Refs. 9, 10, 11, 12).
Self-Sustained and Relaxation Oscillations

\[ e_s = M \frac{di}{dt} \]  
(5-47)

and the plate potential \( e_p \) is given by

\[ e_p = E_b - L \frac{di}{dt} \]  
(5-48)

where \( E_b \) is the battery potential.

Thus far we have made practically no use of the properties of the tube itself. Since the tube is a nonlinear element designed such that the plate current depends, with good accuracy, on a linear combination \( e \) of the grid potential \( e_g \) and the plate potential \( e_p \) we can write

\[ i_p = g(e) \]  
(5-49)

where

\[ e = e_g + ce_p, \quad c > 0 \]  
(5-50)

c being a constant determined by the tube amplification factor. The function \( g(e) \) is sometimes called the tube characteristic and is, in general, nonlinear, since the current \( i_p \) is limited by the rate of production of electrons in the cathode circuit.

From (5-47), (5-48), and (5-50) we can write

\[ e = cE_b + (M - cL) \frac{di}{dt} \]  
(5-51)

and from (5-49) we see that (5-51) is a nonlinear differential equation in which the nonlinearity involves the first derivative. When we introduce the new dependent variable

\[ z = i - g(cE_b) \]  
(5-52)

(5-46) becomes

\[ LC\ddot{z} + F(\dot{z}) + z = 0 \]  
(5-53)

where \( \dot{z} = di/dt \), and

\[ f(\dot{z}) \triangleq g(cE_b) - g[cE_b + (M - cL)\dot{z}] \]  
(5-54)

\[ F(\ddot{z}) \triangleq \frac{L}{R} \dot{z} + f(\dot{z}) \]
We further assume that \((M - cL) > 0\). Without this condition, as we shall subsequently see, self-excited oscillations would not occur. It is also important to be able to design the tube characteristic and adjust the circuit parameters so that \(F(\dot{z}) < 0\) for \(|\dot{z}| \ll 1\) but \(\dot{z}F(\dot{z}) > 0\) for large \(|\dot{z}|\). This means that the "damping" is negative for \(|\dot{z}|\) small and, on the basis of our previous discussions, we would expect the amplitude \(z\) to increase. However, for large \(\dot{z}\) the system dissipates energy and we would expect the amplitude to be limited from above. Consequently, after the transients have died out, one would intuitively expect a steady oscillation of a certain amplitude to exist.

The pioneering work relative to solutions of (5-53) is due to Van der Pol (Refs. 9, 10, 11, 12), who preferred to work with an equivalent equation in \(x\). Letting \(x = \dot{z}\) and differentiating (5-53) with respect to \(t\) yields

\[
LC\ddot{z} + F(\dot{z}) \ddot{z} + \dot{z} = 0 \quad LC\ddot{x} + G(x)\dot{x} + x = 0 \quad (5-55)
\]

where \(G(x) = F(x)\). If the tube physics are such that \(F(\dot{z})\) is of the type indicated in Fig. 5-11a, we see that \(G(x)\) appears as in Fig. 5-11b. For small \(|x|\), \(G(x) < 0\); and for large \(|x|\), \(G(x) > 0\), which indicates that a self-sustained oscillation would occur.

![Fig. 5-11. Characteristic for Feedback Circuit.](image)

To investigate the response of (5-53), we approximate \(F(\dot{z})\) by the linear line segments shown in Fig. 5-12. The nonlinear solution to the differential equation could therefore be approximated for each line segment \(-a \leq \dot{z} \leq a\), \(a \leq \dot{z} \leq \infty\), and \(-\infty \leq \dot{z} \leq -a\). We now write (5-53) as a pair of simultaneous equations

\[
\dot{z} = v \quad \dot{v} = -F(v) - z \quad (5-56)
\]
where $v$ is the velocity of $z(t)$ and we have assumed $LC = 1$—that is, $\omega_0 = 1$. From (5-56) use get

$$v \frac{dv}{dz} = -z - F(v)$$  \hspace{1cm} (5-57)

If we plot our results in the $(z, v)$ phase plane, a family of curves gives the response of the system for any combination of initial values of the position $z(t)$ and velocity $\dot{z}(t)$.

If the damping $F(v)$ were zero, we could integrate the preceding equation directly to give the energy equation $v^2/2 + z^2/2 = E$ with $E$ being the constant of integration. In the phase plane this solution is a set of circles (see Fig. 5-3) for different $E$. If this solution is substituted back into the equations above, we obtain the familiar result $z = A \sin (t + \phi)$ and $v = A \cos (t + \phi)$ for simple harmonic motion. If linear positive damping is added, we know from Section 5-2 that the trajectories spiral into the origin.

For the polygonal damping function of Fig. 5-12, we can piece together spirals in the phase plane corresponding to the three linear segments of $F(v)$, using the method of Section 5-5. In each of the three regions the appropriate trajectories are drawn and joined at the boundaries as shown in Fig. 5-13. In the central band the trajectories spiral outward, whereas in the upper and lower regions they spiral inward. At an intermediate amplitude, there is a closed curve that spirals as much outward as inward. All other trajectories approach this limit cycle, which represents a stable periodic oscillation performed by the system of its own accord. This example serves to illustrate the nonlinear behavior of a simple nonlinear system. Of course, as we increase the complexity of the nonlinearity $F(v)$, we can expect to obtain more complex phase plane plots. One of the first to explore the behavior of such nonlinear systems was Poincaré (Refs. 3, 5, 6) whose work was so far-reaching that it has formed a base for almost all that has followed.
Limit cycles can be classified as stable, unstable, and neutral. A stable limit cycle is a mode of oscillation that will return to its original state after being perturbed; an unstable limit cycle never returns to its original mode after an arbitrarily small perturbation. The neutral limit cycle, on the other hand, depends only on initial conditions and any perturbation will alter it in proportion to the magnitude of the perturbation. In fact, perturbations can be considered as new initial conditions; the frictionless pendulum that preserves information about its past is exemplified here. Stable limit cycles are sometimes classified as hard or soft. Further detailed discussions can be found in the reference material cited at the end of this chapter.

5.8.1 Qualitative Treatment of the Van der Pol Equation

If we now consider the case where $G(x) = -\tau(1 - x^2)$ in (5-55), and $LC = 1$, we have one form of the much-studied Van der Pol equation,

$$\ddot{x} - \tau \dot{x}(1 - x^2) + x = 0$$  \hspace{1cm} (5-58)

This equation has the property that when $x$ is small, the damping is negative, but it becomes positive for large $x$. Trajectories in the phase plane for the Van
der Pol equation can be carried out by means of a computer. Typical phase plane plots of the energy curves are given in Fig. 5-14 for various values of $\tau$. Note the effects as $\tau$ is varied. The energy curves near the origin have the shape of logarithmic spirals, but instead of diverging to infinity they converge.
to a closed curve. Those energy curves originating at infinity converge to the same closed curve, which represents a periodic motion with constant amplitude. For small values of \( \tau \), for example, the closed curve is only slightly different from an ellipse, indicating sinusoidal oscillations; however, its shape varies quite radically with increasing \( \tau \). The corresponding motions for typical plots of the amplitude of \( x(t) \) vs. time for oscillations growing to the limit cycle are shown in Fig. 5-15 where the amplitude \( x(t) \) is plotted as a function of time. It is seen that for \( \tau = 0.1 \) the motion is smooth and very nearly sinusoidal. Whereas for \( \tau = 10 \), the oscillation is made up of sudden transitions between amplitudes of opposite sign; for this reason, Van der Pol called this type of motion a \textit{relaxation oscillation}. He used these to account for the heartbeat. It is noteworthy that increasing the nonlinear damping coefficient \( \tau \) not only causes the transition from sinusoidal oscillations to relaxation oscillations but also lowers the frequency. Contrast this with the case of linear-free oscillations discussed earlier. This is a significant point, as we shall presently see.

The stability of this type of motion can be investigated by the method of the “equations of variations.” Although we do not demonstrate the method, one can show that the motion corresponding to a periodic solution \( x(t) \) is stable if the line integral
Fig. 5-15. Change in Character of the Oscillations with Increase in Nonlinearity.

\[ \int G(x) \, dx < 0 \]  \hspace{1cm} (5-59)

where \( G(x) \) is defined in (5-55).

Important applications of the theory of oscillations with nonlinear damping sketched in this section occur in the field of telecommunication engineering. Outside this field it is known that various periodic biological processes (Ref. 14) are well approximated by equations like (5-55). The psychological response of certain groups of people to changing business conditions undoubtedly shows some analogy to the behavior of self-sustained oscillator systems. There are other relaxation oscillations, not only in electronic technology and biology but in the economy. A tremendous crash in the stock market is a relaxation oscillation—one cycle. A relaxation oscillation is characterized by long intervals of quiescence followed by a very sudden, sometimes catastrophic change over a very short period of time.

The relaxation oscillation also serves as a particularly good example for illustrating the phase-locking mechanism. By varying the amount of nonlinearity (i.e., changing \( \tau \)), one can alter the frequency of the oscillator. By applying an external signal, one can alter the phase and frequency of the relaxation oscillation to the point where it is phase locked to the external signal’s phase and frequency. In the next section we explore this phenomenon quantitatively.

We summarize by noting that we have studied two extreme types of oscillations, sinusoidal and relaxation. As we have seen, a sinusoidal oscillation has a smooth, continuous change of phase in a mechanism that involves an ex-
change of energy from two reservoirs. The spring and mass combination periodically exchanges potential energy and kinetic energy. In the $LC$ oscillator circuit there is a change from capacitive energy to inductive energy. In the oscillations in cellular chemistry, the two reservoirs may be two disparate enzymes.

In relaxation oscillations there are not two reservoirs of energy switching back and forth. There is a single reservoir of energy that gradually fills up. When the reservoir fills to a certain level, a gating valve opens to empty the reservoir, whereupon it proceeds to refill. In the sinusoidal oscillating case, forces gated by feedback push the oscillator to prevent it from decaying, much like pushing a child in a swing. By contrast, in the relaxation oscillation, we do not have a resonant frequency, we have a *threshold effect*. When the threshold is reached, the reservoir is emptied but the source that fills it continues. Such a situation arises in the heart.

5-9 Coupled Nonlinear Oscillations, the Phase-Locked Regenerative Receiver, and the Principle of Synchronization

At this point we turn our attention to a different aspect of the effect of nonlinearity on resonant systems. Interest here centers on the case where nonlinearity gives the system the capability of performing self-sustained oscillations as discussed in the previous section. The question is what happens when such a system is subjected to an external signal having a frequency near the resonant frequency of the system. Instead of imposing an external signal, we ask what happens when two nonlinear oscillators, tuned to neighboring frequencies, are mutually coupled together. The answer to these rather complicated questions exhibits the *principle of synchronization, entrainment* or *phase locking*.

The phenomenon of synchronization or frequency entrainment was among the first of other nonlinear phenomena to be studied. As we shall see in this section, the synchronization effect can readily be observed in electronic circuits where a tuned circuit is oscillating at a frequency of $\omega_0$. If an extraneous signal of frequency $\omega$ is applied, one observes the *beats* of the two frequencies—that is, frequencies $|\omega - \omega_0|$. If $\omega$ is made to approach $\omega_0$, the frequency of the beats decreases; but this happens only up to a certain value of $|\omega - \omega_0|$, after which the beats suddenly disappear and there remains only the frequency $\omega$. Everything happens as if the free oscillation of frequency $\omega_0$ were entrained by the applied signal of frequency $\omega$.

Apparently Huygens (1629-1695) was the first to observe the phenomenon in two clocks hung on a wall. These effects were rediscovered more than two centuries later in electrical circuits by a number of physicists; among them were Lord Rayleigh (Refs. 15, 16, 17), Vincent (Ref. 18), Möller (Ref. 19), Appleton and Van der Pol (Refs. 20, 21), and others. The last two authors
essentially developed the theory. In the following sections we indicate briefly the original Van der Pol version, because it is more appealing to the electrical engineer. We also note the extension by Andronow and Witt (Refs. 22, 23) and Stoker (Ref. 8). Detailed stability arguments are omitted, for they appear at great length elsewhere; for example, see Minorsky (Ref. 24) for an excellent account of nonlinear oscillations.

A typical and important case, which serves to illustrate the synchronization principle, is the electrical system indicated in Fig. 5-16. Here the oscil-

![Fig. 5-16. A Self-Excited, Mutually-Coupled Oscillator System with Excitation $s(t)$.](image)

latory or resonant component of the system is in the grid circuit rather than the anode circuit. A signal source, the external force $s(t)$ of voltage, is also present in the grid circuit. In telecommunications this is analogous to the signal received via an antenna. Since the grid and plate circuits are mutually coupled, the differential equations for the system in terms of the grid current $i$ and the grid potential are

$$L \frac{di}{dt} + Ri + e_g - M \frac{di_a}{dt} = s(t)$$

$$C \frac{de_g}{dt} = i$$

Assuming a tube characteristic of the form

$$i_a = Se_g \left(1 - \frac{e_g^2}{3K^2}\right)$$

with positive constants $S$ and $K$ (frequently $K$ is called the saturation potential and $S$ the steepness of the characteristic) and introducing the quantities

$$x = \frac{e_g}{K}, \quad \omega_0^2 = \frac{1}{LC}, \quad \alpha = \frac{MS - RC}{LC}, \quad \gamma = \frac{MS}{LC}$$

(5-62)
into (5-60), we obtain

\[
\frac{\ddot{x}}{\omega_0^2} - \frac{\alpha}{\omega_0^2} \left(1 - \frac{\gamma x^2}{\alpha}\right) \dot{x} + x = \frac{s(t)}{K}
\]  

(5-63)

which is the forced Van der Pol equation. We further note that \(\omega_0\) is the frequency of the free, linear oscillations of the oscillator tank circuit. When \(s(t) = 0\) and \(MS > RC\), then the oscillator system produces self-sustained nonlinear oscillations. If \(s(t)\) is a stochastic process, then the oscillator system (5-63) produces random nonlinear oscillations and we shall be concerned with such oscillations in later chapters.

### 5-9.1 Basic Equations of Operation

The equations of operation, for the oscillator system described by (5-63), are best determined for the case of greatest interest here by assuming a "quasi-sinusoidal" solution (the grid potential) of the form

\[
x(t) = b_1(t) \sin \omega t + b_2(t) \cos \omega t = A \cos \Phi
\]  

(5-64)

where \(b_1 = -A \sin \phi, b_2 = A \cos \phi, A = \sqrt{b_1^2 + b_2^2}\), and \(\Phi = \omega t + \phi\). Here the \(b_1(t)\) and \(b_2(t)\) are assumed to be slowly varying functions of time; in other words, the motion \(x\) is essentially an oscillation much like a narrowband signal in which \(A\) and \(\phi\) are slowly varying functions when compared with \(\omega\). We note that \(x\) is analogous to the reference signal in a PLL. Assuming that the oscillator system is subject to the external periodic signal \(s(t) = KF \sin \omega t\), then (5-63) can be written as

\[
\frac{\ddot{x}}{\omega_0^2} - \tau \left(1 - \frac{2x^2}{r_0^2}\right) \dot{x} + x = F \sin \omega t
\]  

(5-65)

where \(\tau = \alpha/\omega_0^2, \gamma/\alpha = 2/r_0^2\), and we have assumed that \(MS > RC\). As we shall later see, if we take \(F = 0\), \(r_0\) is the amplitude of the free, nonlinear oscillation of frequency \(\omega_0\) when \(\tau \neq 0\). In fact, the oscillations of Fig. 5-15 are characterized by three parameters when \(F = 0\), namely, \(r_0\) and \(\omega_0\) the amplitude and resonant frequency of its oscillations, and \(\tau\), which characterizes the transient time for the steady-state oscillations to be established.

Now the amplitude \(A\) and the phase \(\phi\), as polar coordinates in the phase plane rotating with angular velocity \(\omega\), are defined by

\[
b_1 = -A \sin \phi = x \sin \omega t + \frac{\dot{x}}{\omega} \cos \omega t
\]

\[
b_2 = A \cos \phi = x \cos \omega t - \frac{\dot{x}}{\omega} \sin \omega t
\]
Coupled Nonlinear Oscillations

\[ \dot{x} = -\omega A \sin \Phi \] Such definitions satisfy the exact equations

\[
\frac{db_1}{dt} = \left(\frac{\ddot{x} + \omega^2 x}{\omega}\right) \cos \omega t \\
\frac{db_2}{dt} = -\left(\frac{\ddot{x} + \omega^2 x}{\omega}\right) \sin \omega t
\] (5-66)

Noting that

\[
\ddot{x} + \omega^2 x = (\omega^2 - \omega_0^2)x + \tau \omega_0^3 \left(1 - \frac{2x^2}{r_0^2}\right) \ddot{x} + \omega_0^3 F \sin \omega t
\]

and substituting for \( x \) and \( \ddot{x} \) into (5-66) produces

\[
2\dot{b}_1 - 2\Omega_0 b_2 - \tau \omega_0^3 \left(1 - \frac{A^2}{r_0^2}\right) b_1 = 0
\]

\[
2\dot{b}_2 + 2\Omega_0 b_1 - \tau \omega_0^3 \left(1 - \frac{A^2}{r_0^2}\right) b_2 = -\omega_0 F
\] (5-67)

when we set \( \Omega_0 = \omega - \omega_0, \omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0) \approx 2\Omega_0 \omega_0 \) and neglect double-frequency terms. In particular, if we set \( F = 0 \), we obtain from (5-67) the results: \( r_0 = A, \Omega_0 = 0 \), since \( b_1 \) and \( b_2 \) cannot both vanish for \( \tau \neq 0 \). This verifies our earlier statement that \( r_0 \) is the amplitude of the free, nonlinear oscillation.

Introducing the expressions \( b_1 = -A \sin \phi \) and \( b_2 = A \cos \phi \) into (5-67), we obtain the differential equations of phase and amplitude operation.

\[
\dot{\phi} = -\left[\Omega_0 - \frac{\omega_0 F}{2A} \sin \phi\right]
\]

\[
\dot{A} = \frac{\tau \omega_0^3}{2} \left(1 - \frac{A^2}{r_0^2}\right) A - \frac{\omega_0 F}{2} \cos \phi
\] (5-68)

The differential equation in \( \phi \) is analogous to the equation of a first-order sinusoidal PLL where \( \theta(t) = 0, \dot{\theta}(t) = \Omega_0 t + \theta_0 \), and noise is absent (see Chapter 3). When \( \dot{\phi} = 0 \), the oscillator is phase locked or synchronized to the input signal. When this occurs, we refer to the oscillator system as the phase-locked regenerative receiver. This example also serves to illustrate the principle of the injection-locked oscillator (Ref. 30) and how one can increase the frequency stability of an oscillator by applying a synchronization voltage from a second oscillator that has a higher degree of frequency stability, even though lower power. The causes of frequency instability are fluctuation of current and voltage in various elements of the oscillator—for example, shot noise effect in anode current, thermal effects, instability of the power supply, or external interference. Such fluctuations can be accounted for by the introduction of a noise source in the grid circuit and by redeveloping the equations of operation. This is left as an exercise for the reader.
It is of further interest to introduce briefly the work of Andronow and Witt (Refs. 22, 23), since it furnishes a beautiful application of the theory of singularities of first-order differential equations to the synchronization problem. Let

\[ X = \frac{b_1}{r_0}, \quad Y = \frac{b_2}{r_0}, \quad Z^2 = X^2 + Y^2, \quad \lambda = \frac{\tau \omega_0^2 t}{2} \]  

in (5-67), we obtain

\[ \frac{dX}{d\lambda} = -\delta Y + (1 - Z^2)Y \quad \frac{dY}{d\lambda} = B + \delta X + (1 - Z^2)Y \]  

where \( B = -F/\tau \omega_0 r_0 \) and \( \delta = 2\Omega_0/\tau \omega_0^3 \). In standard form we have, from (5-70),

\[ \frac{dY}{dX} = \frac{B + \delta X + (1 - Z^2)Y}{-\delta Y + (1 - Z^2)X} = \frac{P(X, Y)}{Q(X, Y)} \]  

Now the singular points lie where \( P \) and \( Q \) vanish—that is,

\[ B + \delta X + (1 - Z^2)Y = 0 \quad -\delta Y + (1 - Z^2)X = 0 \]  

Solving for \( X \) and \( Y \) in terms of \( \delta \) and \( B \) and inserting this into \( Z^2 = X^2 + Y^2 \) gives

\[ Z^2[\delta^2 + (1 - Z^2)^2] = B^2 \]  

which was derived by Van der Pol using another method. In terms of the new variables, the response \( x(t) \) is normalized to

\[ x(t) = r_0(X \sin \omega t + Y \cos \omega t) \]  

so that for each beat frequency (i.e., each \( \delta \)) and for each \( F \), (5-73) furnishes through \( Z \) the amplitudes of \( X \) and \( Y \). In fact, such curves are called the response curves of the oscillation system.

Van der Pol goes further and shows that the conditions for entrainment or synchronization of the oscillations are given by the conditions

\[ Z^2 = \frac{1}{2}, \quad \delta^2 = (1 - Z^2)(1 - 3Z^2) = 0 \]  

We shall elaborate further on this condition later.
5-9.2 Oscillations when the Detuning is Large

When $\Omega_0 = \omega - \omega_0$ is large, then (5-67) reduces to

\[ \dot{b}_1 - \Omega_0 b_2 = 0 \quad \dot{b}_2 + \Omega_0 b_1 = 0 \quad (5-76) \]

so that $b_1(t) = b_f \cos (\Omega_0 t + \phi_0)$ and $b_2(t) = b_f \sin (\phi_0 - \Omega_0 t)$, where $b_f$ is a constant and $\phi_0$ is a constant phase shift. In this case the oscillation (5-64) is amplitude modulated by the signals at the beat notes frequency $\Omega_0$. In fact, using the law for the sine of the difference of two angles, we find from (5-64) and (5-76) that

\[ x(t) = b_f \sin (\omega_0 t - \phi_0) \quad (5-77) \]

From this we see that when $\omega_0$ is large, the externally applied signal has essentially no effect on the response $x(t)$.

5-9.3 Combined Oscillations of Constant Amplitude

We have seen that at the two extremes of large and small detuning, the response $x(t)$ of the Van der Pol oscillator consisted of two simple harmonic oscillations, one with frequency $\omega_0$ and the other with frequency $\omega$. Now let us consider what happens if we analyze the problem from the linear theoretic point of view. In order to get the system to oscillate, we would have to postulate an undamped oscillator—that is, $\tau = 0$. Second, it is intuitively clear that when the external signal is applied, it would superimpose on the output without influencing the existing oscillation and would cause the usual linear resonance phenomena discussed in Section 5-2. Near resonance, the two oscillations would produce a beat note, and this beat frequency could be made as slow as desired by making $\omega$ approach $\omega_0$ as shown in Fig. 5-17a.

![Diagram of Beat Frequency](image)

(a) Beat frequency of a linear oscillator  
(b) Beat frequency of a nonlinear oscillator

**Fig. 5-17.** Strength of Beat Note versus $\omega$ for Two Different Oscillators.
Return now to the quantitative nonlinear analysis. In order to investigate the combination of oscillations quantitatively, it is necessary to assume a solution whose form is different from (5-64). More generally, the primary response (grid potential) of the oscillator will be of the form (at least for small \( \tau \))

\[
x(t) = A \sin(\omega t + \phi) + A_0 \sin(\omega_0 t + \phi_0)
\]

(5-78)

where \( A \) and \( \phi \) are, respectively, the amplitude and phase of the response at the frequency \( \omega \) of the externally applied signal and \( A_0 \) and \( \phi_0 \) are the corresponding quantities for the oscillation of the system's resonant frequency. We can readily eliminate the time variable in (5-63) to produce, in the approximation,

\[
2\left(\frac{A}{r_0}\right)^2 + \left(\frac{A_0}{r_0}\right)^2 = 1
\]

\[
\left(\frac{\delta A}{r_0}\right)^2 + \left(\frac{A}{r_0}\right)^2\left[1 - \left(\frac{A}{r_0}\right)^2 - 2\left(\frac{A_0}{r_0}\right)^2\right] = \left(\frac{F}{\tau\omega r_0}\right)^2
\]

(5-79)

where

\[
\delta = \frac{-\Omega_0}{\tau\omega\omega_0/(\omega + \omega_0)} \approx \frac{-2\Omega_0}{\tau\omega_0^2}
\]

(5-80)

Defining \( z = A/r_0 \) and \( z_0 = A_0/r_0 \) as the normalized response amplitudes of the external signal and the self-oscillations, it is easy to show that

\[
\delta^2 z^2 + z^2(1 - z^2 - 2z_0^2) = \eta^2 \quad 2z^2 + z_0^2 = 1
\]

(5-81)

where \( \eta = F/\tau\omega r_0 \). In Figs. 5-18 and 5-19 the normalized response amplitude has been plotted versus the normalized system detuning \( \delta \) for various values of \( \eta \). For detailed discussion relative to stability of the oscillation, the reader is referred to Minorsky (Ref. 24) and Stoker (Ref. 8).

The response of the oscillator divides itself into two patterns. When \( F/\tau\omega r_0 \) is large, the response of \( A/r_0 \) to the external signal looks very much like a linear resonance curve. However, the amplitude of the natural oscillation \( A_0/r_0 \) experiences a very interesting change. When \( \delta \) is far from resonance, there is little influence of the external signal on \( A_0/r_0 \), but as \( \omega \) approaches \( \omega_0 \), the amplitude \( A_0 \) drops to zero and stays there until \( \delta \) recedes from \( \omega_0 \) on the other side of resonance. Then there is a finite range (the synchronization band) of frequencies near resonance where the oscillator will be locked so that it oscillates exactly at the frequency of the external signal. In this region the beat frequency will disappear completely, as shown in Fig. 5-17b, which should be compared with the linear case of Fig. 5-17a.
Fig. 5-19. Amplitude Response at Frequency of Resonance versus the Detuning.
In other words, if the frequencies of the different oscillators are sufficiently close together, they will be pulled together, rendering phase-locking of the oscillations. The important thing here is the mutual coupling of the frequencies of the oscillations appearing in the external signal with those of the oscillator (VCO) to the synchronized. In fact, in power generation where a number of synchronous generators are feeding a load through two bars, the same phenomenon is noticed—that is, the pulling of the individual frequencies produced on masse into a single well-regulated oscillation.

The oscillator system response represented in Figs. 5-18 and 5-19 exhibits a second pattern of behavior when \( F/\tau \omega_{0} \) is less than \( (\frac{1}{3})^{1/2} \) and \( \delta \) is less than \( (\frac{1}{3})^{1/2} \). Here the response curves are multiple valued and more than one stable state of affairs exists, some with lock-in and some without. Hysteresis and jump phenomena occur. Such behavior is quite different from anything predicted by linear theory and is discussed further in Stoker (Ref. 8). The preceding example illustrates the basic concepts that allow one to establish synchronization in communication networks. From these results it is important to observe that if the frequency of an oscillator can be changed by applying an external signal of a different frequency, the mechanism must be nonlinear. A linear mechanism acting on an oscillator of a given frequency can produce only oscillations of this same frequency, generally with some change of phase and amplitude. This is not true for nonlinear oscillators, which may produce oscillations of frequencies that are the sums and differences of different orders of the frequency of the oscillator and the frequency of the applied signal.

In general, it is possible for the applied signal to displace the frequency of a nonlinear oscillator, and in the case which we have just considered, this displacement will be of the nature of an attraction rendering phase locking of the oscillations. It appears that the brain contains a number of oscillators of frequencies of nearly 10 Hz/sec and that, within limitations, the frequencies can be attracted to one another by visual stimulation (Ref. 25). Under such circumstances, the frequencies are likely to be pulled together (phase locked) into one or more clumps. The phase locking of oscillations occurs in biological systems—in fireflies, crickets, and frogs (Refs. 1, 14). The accurate frequency regulation that makes possible the use of electrical clocks of high accuracy is due to the phase locking or attraction of frequencies in power-generating systems. To use a biological analogy, a parallel combination of generators has a better homeostasis than a single generator because of the pulling together of the line frequencies.

One of the prime problems of biology is to understand the way in which the capital substances constituting genes or viruses, or possibly specific substances producing cancer, reproduce themselves out of materials devoid of this specificity, such as a mixture of amino and nucleic acids. According to Wiener (Ref. 1), it is quite possible that this phenomenon may be regarded as a sort of attractive interaction of frequency. The pulling and binding of frequencies
are also observable in atomic physics (Ref. 26), where molecular spectra are of concern. The planets are oscillators of a nonlinear characteristic; this perturbations represent nonlinear coupling. The same phenomenon occurs in the study of high-energy particles with cyclotrons. Periodic pulling has been directly seen in biological rhythms, such as the motion of the fins of fish (Ref. 27) (called relative coordination) and in almost phase-locked circadian rhythms (Refs. 11, 28), magnetrons (Refs. 28, 29), injection-locked oscillators (Refs. 28, 30), and certain types of plasmas (Ref. 31).

5-10 Electrical Problems Leading to Hill's Equation

As we shall observe later, nonlinear systems producing relaxation oscillations have been studied by methods due to Hill (Ref. 32). The transient solution to the behavior of a phase-locked loop system will depend on our ability to solve a differential equation called Hill's equation (Ref. 32). It is beyond the scope of this book to discuss the general theory of Hill's equation with particular reference to the questions of existence, uniqueness, and boundedness of solutions. Nevertheless, it is necessary here to acquaint the reader with the form of the equation and to refer him to the body of theory that exists in the mathematical literature.

A simple electrical system leading to a Hill's equation consists of a parallel circuit containing a constant inductance \( L \) and a capacitor in which the capacitance \( C(t) \) is assumed to be periodic in \( t \). If \( q \) is the charge on the condenser, we have for \( q \) the differential equation

\[
L \ddot{q} + \frac{1}{C(t)} q = 0
\]  

(5-82)

One could easily give many other examples of physical problems leading to Hill's equations; however, if \( p(t) \) is periodic in \( t \), the linear equation

\[
\ddot{x} + [-\beta + \alpha p(t)] x = 0
\]  

(5-83)

is called a Hill's equation when \( \alpha \) and \( \beta \) are constants. If, for example, we take \( \beta = \alpha \) with \( \alpha > 0 \) and \( p(t) = \cos t \) in (5-83), the result is a special case of Hill's equation called the Mathieu equation. In general, however, Floquet theory for linear differential equations with periodic coefficients may be used to reveal the functional character of the solutions of Hill's equation. This theory, unfortunately, does not decide the stability question, which can usually be solved only by studying the solutions of the given differential equation in considerable detail.
5-11 Further Studies

Since it is impossible in one chapter to relate all the details pertaining to nonlinear oscillation theory, we shall give here a list of references for those more research-oriented readers. One cannot begin further study more delightfully than looking at the original work on nonlinear oscillations due to Van der Pol (Refs. 9, 10, 11, 12), Andronow (Ref. 33), Andronow and Witt (Refs. 22, 23), Lienard (Ref. 34), Haag (Ref. 35), Poincaré (Refs. 4, 5, 6), and Rayleigh (Refs. 15, 16, 17). A rather thorough treatment of obtaining solutions to nonlinear oscillation problems has appeared in the Russian literature—namely, the excellent book by N. Minorsky (Ref. 24), and the translation by S. Lefschetz of books by Kryloff and Bogliuboff (Ref. 36) and by Andronow and Chaikin (Ref. 37). Blaquiere (Ref. 38) also presents an interesting treatment of certain problems pertaining to nonlinear system analysis. The book by Hsu and Meyer (Ref. 39) gives a rather elementary treatment of stability and the notions of singularities. The theory of Liapunov's (Ref. 40) for a discussion of stability problems should not be overlooked, nor should the book by Minorsky. A rather clear and precise treatment of certain problems arising in nonlinear vibrations is given by Stoker (Ref. 8). General references that pertain to differential equations are the books by Forsyth (Ref. 41) and Rainville (Ref. 42). The serious serious researcher will probably want to study the mathematics of Hill's equation and Floquet theory by consulting the book by Magnus and Winkelman (Ref. 32).

For those interested in supplementary reading matter relevant to biological rhythms, one can try for a start the exciting book by Bunning (Ref. 14); also see the books by Milsum (Ref. 43) and Grodins (Ref. 44). Dewan (Refs. 25, 45) gives methods of applying the Van der Pol oscillator model to the interpretation of electroencephalographic data. Walter (Ref. 46) has discussed the phase-lock principle and its relationship with the synchronization of cortical physiological rhythms. Finally, other material of interest are the papers by Dewan and Lashinsky (Ref. 47), and Kobzarev (Ref. 48), and the books by Hayashi (Ref. 49) and Bogolyubov and Mitropolskii (Ref. 50).

The five volumes edited by Lefschetz (Ref. 51), on the theory of nonlinear oscillations, cover a great variety of topics in nonlinear differential equations. The book by McLachlan (Ref. 52) provides engineers and physicists with a practical introduction to the important subject of nonlinear differential equations.

Problems

5-1 For free, linear oscillations described by (5-4), show that (a) the origin of the \((x, v)\) plane is a spiral point of equilibrium provided that the damping is less
than critical—that is, \( r < \sqrt{k/m} \)—and (b) if \( r > \sqrt{k/m} \), the singularity at the origin of the \((x, v)\) plane is a node.

5-2 The flux variations \( \Phi(t) \) in an electrical circuit, containing a condenser and an iron-core inductance coil, are governed by the equation

\[
\frac{d^2 \Phi}{dt^2} + h(\Phi) = 0
\]

where \( h(\Phi) = a\Phi - b\Phi^3/c, a > 0, b > 0, c > 0 \) is the restoring force.
(a) Find the total energy and energy curves in the \((\Phi, v)\) plane if \( v = d\Phi/dt \).
(b) Find the velocity \( v \) of the motion \( \Phi \).
(c) Develop an expression for the potential energy \( U(x) \).

5-3 Consider the class of nonlinear systems governed by the differential equation

\[
\frac{d^3 x}{dt^3} + ax + bx^3 = 0
\]

with \( a < 0 \).
(a) Find the total energy and the energy curves of the motion \( x(t) \).
(b) Using the results of (a), sketch the phase plane and discuss the qualitative nature of the motion \( x(t) \) for \( b > 0 \) (hard spring).
(c) Repeat (b) for \( b < 0 \) (soft spring).

5-4 The motion \( x(t) \) of a current-carrying conductor restrained by springs and subjected to a force from the magnetic field due to another infinitely long fixed parallel wire is given by

\[
md^2 x \quad dt^2 + k \left( x - \frac{\lambda}{a - x} \right) = 0
\]

with \( k > 0, a > 0, \) and \( \lambda \) is dependent on the current.
(a) Where are the singularities?
(b) Find the total energy and the energy curves.
(c) Find the potential energy \( U(x) \).

5-5 Starting from (5-60), develop the equation, (5-63), of operation for the Van der Pol oscillator and justify that it reduces to (5-65).

5-6 Show that the amplitude \( A \) and phase \( \Phi \) satisfy the relationship given in (5-66). Using these expressions, show that the differential equations for the amplitude and phase satisfy (5-68).

5-7 The external signal appearing in the grid circuit of the Van der Pol oscillator is to be characterized by

\[
s(t) = K(F \sin \omega t) + n(t)
\]

where \( n(t) = n_r(t) \cos \omega t - n_i(t) \sin \omega t \) is a narrowband Gaussian process.
(a) Develop the differential equations (analogous to 5-68) of operation of the amplitude $A$ and phase $\phi$ of the oscillator. Comment on your results.

5-8 Justify the mathematical development that leads to (5-79).

5-9 Consider the forced Van der Pol equation (5-65). Is it possible to obtain solutions of the form given in (5-78) which oscillate with two basic frequencies, one due to the forcing function $\sin \omega t$ and one due to the basic frequency of the self-excited oscillations? If such a phenomenon occurs, the solution is said to have combination tones.

5-10 The Van der Pol oscillator considered in Section 5-9.3 operates such that $F_0/\tau_0 r_0 = 1$.
   (a) Find the synchronization band $\Omega_0$.
   (b) At this boundary of synchronization, what is the value of the normalized amplitude response?

5-11 From the response curves, given in Fig. 5-19 for the Van der Pol oscillator, sketch the synchronization band $\Omega_0$ vs. the parameter $F/\tau_0 r_0$. Discuss the sketch in regard to the fundamental behavior of oscillator operation.

5-12 The Van der Pol equation $\ddot{x} - (1 - x^2)\dot{x} + x = 0$ exhibits a stable limit cycle. Use phase-plane techniques to locate approximately this limit cycle.

References


References

5-28 De Vito, P. A., W. J. Kearns, and M. H. Seavey, Injection-Locked Pulsed Magnetrons, NEREM Record, 10 (1968), 210-211.


STOCHASTIC FIELD THEORY
FOR TRANSPORT
PROCESSES

6-1 Introduction

One of the main objectives of this chapter is the development of the equations of stochastic field theory for transport processes in a way that is appealing to the communications engineer. As we shall see, they have a deep analogy with Maxwell’s field equations. Furthermore, the development also shows the interconnection between probabilistic potential theory and Maxwell’s wave equations for stochastic fields. The development will also be used in the next chapter to show the interconnection between the theory of continuous Markov processes of the diffusion type and Maxwell’s wave equations for stochastic fields. It turns out that the theory of Markov processes plays a major role in carrying out the nonlinear analysis of SCSs in the presence of noise.

For our purpose, one of the most useful applications of stochastic field theory is in the physical interpretation and direction of the Fokker-Planck equation—the equation of probability flow. The insight gained by our study allows us to introduce in a novel way the concepts of probability flow, probability current, and diffusion. These concepts are fundamental to the understanding of the nonlinear phenomenon associated with SCSs in the presence of
noise. We assume that the reader has been exposed to the subject of vector analysis. Certain concepts are summarized in Appendix I.

One of the objectives of the physics of modern communications engineering is to approximate nature by a series of mathematical equations. These equations are written so that they incorporate within themselves the aspects of phenomena that have been set down, and they can be solved and used to predict the results of future measurements. The value of a theory (or mathematical model) is its ability to predict the results of future experiments that may be carried out in the laboratory. If the predictions of the mathematical equations are not justified by experiment, a new mathematical model must be sought. All mathematical theories are, to some extent, approximations; nature has been unkind enough to see to that. It is not unreasonable, then, that different mathematical theories are devised to describe one set of physical observations. If the theories are different, the degree of approximation to nature will be different. This does not mean that the theory with the poorer degree of approximation is not a useful one. Often accurate theories are not used because their accuracy is not warranted or their application is too difficult. In general, the more closely the mathematical theory approximates nature, the less tractable and susceptible the equations of that mathematical theory are to solution. With these remarks as preambles to the material contained in this chapter, we proceed with the theoretical developments.

6-2 Probability Flux and Gauss’s Law

The concept of flux passing through an area does not have to be limited to electric phenomena. If \( A \) is a vector function of the space and time variables \( (x, t) = (x_1, x_2, \ldots, x_n, t) \), the product of the magnitude of \( A \) at any point in space and time times an element of area normal to \( A \) may be called the flux of \( A \) passing through that area at that particular instant in time. Thus from Gauss’s law we have that the total flux flowing through a surface \( S \) is given by the surface integral integrated over that surface.

The concept of probability flux is a useful one to consider from the point of view of the vector field that causes this flux to be generated. In electrical systems, probability flux is generated by the random flow of electric charges. If \( \mathcal{D} \) is defined to be the probability flux density, then the probability flux, \( \psi_p \), flowing through some closed surface \( S \) at time \( t \) is just the probability of being in that volume \( V \) enclosed by \( S \). That is,

\[
\psi_p \triangleq \oint_S \mathcal{D} \cdot dS = \iiint_V p(x, t) \, dV
\]

\[
= \iiint_{n\text{-fold}} p(x, t) \, dx
\]  

(6-1)
Divergence of the Probability Flux Density

The symbol \( \oint \) is used to denote that the integration is performed over a closed path, a closed surface, or a definite volume. Obviously, if \( S \) represents that surface which encloses the total probability space \( \Omega \), then

\[
\oint_{\Omega} \mathcal{D} \cdot dS = \int_{\text{n-fold}} p(x, t) \, dx = 1 \tag{6-2}
\]

and we see that the total probability flux enclosed can never be greater than unity for all \( t \). This equation is a statement of the conservation of probability and is analogous to the statement of conservation of energy, mass, etc., familiar from elementary physics. The probability flux density \( \mathcal{D} \) is analogous to the flux density \( \mathbf{D} \) in electromagnetic theory, whereas the probability flux is analogous to the concept of flux in electromagnetics. We shall find this interpretation useful. In conclusion, we write Gauss's law for stochastic fields in words—namely,

| Probability Flux Out of A Closed Surface = Probability Enclosed |

6.3 Divergence of the Probability Flux Density

In this section we shall make use of the divergence theorem (Refs. 1, 2) to develop the first major equation in stochastic field theory. This theorem states that for any closed volume \( V \), the total flux \( \psi \) emerging through the surface \( S \) bounding the volume is given by

\[
\psi = \oint_{S} \mathbf{A} \cdot dS = \oint_{V} \nabla \cdot \mathbf{A} \, dV \tag{6-3}
\]

This equation states that the surface integral of any vector over any surface is equal to the volume integral of the divergence of that same vector over the volume bounded by the surface. This is called the divergence theorem. A special solution of this equation is

\[
\nabla \cdot \mathbf{A} = \frac{d}{dV} \oint_{S} \mathbf{A} \cdot dS \tag{6-4}
\]

If the vector \( \mathbf{A} \) in (6-3) is replaced by the probability flux density \( \mathcal{D} \) from (6-3), we have

\[
\oint_{S} \mathcal{D} \cdot dS = \oint_{V} \nabla \cdot \mathcal{D} \, dV = \oint_{V} p(x, t) \, dV \tag{6-5}
\]
Later it will be convenient to replace \( p(x, t) \) by \( p(x, t|\mathbf{x}_0, t_0) \). This, of course, means the probability that the representative point is at \( x \) at time \( t \) given it was at \( \mathbf{x}_0 \) at \( t_0 \). For the present, however, it is not necessary to make this distinction between the two cases. Since the integrands in (6-5) are taken over the same volume, we have the first major equation in stochastic field theory.

\[
\nabla \cdot \mathcal{D} = p(x, t) \tag{6-6}
\]

which is the relation between the p.d.f. at a point and the way in which the stochastic field \( \mathcal{D} \) varies at a point. The divergence measures the net rate at which probability is being transported away from the neighborhood of a point. This equation states that the total probability flux emerging from a point at time \( t \) surrounded by an incremental volume element \( dV \) is just the total probability of being in \( dV \) at time \( t \). This point equation has a deep analogy with one of Maxwell’s equations, which states that the divergence of the flux density equals the charge density. In one dimension we have

\[
\nabla \cdot \mathcal{D} = \frac{\partial \mathcal{D}_x(x, t)}{\partial x} = p(x, t) \tag{6-7}
\]

so that \( \mathcal{D}_x(x, t) \) is just the cumulative probability distribution function (c.p.d.f.).

6-4 Stoke’s Theorem and the Probability Flux Density

Stoke’s theorem is the companion and complement of the divergence theorem. The theorem is useful in defining the curl of the vector, say \( \mathbf{A} \). Later on in the study of SCSs we shall observe a vector field that possesses a curl, an extremely interesting physical situation. Under certain desirable circumstances, the SCS can be designed such that the curl of this vector field is zero—that is, the vector field is irrotational.

Stoke’s theorem (Refs. 1, 2) represents a statement of the fact that the line integral of a vector over the boundary of any cap is equal to the surface integral of the curl of the vector over the cap; that is,

\[
\oint \mathbf{A} \cdot d\mathbf{r} = \int_{\text{cap}} \nabla \times \mathbf{A} \cdot d\mathbf{S} \tag{6-8}
\]

where \( \mathbf{r} \) is the path of integration taken around the cap. This permits the following definition of the curl, \( \nabla \times \mathbf{A} \), of the vector \( \mathbf{A} \):

\[
\nabla \times \mathbf{A} \cdot \Delta \mathbf{S} = \mathbf{A} \cdot \Delta \mathbf{r} \tag{6-9}
\]
A vector field $\mathbf{A}$ such that $\nabla \times \mathbf{A} = 0$ is often termed \textit{irrotational}. The curl is essentially a measure of the angular velocity of the motion. In a SCS we will observe that the curl gives a measure of loop detuning which introduces a bias in any attempt to make a measurement of the phase of an arriving electromagnetic signal.

If we insert $\mathcal{D}$ into (6–8), we obtain a result that relates the line integral of the probability flux density to the curl of $\mathcal{D}$.

6-5 The Changing Flux Theorem and the Probability Flux Density

In this section we present the changing flux theorem. Not only does this provide us with a powerful tool for rapid development of subsequent theory, its approach also lends insight into the physical relationships expressed by the equations to be developed. The proof of this theorem is given in Appendix II.

\textbf{Theorem.} If the value of a vector $\mathbf{A}$ is changing with respect to time at rate $\partial \mathbf{A}/\partial t$, and the cap is moving with average velocity $\mathbf{v}$, then the total time derivative of the flux $\psi$ of $\mathbf{A}$ over any cap is

$$\frac{d\psi}{dt} = \int_S \frac{d\mathbf{A}}{dt} \cdot d\mathbf{S} = \int_S \left[ \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \nabla \cdot \mathbf{A} + \nabla \times (\mathbf{A} \times \mathbf{v}) \right] \cdot d\mathbf{S} \quad (6-10)$$

This theorem expresses the total time rate of change of the flux across a given cap in terms of a surface integral of the vector function $\mathbf{A}$, even when the surface itself across which the flux is evaluated is in motion. The first term of this expression represents the change in the flux through $\mathbf{S}$ that is caused by the time variations of the vector field. The second term arises from the passage of $\mathbf{S}$ through an inhomogeneous vector field in which flux is generated. The third term represents the flux loss across the boundary of the moving surface.

Now according to Gauss's law,

$$\frac{d\psi}{dt} = \frac{d}{dt} \int_S \mathbf{A} \cdot d\mathbf{S} \quad (6-11)$$

which is recognized as the total rate of change of the surface integral $\int_S \mathbf{A} \cdot d\mathbf{S}$ due to (1) moving $d\mathbf{S}$ and (2) time-changing $\mathbf{A}$. If we now interpret $\mathbf{A}$ as the probability flux density $\mathcal{D}$, then we can write

$$\frac{d\psi_p}{dt} = \frac{d}{dt} \int_S \mathcal{D} \cdot d\mathbf{S} \quad (6-12)$$
where $\psi_p$ is the flux generated due to $\mathcal{D}$. If the surface $S$ is taken as that surface which bounds the total probability space, then integrating over the total probability space yields

$$
\frac{d\psi_p}{dt} = \frac{d}{dt} \int_S \mathcal{D} \cdot dS = \frac{d}{dt} \int_{\Omega} \nabla \cdot \mathcal{D} \, dV \tag{6-13}
$$

since the conservation of probability implies that the integral is constant; that is,

$$
\frac{d}{dt} \int_{\Omega} \nabla \cdot \mathcal{D} \, dV = \frac{d}{dt} \int_{n\text{-fold}} p(x, t) \, dx = 0 \tag{6-14}
$$

This says that all the probability remains trapped in $\Omega$ and no sources or sinks of probability exist outside $\Omega$. Therefore

$$
\int_S \left( \frac{\partial \mathcal{D}}{\partial t} + \mathbf{v} \nabla \cdot \mathcal{D} + \nabla \times \mathcal{D} \times \mathbf{v} \right) \cdot dS = 0 \tag{6-15}
$$

where $S$ is the surface that bounds the probability space $\Omega$.

6-6 Equation of Probability Flow

Now $\nabla \cdot \mathcal{D} = p(x, t)$ and $\mathbf{v} p(x, t)$ is analogous to the electrical current density $\mathbf{J}$ arising in electromagnetic theory. Thus we define the vector

$$
\mathcal{J}(x, t) \triangleq \mathbf{v} p(x, t) \tag{6-16}
$$

to be the probability current density produced by $\mathcal{D}$. Notice that the probability mass $p(x, t) \, dx$ times the velocity $\mathbf{v}$ yields momentum so that $\mathcal{J} \, dx$ represents the movement of probability mass with velocity $\mathbf{v}$. Furthermore, $\mathbf{v} \times \mathcal{D}$ is analogous to the magnetic field strength in electromagnetic theory. Therefore if we define the vector

$$
\mathcal{H} \triangleq \mathbf{v} \times \mathcal{D} \tag{6-17}
$$

as the stochastic magnetic field intensity, we have from (6-15) that

$$
\int_S \left( \mathcal{D} + \mathcal{J} - \nabla \times \mathcal{H} \right) \cdot dS = 0 \tag{6-18}
$$

If the surface is closed, the divergence theorem yields
\[
\begin{align*}
\oint_V \nabla \cdot (\dot{\mathcal{O}} + \mathcal{J} - \nabla \times \mathcal{H}) \, dV &= 0 \quad (6-19) \\
\text{and since } \nabla \cdot \nabla \times \mathbf{A} &= 0 \text{ for any vector } \mathbf{A}, \text{ we have} & \\
\oint_V \nabla \cdot [\dot{\mathcal{O}} + \mathcal{J}] \, dV &= 0 \quad (6-20) \\
\text{and at any point in the probability space we write} & \\
\nabla \cdot \dot{\mathcal{O}} + \nabla \cdot \mathcal{J} &= 0 \quad (6-21) \\
\text{Since } \nabla \cdot \mathcal{O} &= p(x, t), \text{ we have, after interchanging time and space differentia-} \\
\text{tion operators,} & \\
\nabla \cdot \mathcal{J}(x, t) + \frac{\partial p(x, t)}{\partial t} &= 0 \quad (6-22) \\
\text{where} & \\
\nabla \cdot \mathcal{J}(x, t) &\triangleq \sum_{k=1}^{n} \frac{\partial \mathcal{J}_k(x, t)}{\partial x_k} \quad (6-23) \\
\text{This is the equation of probability flow (continuity) for stochastic fields, which} & \\
\text{is the form of the Fokker-Planck equation to be derived in the next chapter.} & \\
\text{Equation (6-22) says that the divergence of the current density produced by a} & \\
\text{probability mass moving at a velocity } \mathbf{v} \text{ in a differential volume } dV \text{ surrounding} & \\
x \text{ plus the time rate of change of the probability of being in } dV \text{ at time } t \text{ equals} & \\
\text{zero. In passing, we point out that Maxwell’s field equations, as well as the} & \\
equation of flow which arises in fluid mechanics, plasma physics, hydrody- & \\
namics, heat and mass flow, and so on, can be derived directly from the chang- & \\
ing flux theorem by substitution of the appropriate field vector. The fact that} & \\
v\text{arious physical phenomena are tied together so closely by this one theorem is} & \\
\text{striking. The solution, of course, to the partial differential equation will gen-} & \\
\text{erally be different due to the different boundary and initial conditions for each} & \\
\text{physical situation.} & \\
\text{It should be noted that the equation of probability flow is a point equa-} & \\
tion (describes the behavior of the field at a point in space and time); hence in & \\
\text{using } p(x, t) \text{ and } \mathcal{J} \text{ we shall be discussing the field from a macroscopic point} & \\
of view. Thus the variables } (x_1, x_2, \ldots, x_n) \text{ are referred to as macroscopic vari-} & \\
\text{ables.} & \\
\text{In our studies to follow, the average vector velocity } \mathbf{v} \text{ will usually consist} & \\
of two components that produce the total probability current density } \mathcal{J}. \text{ The}
first component will be one produced by the transmitted signal, while the second component, which produces a stochastic current flow, or causes diffusion, will be due to the additive noise. Thus

\[ \mathcal{J} = \mathcal{J}_s + \mathcal{J}_d \]

stochastic current component due to random noise
produced by presence of a signal or external force

(6-24)

Later we shall characterize these current components in terms of the p.d.f. \( p(x, t) \). In the steady state (i.e., the case of the existence of a static field)

\[
\lim_{t \to \infty} \nabla \cdot \mathcal{J} = \lim_{t \to \infty} \sum_{k=1}^{a} \frac{\partial \mathcal{J}_k(x, t)}{\partial x_k} = \lim_{t \to \infty} -\frac{\partial p(x, t)}{\partial t} = 0
\]

(6-25)

which says that the divergence of the probability current density equals zero in the steady state.

Another concept of interest is one that represents the probability current density flowing in the direction of the unit vector \( e_k \). We refer to this component of \( \mathcal{J} \) as the probability current density of the \( k \)th projection and define it by the dot product

\[ \mathcal{J}_k(x, t) \triangleq e_k \cdot \mathcal{J}(x, t) \]

(6-26)

we will also need to evaluate the probability current \( \mathcal{J}(x_k, t) \) of the \( k \)th projection of \( \mathcal{J}(x, t) \), which is given by

\[ \mathcal{J}_k(x_k, t) = \int \cdots \int_{\text{n-1 fold}} \mathcal{J}_k(x, t) \, dx'_k \]

(6-27)

where \( dx'_k \triangleq dx_1, dx_2, \ldots, dx_{k-1}, dx_{k+1}, \ldots, dx_n \). The probability current of the \( k \)th projection describes the amount of probability mass crossing the hyperplane \( x_k = \) constant in the positive direction per unit time. A geometric interpretation of (6-27) is also possible. The probability current density \( \mathcal{J}_k(x, t) \) represents the probability current density flowing in the positive \( x_k \) direction, and \( \mathcal{J}_k(x, t) \, dx'_k \) represents the amount of probability surface current flowing through the differential surface area \( dx'_k \). Integrating over the surface gives the total probability current (average number of trajectories) flowing through the hyperplane \( x_k = \) constant. Just as the equation of heat conduction involves a flow of heat, (6-22) involves a flow of probability.
In solving the equation of probability flow, we shall describe $\mathbf{J}$ in terms of the "equations of motions" of the particles and the particle probability density function $p(x, t)$. Then from the equation of probability flow we solve for the p.d.f. (particle density) $p(x, t)$, using the appropriate initial and boundary conditions. From this, the probability current density $\mathbf{J}$ can then be found.

Finally, it is of interest to develop an equation for the curl of the stochastic magnetic field strength. From (6-19) we have the point curl equation,

$$\nabla \times \mathbf{H} = \mathbf{J} + \dot{D} \tag{6-28}$$

In the steady state

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{6-29}$$

if we assume that $\dot{D}$ approaches zero as $t$ approaches infinity.

6-7 Maxwell’s Curl Equations and the Potential Equations for Stochastic Fields

Let the stochastic magnetic flux density be denoted by the vector field $\mathbf{B}$. Then from Gauss's law we know that the magnetic flux $\psi_m$ is the surface integral of $\mathbf{B}$; that is,

$$\psi_m = \int_S \mathbf{B} \cdot dS \tag{6-30}$$

We shall be interested here in stochastic magnetic fields produced by probability currents arising because of the random motion of electric charges. Unlike the case where electric charges in random motion produces probability flux, we postulate that there are no free sources in the probability space producing magnetic flux. Consequently, from the divergence theorem and (6-30),

$$\nabla \cdot \mathbf{B} = 0 \tag{6-31}$$

which says that the stochastic magnetic field has zero divergence everywhere. If we define the dynamic electric fields strength $\mathbf{E}_d$ associated with the stochastic field $\mathbf{B}$ by

$$\mathbf{E}_d \triangleq \mathbf{B} \times \mathbf{v} \tag{6-32}$$

then when $\mathbf{B}$ is inserted into the changing flux theorem, we have

$$\int_S (\dot{\mathbf{B}} + \mathbf{v} \nabla \cdot \mathbf{B} + \nabla \times \mathbf{B} \times \mathbf{v}) \cdot dS = 0 \tag{6-33}$$
As the surface $S$ gets smaller, we note that (6-33) reduces to the point equation

$$\nabla \times \mathbf{E}_d = -\frac{\partial \mathbf{B}}{\partial t}$$  \hspace{1cm} (6-34)

when (6-31) and (6-32) are used.

For static fields, we know from electrostatics that the \textit{static electric field strength} $\mathbf{E}_s$ has zero curl so that

$$\nabla \times \mathbf{E}_s = 0$$  \hspace{1cm} (6-35)

Also, it is known (Refs. 1, 2) that if the curl of a vector vanishes, it can be written as the gradient of an electric scalar \textit{potential}, say $V(x)$. Hence $V(x)$ can be defined implicitly by

$$\mathbf{E}_s = -\nabla V$$  \hspace{1cm} (6-36)

where $V$ is a scalar function. Thus

$$V(x) = -\int \mathbf{E}_s \cdot dr$$  \hspace{1cm} (6-37)

and since (6-35) holds for all $t$, then (6-34) can be written as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$  \hspace{1cm} (6-38)

where

$$\mathbf{E} \triangleq \mathbf{E}_d + \mathbf{E}_s$$  \hspace{1cm} (6-39)

Equations (6-28) and (6-38) are analogous to Maxwell's curl equations for deterministic electric fields.

Since $\nabla \cdot \mathbf{B} = 0$, we can define the \textit{magnetic vector potential} $\mathbf{A}$ for stochastic fields implicitly at any point by

$$\nabla \times \mathbf{A} \triangleq \mathbf{B}$$  \hspace{1cm} (6-40)

Thus the magnetic flux $\psi_m$ over any cap is given by

$$\psi_m = \int_{\text{cap}} \mathbf{B} \cdot dS = \int_{\text{cap}} \nabla \times \mathbf{A} \cdot dS$$  \hspace{1cm} (6-41)
Maxwell’s Curl Equations and the Potential Equations for Stochastic Fields

and from Stoke’s theorem

\[ \psi_m = \oint \mathbf{A} \cdot d\mathbf{r} \]  

(6-42)

Now \( \mathbf{A} \) cannot be expressed as the gradient of a scalar magnetic potential, \( \nabla \psi_m \), since \( \nabla \times \mathbf{A} \neq 0 \). Similarly, \( \mathbf{E}_d \neq \nabla e \) because \( \nabla \times \mathbf{E}_d \neq 0 \); then \( \nabla e \) cannot exist. Moreover,

\[ \oint \mathbf{E} \cdot d\mathbf{r} = \oint_{\text{cap}} \nabla \times \mathbf{E} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \oint_{\text{cap}} \mathbf{B} \cdot d\mathbf{S} \]  

(6-43)

and from (6-40) and (6-43) we have, using Stoke’s theorem,

\[ \oint (\mathbf{E} + \mathbf{A}) \cdot d\mathbf{r} = 0 \]  

(6-44)

so that

\[ \mathbf{E} = -\mathbf{A} + \mathbf{E}_0 \]  

(6-45)

where \( \mathbf{E}_0 \) is a vector such that \( \oint \mathbf{E}_0 \cdot d\mathbf{r} = 0 \). When \( \mathbf{A} = 0 \) (i.e., static fields), then \( \mathbf{E} = \mathbf{E}_s \); but \( \oint \mathbf{E}_s \cdot d\mathbf{r} = 0 \). Hence

\[ \mathbf{E}_0 = \mathbf{E}_s = -\nabla V \quad \mathbf{E} = -\nabla V - \mathbf{A} \]  

(6-46)

Equations (6-40) and (6-46) are the potential equations for stochastic fields.

We can complete the analogy between Maxwell’s wave equations and those given here by relating \( \mathcal{J} \) to \( \mathbf{E} \) through

\[ \mathcal{J} \triangleq g \mathbf{E} \]  

(6-47)

where \( g \) is defined as that factor which \( \mathbf{E} \) must be multiplied by to give the probability current density. Usually \( g \) is referred to as the conductivity. Also, we relate the probability flux density to the electric field strength \( \mathbf{E} \) through

\[ \mathcal{D} = \varepsilon \mathbf{E} \]  

(6-48)

where \( \varepsilon \) is that permittivity function that \( \mathbf{E} \) at any point must be multiplied by \( \varepsilon \) to obtain \( \mathcal{D} \). The nature of the permittivity depends only on the medium in which the field exists. If the medium is linear, homogeneous, and isotropic
throughout the entire probability space, then $\epsilon$ is a scalar constant. If at any point in the probability space $\epsilon$ is not a constant but depends on $\mathcal{E}$ at the point, the medium is said to be nonlinear. The mathematical properties of $g$ and $\epsilon$ are the same. Finally, we can relate $\mathcal{B}$ to $\mathcal{H}$ through $\mu$. That is,

$$\mathcal{B} = \mu \mathcal{H}$$ \hspace{1cm} (6-49)

where $\mu$ is the permeability function of the medium.

In summary, then, the equations of stochastic field theory in point form are

$$\nabla \cdot \mathcal{D} = p(x, t) \quad \nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t}$$

$$\nabla \cdot \mathcal{B} = 0 \quad \nabla \times \mathcal{H} = \mathcal{J} + \frac{\partial \mathcal{B}}{\partial t}$$ \hspace{1cm} (6-50)

where

$$\mathcal{D} = \epsilon \mathcal{E}, \quad \mathcal{J} = g \mathcal{E}, \quad \mathcal{B} = \mu \mathcal{H}$$ \hspace{1cm} (6-51)

These equations may obviously be written in the form of integral equations.

6-8 Stochastic Fields Produced by Brownian Motion

In 1827 an English botanist, Robert Brown, noticed that small particles suspended in fluids perform peculiarly erratic movements. This phenomenon, which can also be observed in gases, is referred to as Brownian motion. Although it soon became clear that Brownian motion is an outward manifestation of the molecular motion postulated by the kinetic theory of matter, it was not until 1905 that Albert Einstein advanced an acceptable theory.

Actually, in connection with the stock market, L. Bachelier (Ref. 3), a French mathematician, discovered the mathematical theory of Brownian motion five years before Einstein’s classic 1905 paper. Bachelier’s study also marked the beginning of the theory of stochastic processes although it went unrecognized for decades.

The theory was considerably generalized and extended by the Polish physicist Marjan Smoluchowski, with further important contributions being made by Fokker, Planck, Burger, Fürth, Ornstein, Uhlenbeck, Chandrasekhar, Kramers, and others (Ref. 4). On the purely mathematical side (see Section 6-12 for references), various aspects of the theory were analyzed by Wiener, Kolmogorov, Feller, Levy, Doob, and Fortet. Einstein considered the case of
the free particle—that is, a particle on which no forces other than those due to the molecules of the surrounding medium are acting.

6-8.1 A Physical Example

Before discussing any basic results we present a physical example that gives considerable insight into the concept of Brownian motion as applied to the process of diffusion. If a puff of smoke is released in the center of a closed volume \( V \) (container such as a room), it is easy to observe that the size of the puff will gradually grow, and the average density of the smoke particles within the puff will decrease with the passage of time until the smoke particles are uniformly distributed throughout the volume \( V \). The mechanism by which the smoke particles are propagated (flow) through the volume is called diffusion. The fundamental idea behind diffusion process is that, on account of their random thermal motion, particles will tend to flow from the regions in \( V \) where they are heavily concentrated to neighboring regions in \( V \) of lower concentration. For example, smoke particles located at the boundary of the smoke puff have an equal probability of leaving the boundary or returning to the boundary on its next free path before collision with another smoke particle. This, of course, assumes that no external forces act upon the puff; for example, the puff is released in a vacuum. Therefore, qualitatively speaking, half the smoke particles leave the puff on their next step. Since there are no particles beyond the boundary (surface within \( V \) that enclosed all particles at time \( t \)) to reenter the puff, the boundary outlining the puff size at time \( t \) increases. The increase in puff size is accompanied by a decrease in the particle density at the boundary, so smoke particles from the next layers within the puff experience a similar situation to those along the surface of the puff at time \( t \). Consequently, they diffuse into the boundary layer. For any particle in the puff, the probability that it will move toward the center of the puff is equal to the probability that it will move toward the boundary layer. However, the density of the particles toward the boundary is always less than the density toward the center. Hence there is an average flow of particles from the center toward the boundary layer.

We will show later that the rate of flow of particles due to diffusion is proportional to the difference in concentration of the particles at adjacent points. Since \( p \triangleq p(x, t) \) gives a measure of the density of particles in a small volume about the point \( x \) at time \( t \), the probability current density, (i.e., the rate at which particles flow across the boundary) is proportional to \(-\nabla p\). The minus sign accounts for the fact that there is a net flow outward when the concentration is decreasing with increasing \( x \). Thus the presence of the concentration gradient, \( \nabla p \), leads to the appearance of a diffusion flux or the stochastic current component \( J_d \). Subsequently we make the preceding concepts quantitative by treating the one dimensional case.
6-8.2 Brownian Motion in the Absence of an External Field Force

Now let us consider the motion of a free particle (in the smoke puff for instance) on a straight line that we refer to as the $x$ axis. The trajectory, say $x(t)$, may be treated as a sample function of the stochastic process $\{x(t)\}$. One desires to find the p.d.f. of the random increments $x(t) - x(0)$, say $p(x, t|x_0, t_0)$—that is, the probability that the particle will lie between $x_1$ and $x_1 + dx_1$ if it was at $x_0 = x(0)$ at time $t = t_0$. In the absence of any external force (analogous to signal absent)—that is, $\mathcal{F}_x = 0$—Einstein was able to show that the probability current flowing in the positive $x$ direction is given by

$$\mathcal{J}_d(x, t) = -D \frac{\partial p}{\partial x}$$  \hspace{1cm} (6-52)

where $D$ is a certain physical constant, the so-called diffusion constant. Thus from (6-22) we have that the p.d.f. $p(x, t|x_0, t_0)$ must satisfy the partial differential equation

$$D \frac{\partial^2 p}{\partial x^2} = \frac{\partial p}{\partial t}$$  \hspace{1cm} (6-53)

where $p \triangleq p(x; t) = p(x, t|x_0, t_0)$. This is the so-called diffusion equation in one dimension. As we shall see, this equation also specifies the performance of a first-order SCS operating in the presence of white Gaussian noise only—that is, in the absence of signal input. If one imposes the boundary conditions

(a) $\lim_{x \to \pm \infty} \mathcal{J}_d(x, t) = 0$

(b) $\int_{-\infty}^{\infty} p(x; t) dx = 1$  \hspace{1cm} (6-54)

and the initial condition

(c) $\lim_{t \to t_0} p(x; t) = \delta(x - x_0)$  \hspace{1cm} (6-55)

it is easy to show that (6-54) and (6-55) imply

$$p(x, t|x_0, t_0) = \frac{1}{\sqrt{4\pi D(t - t_0)}} \exp \left[ -\frac{(x - x_0)^2}{4D(t - t_0)} \right]$$  \hspace{1cm} (6-56)

and that this solution is unique. The variance $\sigma_x^2(t)$ is easily seen to be

$$\sigma_x^2(t) = 2D(t - t_0)$$  \hspace{1cm} (6-57)
and in the limit as \( t \) approaches infinity the variance is unbounded. This says that in the steady state the probability of finding the particle between \( x \) and \( x + dx \) is zero. Such a phenomenon is frequently characteristic of a diffusion-type process. Recall from Chapter 1, Section 1-9, that (6-56) is the transition p.d.f. of the Wiener process. Consequently, the process \( \{x(t)\} \) is sometimes called the Einstein-Wiener process. We will have more to say about such a process in the next chapter. Frequently, it is referred to as the Wiener-Levy process.

The probability current is easily computed using (6-52) and (6-56). Thus

\[
J_s(x, t) = \frac{x - x_0}{2(t - t_0) \sqrt{4\pi D(t - t_0)}} \exp \left[ -\frac{(x - x_0)^2}{4D(t - t_0)} \right]
\]  

(6-58)

This represents the average number of particles moving in the positive direction of the \( x \) axis per unit time.

6-8.3 Brownian Motion in the Presence of an External Field Force

If, on the other hand, an external force \( F(x) = -bx \) (corresponding to the case of signal present) acts in the direction of the \( x \) axis, the probability currents are represented by

\[
J_s(x, t) = -bx \quad J_d(x, t) = -D \nabla p
\]  

(6-59)

The equation of probability flow becomes

\[
\frac{\partial p}{\partial t} = \frac{\partial}{\partial x} \left[ (bx + D \frac{\partial}{\partial x}) p \right]
\]  

(6-60)

The solution, subject to the previously discussed boundary and initial conditions, becomes

\[
p(x, t|x_0, t_0) = \frac{1}{\sqrt{2\pi \sigma_x^2}} \exp \left[ -\frac{(x - \bar{x})^2}{2\sigma_x^2} \right]
\]  

(6-61)

where the conditional mean and variance are given by

\[
\bar{x} = x_0 \exp \left[ -b(t - t_0) \right]
\]

\[
\sigma_x^2(t) = \frac{D}{d} \left[ 1 - \exp \left[ -2b(t - t_0) \right] \right]
\]  

(6-62)
The ratio $D/b$ is analogous to a noise-to-signal ratio and $b$ is proportional to signal strength. Notice that $\bar{x} = 0$ and $\sigma_x^2 = D/b$ in the steady state. These steady state conditions are a consequence of the linear externally applied force which is analogous to the spring force familiar from engineering mechanics. It is also the representation for an elastically bound particle arising in statistical mechanics.

Now let us consider the process $\{x(t)\}$ generated by the stochastic differential equation, sometimes called the *Langevin* equation,

$$\dot{x} + bx = n(t)$$  \hspace{1cm} (6-63)

where $\{n(t)\}$ is a white Gaussian noise process with $E(nn_\tau) = K_\alpha \delta(\tau)$. To study what happens when the process $\{x(t)\}$ is switched on, we imposed the initial condition $x(t_0) = x_0$. The conditional mean value and correlation function of $\{x(t)\}$ are respectively given by

$$\bar{x} = x_0 \exp \left[-b(t - t_0)\right]$$  \hspace{1cm} (6-64)

$$R_x(t_1,t_2) = \frac{K_\alpha}{2b} \left[\exp \left[-b|t_2 - t_1|\right] - \exp \left[-b(t_1 + t_2 - 2t_0)\right]\right]$$  \hspace{1cm} (6-65)

while the variance is given by

$$\sigma^2_x(t) = \frac{K_\alpha}{2b} \left[1 - \exp \left[-2b(t - t_0)\right]\right]$$  \hspace{1cm} (6-66)

Furthermore, since (6-63) represents a linear operation on the p.d.f. of $\{n(t)\}$, the p.d.f. of $\{x(t)\}$ is Gaussian. As $t - t_0$ increases, we see from (6-65) that the stationary correlation function becomes established.

It is important to note how the restoring force $F(x) = -bx$ in (6-63) enters into the determination of $J(x,t)$ in (6-59) and that the intensity coefficient $K_\alpha$ is related to $D$ in (6-59). In fact, comparing (6-62) and (6-66), we see that $D = K_\alpha/2$. From this simple example it is easy to see that the components of the equation of probability flow can be determined directly from the stochastic differential equation generating the process. In the next chapter we consider the problem of determining the equation of probability flow for dynamical systems described by a set of stochastic differential equations—that is, an $n$-dimensional generalization of the above.

The derivation of the equation of probability flow via the changing flux theorem is phenomenological. As such, it is difficult to determine the velocity $v$ needed in the characterization of the probability current density $J$. The derivation of the equation of probability flow is common to various *random walk* and diffusion-type stochastic processes. In the next chapter we shall consider the derivation of the Fokker-Planck equation from another point of view.
As we shall see, the characterization of \( v \) is based on the concepts of Markov processes. In probability theory a stochastic process having the property that its future behavior depends only on the present, but not on the past, state of the system is termed Markovian. As we shall see, two coefficients will arise: the so-called intensity coefficient of dynamical friction, which when multiplied by \( p(x, t) \) serves to characterize \( \mathcal{J}_s \), and the intensity coefficient of the diffusion-in-velocity tensor, which when combined with \( p(x, t) \) serves to characterize \( \mathcal{J}_d \).

### 6-9 Poisson's and Laplace's Equations for Static Stochastic Fields

Since \( \mathcal{E}_d = 0 \) we can write from (6-35) and (6-48)

\[
\nabla \cdot \mathcal{E}_s = \frac{1}{\epsilon} p_{ss}(x) = f(x)
\]  

(6-67)

if \( \epsilon \) does not depend on the space variables and \( p_{ss}(x) \) is defined to be the static (steady-state) p.d.f. That is,

\[
p_{ss}(x) \triangleq \lim_{t \to \infty} p(x, t)
\]  

(6-68)

From (6-36) and (6-67) we have

\[
\nabla^2 V(x) = -\frac{p_{ss}(x)}{\epsilon} \triangleq f(x)
\]  

(6-69)

where \( \nabla^2 \) is the Laplacian operator defined in Appendix I and \( V(x) \) is the potential associated with \( f(x) \). Equation (6-69) is Poisson's equation for stochastic fields. If, in the steady state, \( p_{ss}(x) \) approaches zero [the case of Brownian motion discussed in Section 6-8.2, (6-56)] throughout the probability space, then

\[
\nabla^2 V(x) = 0
\]  

(6-70)

which is Laplace's equation for stochastic fields.

Regardless of one's familiarity or lack of familiarity with partial differential equations, knowing about Poisson's and Laplace's equations for stochastic fields is helpful because it opens to one's resources the vast literature of potential theory and the branch of applied mathematics called harmonic functions. There is no particular point, in this age, for most of us to seek new and independent solutions if we know where to find those that are available in this extensive product of some of the finest mathematical minds.
Poisson’s equation for stochastic fields is Gauss’s law in the form of a second-order partial differential equation in which the scalar potential is the dependent variable. In Poisson’s equation for stochastic fields, the scalar concepts of probability and potential theory are tied together. Further discussion of the subject of probabilistic potential theory is beyond the scope of this work; however, the interested reader who wishes to dig deeper will find it enlightening to begin with the book edited by Chover (Ref. 5).

6-10 Transport Processes and Carriers

There are various transport processes, besides the transport of probability, which have similar or analogous features. This analogy is best manifested by setting up the equations of motion. In practice, there are various transport systems or processes: probability flow, momentum flow, heat flow, electric current flow, magnetic flux flow, information flow, elastic displacement, and diffusive flow. In this section we point out certain physical aspects of these analogous transport processes.

In any transport process a necessary condition is that there must be suitable carriers free to move and transfer some property. Furthermore, in any transport process there must be some driving action such that something can be carried from one position to another. In electric current flow in a

<table>
<thead>
<tr>
<th>Carrier</th>
<th>Involved in</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluid stream</td>
<td>Fluid flow, heat transfer, momentum exchange, diffusion</td>
</tr>
<tr>
<td>Molar aggregates or eddies</td>
<td>Fluid flow, heat transfer, momentum exchange, diffusion</td>
</tr>
<tr>
<td>Molecules</td>
<td>Momentum exchange (laminar flow), heat transfer, diffusion</td>
</tr>
<tr>
<td>Electrons</td>
<td>Electric current, heat transfer</td>
</tr>
<tr>
<td>Ions</td>
<td>Electric current, momentum exchange, heat transfer, diffusion</td>
</tr>
<tr>
<td>Steady electromagnetic field</td>
<td>Electric current, magnetic flux</td>
</tr>
<tr>
<td>Alternating electromagnetic field</td>
<td>Electric current, magnetic flux change, momentum exchange, heat transfer</td>
</tr>
<tr>
<td>Steady elastic field</td>
<td>Elastic deformation</td>
</tr>
<tr>
<td>Alternating elastic field</td>
<td>Elastic deformation, momentum exchange, heat transfer</td>
</tr>
</tbody>
</table>
metal, the free electrons serve as the carriers. In ionic conduction, positive and negative ions of different types serve as carriers of electric charge. In turbulent flow there is momentum exchange due to two kinds of carriers—molecules and fluid molar aggregates. In heat conduction, the carriers may consist of all chemical types and parts of molecules. In heat convection, the carriers are molar assemblies, such as eddies in a fluid stream. Electromagnetic quanta are involved as the carriers in thermal radiation. An electromagnetic wave serves as the carrier of information in a communication system. In the diffusion of water vapor in a gaseous system, the water may be carried by the individual water molecules moving with random thermal motion, by a fluid stream or by the motion of molar aggregates or eddies holding water. Table 6-1 lists some carriers in different transport processes. Notice that in some transport processes one or more carriers may be involved.

6-10.1 Analogous Properties of Various Transport Processes

As noted in Section 6-2, the concept of flux passing through an area does not have to be limited to electric phenomena. One can define flux as the charge per unit time. For a fluid, with volume as the charge, flux would be the volume rate of flow. For electric current, with charge as the charge, flux would be coulombs per unit time. In probability flow, probability is the charge, and flux would be probability per unit time. Table 6-2 presents the interpretation of charge and flux density in various transport processes.

<table>
<thead>
<tr>
<th>Process</th>
<th>Potential</th>
<th>Gradient of Potential</th>
<th>Charge</th>
<th>Flux Density</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrical</td>
<td>Electromotive force (emf.)</td>
<td>emf gradient</td>
<td>Electric charge</td>
<td>Electric charge flow per unit area</td>
</tr>
<tr>
<td>Magnetic</td>
<td>Magnetomotive force (m.m.f.)</td>
<td>m.m.f. gradient</td>
<td>Magnetic flux</td>
<td>Magnetic flux rate of change per unit area</td>
</tr>
<tr>
<td>Diffusion</td>
<td>Concentration (p.d.f.)</td>
<td>Concentration gradient</td>
<td>Masses of the diffusing component (probability)</td>
<td>Masses passing through unit area per unit time (probability current density)</td>
</tr>
<tr>
<td>Momentum</td>
<td>Velocity</td>
<td>Velocity gradient</td>
<td>Momentum</td>
<td>Stress</td>
</tr>
<tr>
<td>Thermal</td>
<td>Temperature</td>
<td>Temperature gradient</td>
<td>Entropy quantity</td>
<td>Entropy rate per unit area</td>
</tr>
<tr>
<td>Fluid</td>
<td>Pressure</td>
<td>Pressure gradient</td>
<td>Fluid volume</td>
<td>Volume rate per unit area</td>
</tr>
</tbody>
</table>
6-11 Further Studies

APPENDIX I

VECTOR FIELD CONCEPTS

I-1 The Gradient Field

Without loss in generality we restrict ourselves to the case where \( n = 3 \)—that is, \( e_k = 0 \) if \( k \geq 4 \). If \( \mathbf{r} = e_1x + e_2y + e_3z \), and \( u \) is any scalar function of \( x, y, \) and \( z \), then the derivative of \( u \) with respect to the vector \( \mathbf{r} \) is defined as a vector having the direction of the greatest space rate of increase of \( u \) and a magnitude equal to this rate of increase. The derivative of \( u \) with respect to the vector \( \mathbf{r} \) is called the gradient of \( u \) and is written \( \text{grad } u \).

If \( u \) changes along the \( x \) axis only, \( \text{grad } u = e_1(\partial u/\partial x) \).

If \( u \) changes along the \( y \) axis only, \( \text{grad } u = e_2(\partial u/\partial y) \).

If \( u \) changes along the \( z \) axis only, \( \text{grad } u = e_3(\partial u/\partial z) \).

If \( u \) changes without restriction, the three components of \( \text{grad } u \) act independently. Hence

\[

\nabla u = \text{grad } u = e_1 \frac{\partial u}{\partial x} + e_2 \frac{\partial u}{\partial y} + e_3 \frac{\partial u}{\partial z} \tag{I-1}

\]

Note that \( \text{grad } u \) is a vector normal to the surface \( u = \text{constant} \), and that \( u \) will change at maximum rate along the normal. In \( n \) space, the gradient of \( u \) is given by

\[

\nabla u = \sum_{j=1}^{n} e_j \frac{\partial u}{\partial x_j} \tag{I-2}

\]

277
Other comments that can be made about the gradient vector is that it is normal to the tangent plane at \((x, y, z)\) (see Fig. 16-1). The fact that the magnitude of the gradient vector is maximum along the normal to the surface \(u = \text{constant}\) is easily observed by noting that the directional derivative of the scalar \(u\) in the direction of the unit vector \(v = \cos \alpha e_1 + \cos \beta e_2 + \cos \gamma e_3\) is given by

\[
\nabla u \cdot v = \cos \alpha \frac{\partial u}{\partial x} + \cos \beta \frac{\partial u}{\partial y} + \cos \gamma \frac{\partial u}{\partial z}
\]

Fig. 16-1. Gradient Vector \(u(x, y, z)\) and Surface \(u = \text{Constant}\).

where \(\cos \alpha, \cos \beta, \cos \gamma\) are the direction cosines. This indicates that the magnitude of \(\nabla u\) is maximum in the direction \(\nabla u\). In particular, \(\nabla u\) points in the direction of the maximum increase of \(u\).

I-2 Differential Operators

Differentiations with respect to time and space are so frequently performed that special abbreviated forms, called **differential operators**, are used to indicate them. The differential operator for time (the dot operator) is a dot over the quantity differentiated; that is,

\[
\frac{du}{dt} = \dot{u} \quad \frac{dA}{dt} = \dot{A}
\]

\[
\frac{d^2 A}{dt^2} = \ddot{A} = \dddot{A}
\]

(I-3)
The differential operator for space, the \textit{del operator}, is

\[ \nabla = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j} \]  

(I-4)

The \textit{Laplacian operator} in rectangular coordinates is defined by

\[ \nabla \cdot \nabla = \nabla^2 = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \]  

(I-5)

I-3 The Line Integral

If \( A \) varies along any path, divide the path into cords \( \delta r_1, \delta r_2, \ldots, \delta r_k, \ldots, \delta r_n \), and let \( A_1, A_2, \ldots, A_k, \ldots A_n \) be the values of \( A \) near the center of each cord (Fig. A6-2). Then the \textit{line integral} of \( A \) along the path is defined as

\[ \lim_{A \to 0} \sum_{k=1}^{n} |A_k||\delta r_k| \cos \alpha_k \triangleq \int_r A \cdot dr \]  

(I-6)

![Fig. I6-2. The Line Integral Path.](image)

where \( \alpha_k \) is the angle between \( A_k \) and \( \delta r_k \). Work or energy is the line integral of the force vector \( F \).

I-4 The Surface Integral

If \( A \) varies over any surface, divide the surface into elements \( S_1, S_2, \ldots, S_k \) and let \( A_1, A_2, \ldots, A_k \) be the values of \( A \) near the center of each element (Fig. I6-3). Then the \textit{surface integral} of \( A \) over the surface \( S \) is defined
as

\[
\lim_{\delta S_k \to 0} \sum_{k=1}^{n} |A_k||\delta S_k| \cos \alpha_i = \int_S A \cdot dS
\]  

(I-7)

The direction of the elements \(\delta S_k\) are taken along the outward normal to \(S\) at the point in question and \(\alpha_i\) is the angle between \(A_k\) and \(\delta S_k\).

I-5 The Volume Integral

*Volume integrals* can be formed for either vectors or scalars and are defined in a manner analogous to surface integrals as follows:

\[
\int_V A \, dV = \lim_{\delta V_k \to 0} \sum_{k=1}^{n} A_k \, \delta V_k
\]  

(I-8)

If

\[
A = \frac{dB}{dV} \quad \text{then} \quad B = \int_V A \, dV
\]  

(I-9)

The converse is not necessarily true. \(A\) is called the *volume density* of \(B\). Similarly, if \(w = du/dV\), then \(u = \int_V w \, dV\) and again the converse is not necessarily true. In \(n\) space, \(dV = dx_1, \ldots, dx_n = dx\).
APPENDIX II

PROOF OF THE CHANGING FLUX THEOREM

Figure II6-4 shows an infinitesimal part of the path through which the vector $A$ is moving. The vector moves with the cap $S_1$ at velocity $v$ and occupies the position $S_2$ after time $dt$, which is treated as a temporary constant during the development of the proof. The change in flux in time $dt$ through the moving surface is

$$d\psi = \psi_2 - \psi_1$$

$$= \int_{S_2} A(r_2, t + dt) \cdot dS_2 - \int_{S_1} A(r_1, t) \cdot dS_1$$  \hspace{1cm} (II-10)

Fig. II6-4. Infinitesimal Path of the Vector $A$ Moving with Velocity $v$.  

281
Now, using the Taylor series, we write

\[ A(r_2, t + dt) = A(r_2, t) + \frac{\partial}{\partial t} A(r_2, t) dt \]  

so that

\[ dy = \int_{S_2} A(r_2, t) \cdot dS_2 - \int_{S_1} A(r_1, t) \cdot dS_1 + dt \int_{S_2} \frac{\partial}{\partial t} A(r_2, t) \cdot dS_2 \]  

(II-12)

Adding and subtracting \( \int_{S_3} A \cdot dS_3 \) to (A-12), we obtain

\[ dy = dt \int_{S_2} \frac{\partial}{\partial t} A(r_2, t) \cdot dS_2 + \oint A \cdot dS_4 - \int_{S_2} A \cdot dS_3 \]  

(II-13)

where

\[ \int_{S_2} A(r_2, t) \cdot dS_2 - \int_{S_1} A(r_1, t) \cdot dS_1 + \int_{S_3} A \cdot dS_3 = \oint_{S_4} A \cdot dS_4 \]  

(II-14)

and \( dS_4 \) is any element of the whole exterior surface of the generated volume. In evaluating \( dS_4 \), remember to take all the elements of area along the outward normal. This accounts for the change in sign of \( dS_1 \). From the divergence theorem,

\[ \oint S A \cdot dS = \oint_V \nabla \cdot A \, dV \]  

(II-15)

and \( dV = (v \, dt) \cdot dS_1 \) so that

\[ \oint_{S_1} A \cdot dS_4 = dt \int_{S_1} (\nabla \cdot A)(v \cdot dS_1) \]  

(II-16)

Note the change in limit accompanying the change in variable. From Fig. II6-4 we observe

\[ dS_3 = -(v \, dt) \times dr_1 \]  

(II-17)

so that
From Stoke's theorem, we write

\[ dt \oint_{\text{cap}} \mathbf{A} \times \mathbf{v} \cdot d\mathbf{r}_1 = dt \int_{S_1} \nabla \times (\mathbf{A} \times \mathbf{v}) \cdot dS_1 \quad \text{(II-19)} \]

Combining terms, we write

\[
d\psi = dt \left\{ \int_{S_1} \frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}_2, t) \cdot dS_2 + \int_{S_1} \nabla \times (\mathbf{A} \times \mathbf{v}) \cdot dS_1 + \int_{S_1} (\nabla \cdot \mathbf{A})(\mathbf{v} \cdot dS_1) \right\} \quad \text{(II-20)}
\]

Dividing both sides by \( dt \) and allowing \( dt \) to approach zero yields

\[
\frac{d\psi}{dt} = \int_{S} \left[ \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \nabla \cdot \mathbf{A} + \nabla \times (\mathbf{A} \times \mathbf{v}) \right] \cdot d\mathbf{s} \quad \text{(II-21)}
\]

since the subscripts on \( dS \) may be dropped as we consider any cap. This is the changing flux theorem.

### Problems

6-1 (a) By direct substitution verify that (6-56) satisfies the diffusion equation (6-53).

(b) Sketch (6-56) as a function of \( x \) and \( t \).

(c) Develop an expression for the probability flux density \( \mathcal{J}(x, t) \).

6-2 Evaluate the probability current \( \mathcal{J}_d(x, t) \) due to the Brownian motion of a particle in the absence of an external field force. Sketch your result.

6-3 Let the position of a particle in Brownian motion be specified by the coordinates \( x = (x_1, x_2, x_3) \). Einstein showed that the position vector \( \mathbf{x} \) undergoes a Markov process with transition p.d.f.

\[
p(x, t|x_0, t_0) = \frac{1}{[4\pi D(t-t_0)]^{3/2}} \exp \left[ -\frac{(x - x_0)^2}{4D(t-t_0)} \right]
\]

where \( x^2 = x_1^2 + x_2^2 + x_3^2 \).
(a) Show that this p.d.f. satisfies the three-dimensional diffusion equation

\[ \frac{\partial p}{\partial t} = D \left( \frac{\partial^2 p}{\partial x_1^2} + \frac{\partial^2 p}{\partial x_2^2} + \frac{\partial^2 p}{\partial x_3^2} \right) \]

where \( p = p(x, t|x_0, t_0) \).

6-4 Assume that the probability current of a particle moving along the \( x \) axis is defined by

\[ \mathcal{J} = v \mathcal{P} = \left( \frac{1}{x} - x \right)p + \frac{\partial p}{\partial x} \]

where the first term is due to the presence of an external force (gives rise to the drift component) and the second term is due to random noise (gives rise to diffusion).

(a) What is the equation of probability flow?

(b) Show that

\[ p(x, t|x_0, t_0) = \frac{x}{1 - \mu^2} \exp \left[ -\frac{x^2 + \mu^2 x_0^2}{2(1 - \mu^2)} \right] I_0 \left( \frac{\mu x x_0}{1 - \mu^2} \right) \]

satisfies the equation obtained in part (a) when \( x \geq x_0 \geq 0 \). Here \( \mu = \exp \left[ -(t - t_0) \right] \) and \( I_0(x) \) is the imaginary Bessel function of order zero argument \( x \). Note that at \( t = t_0, x = x_0 \).

(c) Evaluate the probability current \( \mathcal{J}(x, t) \) using the expression for \( p(x, t|x_0, t_0) \).

(d) Develop an expression for the probability flux density \( \mathcal{D} \).

References

6-1 Kaplan, W., *Advanced Calculus*. Addison-Wesley, Reading, Mass., 1952, chs. 1, 2, 3, and 5.


7-1 Introduction

Since the late 1930s the venerable branch of physics known as statistical mechanics and plasma physics has gradually developed even stronger ties with the communication and information sciences in electrical engineering. This trend is an understandable one, for information is communicated by an electromagnetic wave that has usually passed through a plasma—for example, the ionosphere or troposphere or the shock wave-induced plasma that surrounds the body of a space vehicle during high-speed entry of the Earth’s atmosphere. Because a plasma exhibits nonlinear properties, the mathematics used to effect an understanding of the nonlinear medium has many applications in other areas of science—for instance, nonlinear filtering or the study of resonance and relaxation in solid-state materials. Perhaps the strongest tie between the two disciplines lies in the similar mathematics that can be used to describe the respective systems of interest, namely, the equations of mathematical physics. It is particularly important to recognize that the similarity of the mathematical structures can be exploited not only for analysis and synthesis purposes but also for the purpose of gaining needed physical insight; particularly, if the problem is of a nonlinear nature.
In this chapter we show how stochastic field theory is related to the theories of Markov processes, statistical mechanics, plasma physics, and electromagnetic theory. The common mathematical feature of these subjects is the partial differential equation of flow, which under certain assumptions, becomes the Fokker-Planck equation. The common physical feature among these various subjects is the concept of Brownian motion which produces the phenomenon of diffusion. Interestingly enough, the theory of synchronization, tracking, and coherent demodulation by means of a SCS is embedded in solutions to a set of partial differential equations that may also be written as integral equations. These equations have deep analogies with Maxwell’s equations for stochastic fields, as well as the equations of statistical mechanics and plasma physics. Thus the study of the theory of tracking, synchronization, and coherent demodulation reduces in a novel way to the so-called boundary-valued problem arising in attempting to find (1) solutions to certain partial differential equations or (2) to obtaining solutions to the integral equation form of these point equations.

Before proceeding we note that Sections 7-1 to 7-8 present the theory of the first-order Markov process required in later chapters. The remaining sections of this chapter deal primarily with the vector Markov process theory needed for later studies. On a first reading, the reader may only wish to study the first eight sections and return to the last sections when he finds the need for them.

7-2 Random Walk Principles

We derive in this section fundamental concepts taken from the subject of random walk (Ref. 1). Random walk concepts are related to the gambler’s ruin problem described in elementary probability theory. First we shall be concerned with the discrete approach because of the fundamental understanding it provides to the concept of diffusion and the behavior of a first-order digital SCS. The subject of diffusion is fundamental to the analysis and understanding of the properties of the phase error process (among other things) generated by a SCS. In fact, we shall see later that the phase error process has a deep analogy with diffusion type processes, arising in heat flow, stellar dynamics, the study of the generation of stable laser beams, colloid chemistry, plasma physics, and so on.

In order to become thoroughly familiar with the basic idea, we shall begin by describing in a qualitative way an idealized, one-dimensional example of a discrete diffusion-type process. The example is known as a random walk in mathematical statistics, for reasons that will be presently apparent.

Suppose that a man stands in the middle of a long street. Let his initial position be \( x = 0 \). He flips a fair coin to decide which way along the street to take his first step. After taking this step, he again flips the fair coin to decide
whether his next step shall be in the same direction as the last one, or in the opposite direction, and similarly as the end of each step. For the present and for simplicity, we will suppose that the steps are of a uniform size, that each step is completed in a uniform time interval, and that the time needed for flipping the coin can be neglected.

We are now interested in computing the probability that the man will get a certain number of steps away from \( x = 0 \) after a certain time has elapsed. The computation is readily done with the aid of the simple construction shown in Fig. 7-1. At the beginning of his random walk \( t_0 = 0 \) and the man is at \( x = 0 \). He takes steps size \( l \). On his first step, he is equally likely to go right or left. Hence before the first step is taken, his probability of ending up at \( x = l \) is \( \frac{1}{2} \) and the probability of ending up at \( x = -l \) is \( \frac{1}{2} \).

Now consider the second step. There are two possibilities. If the man ended up at \( x = l \) after the first step, then on the second step he is equally likely to end up at \( x = 2l \) or \( x = 0 \). On the other hand, if the man ended up at \( x = -l \) after the first step, then on the second step he is equally likely to end up at \( x = -2l \) or \( x = 0 \). Thus, after two steps, there are three places where the man could be: \( x = -2l, x = 0, x = 2l \). There are two ways for him to end up at \( x = 0 \) (first step left, second step right and vice versa) and one way each for him to end up at \( x = 2l, x = -2l \). If we wish to place a bet, before the proceedings begin, that the man will reach \( x = 2l \), we should ask for at least four to one odds, because the probability of reaching \( x = 2l \) is \( \frac{1}{4} \) (see Fig. 7-1).

Before proceeding with more steps, we note that the probability values for the man's position after the second step may readily be derived from his first-step situation by applying a "random walk" construction to each position. The construction is indicated in Fig. 7-1a and consists of multiplying the probability of reaching \( x = l \) by one-half and assigning these numbers (one-fourth, one-fourth) to the lines going to \( x = 0, x = 2l \). Similarly for \( x = -l \). The results are then added to give the total probabilities for reaching \( x = -2l, 0, 2l \). Using this idea, the reader can construct the remainder of Fig. 7-1a. Figure 7-1b is a three-dimensional representation of the random walk problem, in which the height of each line is equal to the probability of reaching the particular point at the designated time. Note that for each unit of time the probability is conserved; that is, the discrete p.d.f. has unit area.

To make the random walk more analogous to free particles undergoing diffusion, we may begin with \( N \) men, situated at identical locations on \( N \) parallel streets. Each man performs his random walk entirely independent of all others. Then, after six steps, we expect to find \( \frac{N}{64} \) of the men at \( x = 0 \), \( \frac{15}{64} \) of them at \( x = \pm 2l \), \( \frac{1}{64} \) of them at \( x = \pm 4l \), and \( \frac{1}{64} \) of them at \( x = \pm 6l \). If \( N \) is small (say, 128), the actual distribution of men may be quite different from this. On the other hand, if \( N \) is \( 10^{13} \), the relative proportions of men in the different locations will be insignificantly different from that calculated above.
The foregoing concepts can be discussed from a more quantitative point of view. A free particle suffers displacements along a straight line in the form of a series of steps of length \( l \), each step being to the right or to the left. At this point we assume that each step takes unit time. After taking \( N \) such steps, the particle could be at any one of the points \( -N, -N + 1, \ldots, -1, 0, 1, \ldots, N - 1, N \). We ask what is the probability \( P(n|N) \) that the particle arrives at the point \( n \) after suffering \( N \) displacements.
First we assume that steps to the right occur with probability $p$, while steps to the left occur with probability $q = 1 - p$. We also assume that each individual step taken either to the right or to the left is independent of the direction of all the preceding ones (the Markov property); that is, the future position depends only on the present position. Hence the probability of any given sequence of $N$ steps in which $R$ steps are to the right and $L$ steps are to the left is $p^R(1 - p)^L$. But in order to arrive at $n$ after $N$ steps, some $R = (N + n)/2$ steps should have been taken to the right and the remaining $L = (N - n)/2$ steps would be taken to the left. (Notice that $n$ can be even or odd, depending on whether $N$ is even or odd.) The number of such distinct sequences leading to $n$ after $N$ steps is clearly

$$C^N_L = C^N_R = \frac{N!}{R!L!}$$  (7-1)

Hence

$$P(n|N) = \frac{N!}{R!L!} p^{(N + n)/2}(1 - p)^{(N - n)/2}$$  (7-2)

where $R = (N + n)/2$ and $L = (N - n)/2$. In other words, we have a binomial or Bernoulli distribution. As is well known, $E(n|N) = Np$ and $E(n^2|N) = pN(q + pN)$ so that $\sigma_n^2 = pqN$, where we have treated $n$ as a discrete r.v.

Instead of $P(n|N)$, let us now generalize somewhat and evaluate $P(nl, N\tau|ml)$, which is the probability that the particle arrives at $nl$ at time $t = N\tau$, given that at $t = t_0$, it was at $ml$. Here $\tau$ is the time per step. This is also the probability that after $N$ games of “heads or tails” the gain of the player is $v = n - m$. In this case

$$P(nl, N\tau|ml) = \frac{N!}{[(N + |v|)/2]![N - |v|)/2]!} p^{(N + |v|)/2}(1 - p)^{(N - |v|)/2}$$  (7-3)

if $|v| \leq N$ and $|v| + N$ is even; otherwise $P(nl, N\tau|ml) = 0$. Suppose now that $p = q = \frac{1}{2}$ and that $l$ and $\tau$ approach zero in such a way that

$$\lim_{(l, \tau) \to 0, N \to \infty} D \triangleq \frac{l^2}{2\tau}$$  \hspace{1cm} nl \rightarrow x, \hspace{1cm} ml \rightarrow x_0, \hspace{1cm} N\tau \rightarrow t$$  (7-4)

then it follows from the classical de Moivre-Laplace limit theorem (Ref. 2) that

$$P[x \leq w] = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{w} \exp \left[ -\frac{(x - x_0)^2}{4Dt} \right] dx$$  (7-5)
and

\[ p(x, t|x_0, t_0) = \frac{1}{\sqrt{4\pi Dt}} \exp \left[ -\frac{(x - x_0)^2}{4Dt} \right] \quad (7-6) \]

which is in agreement with (6-56) of Chapter 6 with \( t_0 = 0 \). We also note that the continuous version of (7-2) is given by (7-6) when \( p = q = \frac{1}{2} \). Thus the Einstein-Wiener process discussed in Chapters 1 and 6 is the analog, for continuous random variables, of the discrete random walk process. Notice that in the steady state \( p(x, t|x_0, t_0) \) approaches zero for all \( x \), since \( \sigma^2_x = 2Dt \) approaches infinity. This phenomenon is characteristic of processes undergoing diffusion in the absence of any restoring field force.

It is both important and instructive to point out a striking formal connection between the discrete random walk and its continuous version. Assuming that the particle starts from \( ml \geq 0 \) (\( m \) an integer), we notice that, for \( n \geq 2 \), \( P(nl, N\tau|ml) \) satisfies the difference equation

\[ P[nl, (N + 1)\tau|ml] = qP[(n - 1)l, N\tau|ml] + pP[(n + 1)l, N\tau|ml] \quad (7-7) \]

and that for \( n = 1 \) we have

\[ P[l, (N + 1)\tau|ml] = qP(0, N\tau|ml) + pP(2, N\tau|ml) \quad (7-8) \]

while for \( n = 0 \) we have

\[ P[0, (N + 1)\tau|ml] = pP(l, N\tau|ml) \quad (7-9) \]

with the initial condition \( P(nl, 0|ml) = \delta(n - m) \). Setting \( p = q \), without loss in generality, (7-7) can be written in the form

\[ \frac{P[nl, (N + 1)\tau|ml] - P(nl, N\tau|ml)}{\tau} = \frac{p^2}{2\tau} \left\{ P[(n + 1)l, N\tau|ml] - 2P(nl, N\tau|ml) + P[(n - 1)l, N\tau|ml] \right\} \quad (7-10) \]

In the limit (7-4), this difference equation (7-10) goes over formally into the diffusion equation (6-53). A more general approach to problems of the discrete nature can be based on the theory of Markov chains (Refs. 3, 4). This subject is beyond the scope of the present exposition; however, in the study of first-order digital PLLs (7-7) arises (Ref. 5).
7-3 The Law of Diffusive Flow

We now wish to carry the random walk concept over to the problem of determining the average rate at which particles will diffuse across a given surface. To do this, we have drawn an arbitrary continuous curve in Fig. 7-2 to represent an instantaneous, one-dimensional p.d.f. of diffusing particles. We shall assume for simplicity that all particles have the same free-path length \( l \) and step time \( \tau \), although they have been distributed densely along the \( x \) axis at random; that is, not just at the positions \( x = 0, l, 2l \), etc. The p.d.f. function is intended to be truly continuous.

![Fig. 7-2. Development of the Law of Diffusive Flow.](image)

We now wish to calculate how many particles cross the plane \( x_0 \) from left to right per unit time per unit area. Now, only those particles that are within a distance \( l \) of the plane at \( x_0 \) can cross it on their next free path. If the probability density of particles is denoted by \( p(x) \), then there are approximately \( lp(x_0 - l/2) \) particles per unit area in a strip of length \( l \) on the left of the plane at \( x_0 \) (see Fig. 7-2) and \( lp(x_0 + l/2) \) particles per unit area in the strip on the right. In a step time \( \tau \), the total number \( M \) of particles that cross the plane \( x = x_0 \) per unit area, will be

\[
M_{|x_0} = \frac{1}{2} \left[ \frac{1}{l} p\left(x_0 - \frac{l}{2}\right) - \frac{1}{l} p\left(x_0 + \frac{l}{2}\right) \right] \tag{7-11}
\]

since on each move one half go to the left and one half move to the right. Using the first term of a Taylor series (7-11) can be written as

\[
M_{|x_0} \approx -\frac{1}{2} \left. \frac{\partial p}{\partial x} \right|_{x=x_0} \tag{7-12}
\]
and since \( \frac{\partial p}{\partial x} \) and \( M \) are measured at the same plane, we may drop the subscript \( x_0 \).

\[
M_x \approx -\frac{1}{2} l^2 \frac{\partial p}{\partial x} = -\frac{1}{2} l^2 \nabla p
\]  
(7-13)

Since \( M \) particles per unit area cross the plane in a time \( \tau \), the average flow per unit area or probability current density generated by diffusion is

\[
\mathcal{J}_d \approx -\frac{l^2}{2\tau} \frac{\partial p}{\partial x} = -D \nabla p
\]  
(7-14)

This says that the diffusion probability current density is proportional to the gradient (slope) of the p.d.f. The minus sign accounts for the fact that particles will diffuse from regions of high probability to regions of lower probability in accordance with the direction of the gradient vector.

For the present problem, where all particles have the same step size and step time, the diffusion constant is \( D = l^2/2\tau \), which is equivalent to that found in the random walk problem. The approximation sign in (7-13) can be replaced by an equality by the artifice of assuming that \( l \) and \( \tau \) approach zero in such a way that \( D \) remains a definable constant as in (7-4). Of course, the free particles, in doing a random walk, do not always take step sizes of a fixed length in a fixed time. Instead, they have p.d.f.'s of various step sizes and free times. When these p.d.f.'s exist, we must calculate \( D \) by the formula \( D = \bar{l}^2/2\bar{\tau} \). The fact that this is true will be made evident later in this chapter.

7-4 Random Walk with Absorbing Boundaries

We continue the discussion of the problem of random walk in one dimension but with restrictions placed on the motion of the particle introduced by the presence of an absorbing boundary or wall. The reason why we are interested in this aspect of the random walk problem is that, in evaluating the mean time to first loss of phase synchronization (the first passage time) in a SCS we will gain physical insight into what happens at the synchronization boundary. Studying the simpler problem here will enable us to characterize with ease the boundary conditions in the more complicated case. Our discussion parallels that given by S. Chandrasekhar (Ref. 6).

For the discussion of this problem it is convenient to trace the course of this particle in an \((n, N)\)-plane as in Fig. 7-3. In this diagram the displacement of a particle by a step means that the representative point moves upward by one unit while at the same time it suffers a lateral displacement also by one unit, either in the positive or in the negative direction. In what follows, we denote by \( P(n|N; n_1) \) the probability that the particle will arrive at \( n \leq n_1 \) after
$N$ steps. We consider the case when there is a perfectly absorbing barrier at $n = n_1$. The interposition of the perfect absorber at $n_1$ means that whenever the particle arrives at $n_1$ it becomes incapable of suffering further displacements. There are two questions that we should like to answer under these circumstances. The first is the analog of the problems we have considered so far, namely, the probability that the particle arrives at $n \leq n_1$ after taking $N$ steps. The second question, which is characteristic of the present problem, concerns the average rate at which the particle will deposit itself on the absorbing wall.

Considering first the probability $P(n|N; n_1)$; it is clear that in counting the number of distinct sequences of steps leading to $n$, we should be careful to exclude all sequences that include even a single arrival to $n_1$. In other words, if we first count all possible sequences leading to $n$ in the absence of the absorbing wall, we should then exclude a certain number of "forbidden" sequences. It is evident, on the other hand, that every such forbidden sequence uniquely defines another sequence leading to the image $(2n_1 - n)$ of $n$ on the line $n = n_1$ in the $(n, N)$-plane (see Fig. 7-3) and conversely. By reflecting about the line $n = n_1$ the part of a forbidden trajectory above its last point of contact with the line $n = n_1$ before arriving at $n$, we are led to a trajectory leading to the image point. Conversely, for every trajectory leading to $2n_1 - n$, we obtain by reflection a forbidden trajectory leading to $n$ (since any trajectory leading to $2n_1 - n$ must necessarily cross the line $n = n_1$). Hence

$$P(n|N; n_1) = P(n|N) - P(2n_1 - n|N) \quad (7-15)$$

Similarly, employing the limiting argument (7-4) with $n, l \to x_i$ and again using the de Moivre-Laplace limit theorem to get to the continuum, we arrive at
\[ p(x, t|x_0, t_0) = \frac{1}{\sqrt{4\pi Dt}} \left\{ \exp \left( -\frac{x^2}{4Dt} \right) - \exp \left[ \frac{(2x_1 - x)^2}{4Dt} \right] \right\} \quad (7-16) \]

where \( x_1 \) is the position of the absorbing wall and we assume that \( x_0 = 0 \) at \( t = t_0 = 0 \). According to this equation, we note that at the wall or boundary

\[ p(x, t|x_0, t_0)|_{x=x_1} = 0 \quad (7-17) \]

This is an important result and we will have occasion to use this as a boundary condition in studying the first passage time problem associated with SCSs. The problem will be the calculation of the mean time to first attainment of the boundary, which is analogous to the classical problem of the attainment of a boundary in which it is required to solve a nonstationary partial differential equation with the zero boundary condition given by (7-17).

Turning next to our second question concerning the probable rate at which the free particle deposits itself on the absorbing wall, we first formulate the problem more specifically. What we wish to know is simply the probability \( Q(n_1; N) \) that after taking \( N \) steps the particle will arrive at \( n_1 \) without ever having touched or crossed the line \( n = n_1 \) at any earlier step.

First, it is clear that \( N \) should have to be even or odd, depending on whether \( n_1 \) is even or odd. We shall suppose that this is the case. Suppose now that there is no absorbing wall. Then the arrival of the particle at \( n_1 \) after \( N \) steps implies that its position after \((N - 1)\) steps must have been either \((n_1 - 1)\) or \((n_1 + 1)\) (see Fig. 7-3). But every trajectory that arrives at \((n_1, N)\) from \((n_1 + 1, N - 1)\) is a forbidden one in the presence of the absorbing wall, since such a trajectory must necessarily have crossed the line \( n = n_1 \). It does not follow, however, that all trajectories arriving at \((n_1, N)\) from \((n_1 - 1, N - 1)\) are permitted ones because a certain number of these trajectories will have touched or crossed the line \( n = n_1 \) earlier than its last step. The number of such trajectories arriving at \((n_1 - 1, N - 1)\) but having an earlier contact with, or a crossing of, the line \( n = n_1 \) is equal to those arriving at \((n_1 + 1, N - 1)\). The argument is simply that by reflection about the line \( n = n_1 \) we can uniquely derive from a trajectory leading to \((n_1 + 1, N - 1)\) another leading to \((n_1 - 1, N - 1)\) which has a forbidden character, and conversely. Thus the number of permitted ways of arriving at \( n_1 \) for the first time after \( N \) steps is equal to all the possible ways of arriving at \( n_1 \) after \( N \) steps in the absence of the absorbing wall minus twice the number of ways of arriving at \((n_1 + 1, N - 1)\) again in the absence of the absorbing wall. That is,

\[ \binom{N}{n_1} \binom{N - 1}{n_1 - 1} = \frac{n_1}{N} \frac{N!}{[\frac{1}{2} (N - n_1)] ![\frac{1}{2} (N + n_1)] !} \quad (7-18) \]
The required probability $Q(n_1; N)$ is therefore given by

$$Q(n_1; N) = \frac{n_1}{N} P(n_1|N)$$  \hspace{1cm} (7-19)$$

Notice that when $n_1 = N$

$$P(n_1|N) = Q(n_1; N)$$  \hspace{1cm} (7-20)$$

and that going to the limit given in (7-4), (7-19) reduces to

$$Q(x_1; t) = \frac{\tau x_1}{lt} \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x_1^2}{4Dt}\right)$$  \hspace{1cm} (7-21)$$

when $x_0 = 0$ and $n_1l \to x_1$.

Finally, we ask for the probability $q(x_1; t) \Delta t$ that the particle arrives at $x_1$ during $t$ and $t + \Delta t$ for the first time. This probability is related to $Q(x_1; t)$ through

$$q(x_1; t) \Delta t = Q(x_1; t) \frac{\Delta t}{\tau}$$  \hspace{1cm} (7-22)$$

Thus

$$q(x_1; t) = \frac{x_1}{t} \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x_1^2}{4Dt}\right)$$  \hspace{1cm} (7-23)$$

We can interpret (7-23) as giving the fraction of a large number of particles initially at $x = 0$, which are deposited on the absorbing wall per unit time, at time $t$.

Using (7-14) and (7-16), we readily verify that $q(x_1; t)$ as defined by (7-23) satisfies the relation

$$q(x_1; t) = -D \frac{\partial p(x_1, t|x_0, t_0)}{\partial x} \bigg|_{x = x_1} = -D \nabla p|_{x = x_1}$$  \hspace{1cm} (7-24)$$

which is just the probability current passing through the plane $x = x_1$ at $t$. Thus the random walk model has provided us with an alternate proof of the law of diffusive flow discussed in the previous section.

From the preceding discussion it is clear that if one desires to generate a coherent laser beam, then the rate at which diffusion in the beam is taking place must be controlled. This is one reason that laser beams are sometimes pulsed. The width of the pulse of energy must be small when compared with
the diffusion rate. The same thing is also true in a SCS; that is, if the SCS is to possess the capability of making accurate measurements, the rate at which the phase error undergoes diffusion must be controlled by proper design.

7-5 Markov Processes and Applications

Synchronization, tracking, and coherent demodulation theory deals mainly with random or stochastic processes which are functions of time that cannot be specified in detail but must be described statistically, using the theory of random processes. Markov processes are convenient mathematical abstractions of physically realizable random processes and will prove to be useful in the analysis and synthesis of SCSs. The random processes encountered in synchronization, tracking, and coherent demodulation theory are not exactly Markovian; however, we shall approximate the actual process at work in the real world by a Markov process. In fact, not being able to invoke such an assumption would render the nonlinear analysis of SCSs mathematically nontractable. The question then arises as to how close do the analytical results obtained under the Markov assumption compare with those encountered by direct measurement of the phenomenon of interest in the laboratory. If the results compare favorably to within some tolerable measurement error (i.e., for all practical purposes, theory and measurement agree), then the assumptions involved are valid to a good engineering approximation.

A fecund and often used model associates the random process with a system whose condition or state at any time can be specified by a finite number of random variables. With the passage of time, the state of the system changes in a way that is more or less random; yet by knowing the state of the system at any time, one can determine the probability distributions of the state variables at all times in the future. Such a system is called Markovian, and the collection of state variables describing it is said to be a Markov process. We are actually supposing that the course or trajectory that \( x(t) \) will take depends only on the instantaneous values of its physical parameters and is entirely independent of its whole previous history. Stated another way, what happens in the future depends only on the state of the system at the present.

Consider, first the random process \( \{x(t)\} \), with sample values \( x(t_1) = x_1, \ldots, x(t_n) = x_n \) where \( t_1 > t_2 > \cdots > t_n \), and the conditional p.d.f. of the value of \( \{x(t)\} \) at time \( t = t_1 \)—that is,

\[
p(x_1|x_2, \ldots, x_n) = \frac{p(x_1, x_2, \ldots, x_n)}{p(x_2, \ldots, x_n)} \tag{7-25}
\]

where \( n \) is arbitrary. If the process of the first-order Markov type, then

\[
p(x_1|x_2) = p(x_1|x_2, \ldots, x_n), \quad n \geq 2 \tag{7-26}
\]
represents the conditional p.d.f. If the process is Markov, \( p(x_1|x_2) \) is frequently called the \textit{transition} p.d.f. The definition of conditional p.d.f. implies the formula

\[
p(x_1, x_2, \ldots, x_n) = p(x_1|x_2, \ldots, x_n)p(x_2|x_3, \ldots, x_n) \ldots p(x_{n-1}|x_n)p(x_n)
\]

(7-27)

from which it follows from (7-26) that, for a first-order Markov process, the multidimensional p.d.f. factors into products of transition p.d.f.'s,

\[
p(x_1, x_2, \ldots, x_n) = p(x_1|x_2)p(x_2|x_3) \ldots p(x_{n-1}|x_n)p(x_n)
\]

(7-28)

This equation points out the remarkable fact that if we know the marginal p.d.f. and the transition p.d.f. of \{x(t)\}, we can write any multivariate p.d.f. of \{x(t)\}; that is, the transition p.d.f. \( p[x(t)|x(t')] \) and the marginal p.d.f. \( p[x(t)] \) completely characterized (see Chapter 1) a first-order Markov process. Consequently, Markov processes are frequently referred to as processes without \textit{after-effect}. We will actually observe these facts in the study of nonlinear SCS theory. In what follows, we shall refer to a first-order Markov process by dropping the redundant term first-order.

Systems of this kind are very common in practice. For example, a mechanical system is described by Hamilton’s dynamical equations, which give the rates of change of a finite number \( N \) of coordinates \( q_i \) and momenta \( p_i \) as derivatives of a certain function \( H(q_1, \ldots, q_N; p_1, \ldots, p_N) \) related to the energy of the system called the \textit{Hamiltonian}.

\[
\frac{\partial q_i}{\partial t} = \frac{\partial H}{\partial p_i}, \quad \frac{\partial p_i}{\partial t} = -\frac{\partial H}{\partial q_i}, \quad 1 \leq i \leq N
\]

(7-29)

Given the values of all the coordinates and momenta at a finite time \( t_0 \), one can, in principle, determine their values at future times \( t > t_0 \) by solving these differential equations.

However, we are stopped at the very beginning if we try to solve (7-29). We are in the presence of a system of \( 6N \) nonlinear differential equations, \( N \) being of the order of \( 6 \times 10^{23} \). It must be realized that the difficulty is not only a practical one. Even if we could imagine a perfect computing machine that could solve (7-29), the solution would be absolutely useless. Indeed, to give a meaning to (7-29) they must be supplemented by a set of \( 6N \) initial conditions. It is absolutely inconceivable for a human observer to measure simultaneously the positions and momenta of \( 10^{23} \) particles at a given time or to prepare the system in such a way that these variables have prescribed values. Therefore the exact formal solution of (7-29) would be of no use in predicting any physical process. One needs a concept that is closer to accessible macro-
scopic facts. This concept is provided by the idea to an ensemble, introduced by Gibbs. Instead of considering a unique system, we study a set of systems that are dynamically identical (i.e., same Hamiltonian) but that differ in their initial conditions. The natural framework for the description of such a system is a $6N$-dimensional space, called the phase space, the coordinates of which are the positions and momenta of the $6N$ particles. A particular system in a given state of motion will be described by a representative point in the phase space. The dynamical evolution of the complete system is described by the motion of this point along a trajectory in phase space. The ensemble representing the real system therefore corresponds to a cloud of representative points, which is usually assumed to be continuous. Its mathematical description will be given by a function representing the density of the cloud at each point in phase space; namely, $p(q_1, \ldots, q_{N}, p_1, \ldots, p_{N}, t)$. This function is somewhat analogous to the p.d.f. $p(x; t)$ discussed in Chapter 6.

The electromagnetic field in space is described by an infinite number of variables, the components of the electric and magnetic field intensities at each point. The evolution of the field is specified by Maxwell's equations, which relate the rate of change of the time derivatives of the field to the spatial derivatives of the field components. If the values of the electric and magnetic field are given at any time $t_0$, their future values are completely determined and can be found by solving Maxwell's equations with the proper boundary conditions. The same is true in quantum mechanics, where the wave equation for all future states of the system evolves from the wave function describing the state of an initial time $t_0$.

That we can idealize and model a broad class of real world-type stochastic processes as Markov processes is remarkable. But we should be careful not to conclude too hastily that every stochastic process is necessarily of the Markov type. For it can happen that the future course of a system is conditioned by its past history—that is, what happens at a given instant of time $t$ may depend on what has already happened during all time preceding $t$.

### 7-6 First-Order Markov Processes and the One-Dimensional Fokker-Planck Equation

In this section we study in detail the one-dimensional Fokker-Planck equation corresponding to a Markov process. Such a process occurs later on in our study of an idealized first-order SCS. Because of its special place in the theory, it merits separate treatment and recognition. First we present a derivation of the one-dimensional Fokker-Planck equation, using the method of characteristic functions, and then study its various solutions. We follow by generalizing the following development to a vector Markov process via the Smoluchowski equation.

The marginal p.d.f. $p(x_1, t_1)$ and the transition p.d.f. $p(x_1, t_1|x_2, t_2)$ are
related to the marginal p.d.f. \( p(x_2, t_2) \) through

\[
p(x_1, t_1) = \int_{-\infty}^{\infty} p(x_1, t_1 | x_2, t_2) p(x_2, t_2) \, dx_2 \quad (7-30)
\]

First, a word about the new notation is in order. Here the notation \( p(x_2, t_1) \triangleq p[x(t_1)], p[x_1, t_1 | x_2, t_2] \triangleq p[x(t_1) | x(t_2)] \) and \( p(x_2, t_2) \triangleq p[x(t_2)] \) is used in order that the time origin be kept definite with respect to the dummy variables \( x_1 \) and \( x_2 \). We also find this notation convenient later.

Consider the conditional c.f. \( C_{\Delta x}(\omega) \) of the random increment \( \Delta x = x_1 - x_2 \) given by (see Chapter 1, (1-7), and Section 1-9)

\[
C_{\Delta x}(\omega) = \int_{-\infty}^{\infty} \exp [i\omega (x_1 - x_2)] p(x_1, t_1 | x_2, t_2) \, dx_1 \quad (7-31)
\]

which occurs during the time interval \((t_1, t_2)\), given that \( x(t_2) = x_2 \). The inverse Fourier transform gives

\[
p(x_1, t_1 | x_2, t_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp [-i\omega (x_1 - x_2)] C_{\Delta x}(\omega) \, d\omega \quad (7-32)
\]

Substitution of (7-32) into (7-30) gives

\[
p(x_1, t_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p(x_2, t_2) \, dx_2 \int_{-\infty}^{\infty} \exp [-i\omega (x_1 - x_2)] C_{\Delta x}(\omega) \, d\omega \quad (7-33)
\]

According to (1-11), the c.f. \( C_{\Delta x}(\omega) \) is defined by

\[
C_{\Delta x}(\omega) = \sum_{q=0}^{\infty} \frac{(i\omega)^q}{q!} m_q(\Delta x) \quad (7-34)
\]

where \( m_q(\Delta x) = E[(x_1 - x_2)^q] \) is the \( q \)th moment function of the random increment \( x_1 - x_2 \). It follows that

\[
p(x_1, t_1) = \sum_{q=0}^{\infty} \frac{1}{2\pi q!} \int_{-\infty}^{\infty} p(x_2, t_2) \, dx_2 \int_{-\infty}^{\infty} m_q(\Delta x) \exp [-i\omega (x_1 - x_2)] (i\omega)^q \, d\omega \quad (7-35)
\]

But

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp [-i\omega (x_1 - x_2)] (i\omega)^q \, d\omega = \frac{1}{2\pi} \left( -\frac{\partial}{\partial x_1} \right)^q \int_{-\infty}^{\infty} \exp [-i\omega (x_1 - x_2)] \, d\omega = \left( -\frac{\partial}{\partial x_1} \right)^q \delta(x_1 - x_2) \quad (7-36)
\]
so that (7-35) becomes

\[
p(x_1, t_1) = \sum_{q=0}^{\infty} \frac{1}{q!} \int_{-\infty}^{\infty} m_q(\Delta x) \left(-\frac{\partial}{\partial x_1}\right)^q \delta(x_1 - x_2) p(x_2, t_2) \, dx_2
\]

\[
= p(x_1, t_2) + \sum_{q=1}^{\infty} \frac{1}{q!} \left(-\frac{\partial}{\partial x_1}\right)^q m_q(\Delta x) p(x_1, t_2)
\]

(7-37)

It is now evident why we have kept track of the time origin. If \(p(x_1, t_1) = p(x_1, t_2) = p(x_1, t)\) for all \(t_1 = t_2 = t\), then the process is stationary of order one and (7-37) reduces to

\[
\sum_{q=1}^{\infty} \frac{1}{q!} \left(-\frac{\partial}{\partial x_1}\right)^q m_q(\Delta x) p(x_1, t) = 0
\]

(7-38)

On the other hand, if \(p(x_1, t_1|x_2, t_2)\) depends only on \(\tau = t_1 - t_2\), the process is said to be temporally homogeneous.

Returning to (7-37) and setting \(t_2 = t, t_1 = t + \tau, x_2 = x, x_1 = x_\tau\), we can write

\[
p(x_\tau, t + \tau) - p(x_\tau, t) = \sum_{q=1}^{\infty} \frac{1}{q!} \left(-\frac{\partial}{\partial x_\tau}\right)^q E[(x_\tau - x)^q] p(x_\tau, t)
\]

(7-39)

Dividing by \(\tau\) and passing to the limit as \(\tau \to 0\), we obtain

\[
\frac{\partial p(x, t)}{\partial t} = \sum_{q=1}^{\infty} \frac{1}{q!} \left(-\frac{\partial}{\partial x}\right)^q K_q(x) p(x, t)
\]

(7-40)

where

\[
K_q(x) \triangleq \lim_{\tau \to 0} \frac{E[(x_\tau - x)^q]}{\tau}
\]

(7-41)

provided the limit exists. Equation (7-40) is sometimes called the stochastic or kinetic equation and arises in the study of the synchronization of optical communication systems as well as SCSs in the presence of impulse noise. If the intensity coefficients \(K_q = 0, q \geq 3\), then (7-40) becomes we have the so-called one-dimensional Fokker-Planck equation, the Kolmogrov equation, or the diffusion equation; that is,*

\[
\nabla \cdot \mathcal{J}(x, t) + \frac{\partial p(x, t)}{\partial t} = 0
\]

(7-42)

*Here \(\nabla \cdot \mathcal{J}(x, t) = \frac{\partial \mathcal{J}(x, t)}{\partial x}\) since we are treating the one-dimensional case.
where we have introduced the probability current flowing in the positive direction of the \( x \) axis

\[
\mathcal{J}(x, t) \triangleq \left[ K_1(x) - \frac{1}{2} \frac{\partial}{\partial x} K_2(x) \right] p(x, t)
\]  \hspace{1cm} (7-43)

and \( \nabla = \partial/\partial x \). Notice that (7-42) is identical with the equation of probability flow derived in Chapter 6, where

\[
\mathcal{J}(x, t) = \nu p(x, t)
\]  \hspace{1cm} (7-44)

and

\[
\nu \triangleq \left[ K_1(x) - \frac{1}{2} \frac{\partial}{\partial x} K_2(x) \right]
\]  \hspace{1cm} (7-45)

is the one-dimensional field velocity.

It is instructive to dissect further the probability current \( \mathcal{J}(x, t) \) into the two components—the drift component, or the component due to the signal, and the diffusion component

\[
\begin{align*}
\mathcal{J}_s(x, t) & \triangleq K_1(x) p(x, t) \\
\mathcal{J}_d(x, t) & \triangleq -\frac{1}{2} \nabla \cdot [K_2(x) p(x, t)]
\end{align*}
\]  \hspace{1cm} (7-46)

discussed in Chapter 6. In correspondence with (7-45), the average particle velocity at a chosen point on the \( x \) axis due to a restoring field force equals \( K_1(x) \). By multiplying this by the p.d.f. \( p(x, t) \), we obtain the convection probability current \( \mathcal{J}_s(x, t) \). The presence of the concentration gradient \( \nabla K_2(x) p(x, t) \) also leads to a diffusion current \( \mathcal{J}_d(x, t) \), which is proportional to the negative of the concentration gradient. The negative sign is due to the fact that the particles are tending to diffuse to lower-probability regions. Thus the total probability current across section \( x \) in unit time is just \( \mathcal{J}(x, t) \). In fact, the term \( \mathcal{J}_d(x, t) \) represents a generalization of the law of diffusive flow, and if \( K_2(x) \) is constant, then

\[
\mathcal{J}_d(x, t) = -\frac{K_2}{2} \nabla \cdot p(x, t) = -D \frac{\partial p}{\partial x}
\]  \hspace{1cm} (7-47)

which is identical with (7-14) and (7-24). In the steady state

\[
\lim_{t \to \infty} p(x, t) \longrightarrow p_{ss}(x)
\]  \hspace{1cm} (7-48)

so that \( \nabla \cdot \mathcal{J}(x) = 0 \) when the limit exists.
7-6.1 Boundary and Initial Conditions

In order to obtain solutions of the Fokker-Planck equation (7-42), we must prescribe the initial conditions and boundary conditions that the solution must satisfy. These conditions are usually determined by the physics of the problem. For example, if at some initial time $t = t_0$ it is known that the p.d.f. of the particle's position is given by an arbitrary initial distribution $p(x, t_0) \triangleq p_0(x)$, we can find the subsequent evolution of this distribution [i.e., the function $p(x, t)$ for $t > t_0$] by solving (7-42) subject to certain boundary conditions. In SCS theory this initial condition on the p.d.f. $p(x, t_0)$ could well be the p.d.f. of the phase error at $t = t_0$. If the initial distribution is a delta function, indicating absolute certainty about the position of $x$ at $t = t_0$, then the resulting p.d.f. is just the transition p.d.f. $p(x, t|x_0, t_0)$. Thus the transition p.d.f. can be found as the solution of the equation

$$\nabla \cdot \mathcal{J}(x; t) + \frac{\partial p(x; t)}{\partial t} = 0$$  \hspace{1cm} (7-49)

where $p(x; t) \triangleq p(x, t|x_0, t_0)$.

Having discussed the initial conditions, we now consider the boundary conditions that have to be taken into account when solving the Fokker-Planck equation (7-42) or (7-49). If the sample function of the process $\{x(t)\}$ can take on all possible values from $-\infty$ to $\infty$, the Fokker-Planck equation is valid over the infinite $x$ axis. Then the boundary conditions take the form of conditions at $\pm \infty$. If, for example, we integrate (7-49) from $-\infty$ to $\infty$ and use the fact that probability must be conserved for all $t$, that is,

$$\int_{-\infty}^{\infty} p(x, t|x_0, t_0) \, dx = 1$$  \hspace{1cm} (7-50)

we have

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} p(x, t|x_0, t_0) \, dx = -\int_{-\infty}^{\infty} \nabla \cdot \mathcal{J}(x; t) \, dx$$  \hspace{1cm} (7-51)

or

$$\mathcal{J}(x; t) \bigg|_{-\infty}^{\infty} = 0$$  \hspace{1cm} (7-52)

This says that

$$\mathcal{J}(\infty; t) = \mathcal{J}(-\infty; t)$$  \hspace{1cm} (7-53)

However, in addition to (7-53), the stronger conditions
First-Order Markov Processes and the Fokker-Planck Equation

\[ \mathcal{J}(\infty; t) = \mathcal{J}(-\infty; t) = 0 \quad \forall t \quad (7-54) \]

and

\[ p(\infty, t|x_0, t_0) = p(-\infty, t|x_0, t_0) = 0 \quad (7-55) \]

are sometimes satisfied. Equation (7-54) is frequently true because in physical situations probability is not allowed to build up at plus or minus infinity. This is equivalent to the statement that representative points [paths \( x(t) \) of the particles] of the process \( \{x(t)\} \) cannot appear at infinity or leave at infinity. Equation (7-55) can be regarded as true because the area under the transition p.d.f. \( p(x, t|x_0, t_0) \) must be unity; a fact that forces \( p(x, t|x_0, t_0) \) to approach zero faster than \( 1/x^{1+\epsilon} \), \( \epsilon > 0 \), as \( x \) approaches infinity. Physically speaking, (7-55) is a statement of the fact that the probability that any trajectory arrives at \( \pm \infty \) is zero.

Frequently sample functions \( x(t) \) of the process \( \{x(t)\} \) can only take bounded values being in some interval, say \( [x_1, x_2] \). This is the case, as we shall see, in a SCS. In particular, we shall be concerned with the case where the boundary condition takes the form of the periodicity condition \( p(x, t|x_0, t_0) = p(x + nT_x, t|x_0, t_0); n = \pm 1, \pm 2, \ldots; T_x \) is the period. Consequently, when we are faced with such a process, we only consider the solution in the region of one period conditioned upon \( n \). Again integrating (7-49) from \( x_1 \) to \( x_2 \) and using the concept of conservation of probability yields

\[ \mathcal{J}(x_1; t) = \mathcal{J}(x_2; t) \quad (7-56) \]

However, the question as to whether

\[ \mathcal{J}(x_1; t) = \mathcal{J}(x_2; t) = 0 \quad (7-57) \]

will depend on whether the curl of the stochastic magnetic field intensity vanishes at the boundaries. If the currents turn out to be equal to zero at \( x_1 \) and \( x_2 \), then we say that there is no flow of probability across the boundary. In particular, no random trajectory can enter the region \( [x_1, x_2] \) by crossing the boundary and every random trajectory terminates when it arrives at the boundary. The question of boundary conditions is approached differently (recall the random walk problem with absorbing boundaries discussed earlier in this chapter) in first passage time problems. We postpone further discussion of this problem until the appropriate time in the development of the theory. In a later section we shall consider methods of solving (7-49).

Before proceeding we give a graphic physical interpretation of the Fokker-Planck equation that will prove useful and somewhat picturesque. For every sample function of the Markov process \( \{x(t)\} \), the trajectory \( x(t) \) has a
very complicated form, and each sample function can be thought of as representing the position of a particle in a one-dimensional “smoke-puff” that moves along the \( x \) axis with the passage of time. If we take a large number of the sample functions, each representing the position of a particle in the puff, the motion of all sample functions appears as a cloud that undergoes diffusion. The density of this cloud at any point is proportional to the transition p.d.f. Thus, in this model, we can talk about the number of moving points (the cloud) in a particular interval by considering

\[
Pr[x_1 \leq x \leq x_2] = \int_{x_1}^{x_2} p(x; t) \, dx
\]  

(7-58)

as the number of points in the interval \( x_1 \leq x \leq x_2 \). The fraction of time that the particle (particles) spends in any region of the probability space is proportional to the probability in that region.

### 7.7 Time Dependent Solutions to the One-Dimensional Fokker-Planck Equation

In some cases the Fokker-Planck equation can be solved by applying the known method of separation of variables; that is, we attempt to find a solution of the form

\[
p(x, t|x_0, t_0) = X(x)T(t)
\]  

(7-59)

(We point out that only in certain special situations can a solution of this form be found.) The question of existence of such a form immediately arises, and any discussion of this point is beyond the scope of the present presentation. The desired solution to (7-49) is a sum of terms, each of which has the form \( X_n(x)T_n(t) \). In the method of separation of variables, the idea is to construct functions of the form \( X_n(x)T_n(t) \), \( n \) an integer), which satisfy the partial differential equation (7-49) and the boundary conditions. By superposition of these functions, one then satisfies the initial conditions. Substituting (7-59) into (7-49) and dividing by \( X(x)T(t) \) yields

\[
\frac{\dot{T}(t)}{T(t)} = \frac{1}{X(x)} \left[ -\frac{\partial}{\partial x} (K_1(x)X(x)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (K_2(x)X(x)) \right] = -\lambda
\]  

(7-60)

since each side must be equal to the constant \( \lambda \) independent of \( x \) and \( t \). For certain prescribed boundary conditions, we have

\[
\frac{1}{2} \frac{d^2}{dx^2} [K_2(x)X(x)] - \frac{d}{dx} [K_1(x)X(x)] + \lambda X(x) = 0
\]  

(7-61)
which has solutions consisting of the sequence of eigenfunctions \( X_0(x), X_1(x), \ldots \), with corresponding *discrete* eigenvalues \( \lambda_0, \lambda_1, \ldots \). If we integrate (7-61) with respect to \( x \), we obtain

\[
\lambda_m \int_{-\infty}^{\infty} X_m(x) \, dx = 0 \quad (7-62)
\]

because the *difference* of the probability currents

\[
\mathcal{J}[X_m(x)] = K_1(x)X_m(x) - \frac{1}{2} \frac{d}{dx} [K_2(x)X_m(x)]
\]

(7-63)

evaluated at the boundaries vanishes. It follows from (7-62) that either \( \lambda_m = 0 \) or

\[
\int_{-\infty}^{\infty} X_m(x) \, dx = 0 \quad (7-64)
\]

Setting \( \lambda_0 = 0 \) in (7-61) produces the steady-state solution. Thus the eigenfunction \( X_0 \) corresponding to \( \lambda_0 = 0 \) is just the steady-state p.d.f. \( p_{ss}(x) \).

\[
p_{ss}(x) = \lim_{t \to \infty} p(x, t|x_0, t_0) = X_0(x) \quad (7-65)
\]

If the other eigenfunctions satisfy (7-62) and the eigenvalues are distinct and discrete, we can write the solution of (7-49) as

\[
p(x, t|x_0, t_0) = T_0p_{ss}(x) + \sum_{n=1}^{\infty} T_nX_n(x) \exp \left[ -\lambda_n(t - t_0) \right] \quad (7-66)
\]

where the \( T_n \)'s are determined by using the initial condition. Here we have made use of the fact that \( T_n(t) = T_n \exp \left[ -\lambda_n(t - t_0) \right] \). To evaluate \( T_n \) it is convenient to use the orthogonality of the eigenfunctions with respect to the weight \( 1/p_{ss}(x) \).

\[
\int_{-\infty}^{\infty} \frac{X_n(x)X_m(x) \, dx}{p_{ss}(x)} = \delta_{nm} \quad (7-67)
\]

where \( \delta_{nm} = 0 \) if \( n \neq m \) and 1 if \( n = m \).

The coefficients \( T_m \) can now be expressed in terms of the initial p.d.f. \( p(x, t_0) \). Multiplying both sides of (7-66) by \( X_m(x)/p_{ss}(x) \), and integrating with respect to \( x \) over the probability space yields the expression for \( T_m \)

\[
\exp \left[ -\lambda_m(t - t_0) \right] T_m = \int_{-\infty}^{\infty} \frac{p(x, t|x_0, t_0)X_m(x)}{p_{ss}(x)} \, dx \quad (7-68)
\]
where we have made use of (7-67). If in the limit as \( t \) approaches \( t_0 \), \( p(x, t|x_0, t_0) \) approaches \( \delta(x - x_0) \), then

\[
T_m = \frac{X_m(x_0)}{p_{zz}(x_0)}
\]

(7-69)

so that the transition p.d.f. becomes

\[
p(x, t|x_0, t_0) = \sum_{n=0}^{\infty} \frac{X_n(x_0)X_n(x)}{p_{zz}(x_0)} \exp \left[ -\lambda_n(t - t_0) \right]
\]

(7-70)

for \( t > t_0 \). If the process is temporally homogeneous (homogeneous with respect to time), then

\[
p(x, t|x_0, t_0) = p(x, t-t_0|x_0, 0) = p(x, \tau|x_0)
\]

(7-71)

and the steady-state, joint p.d.f. \( p_{ss}(x, x_0, \tau) = p(x, \tau|x_0)p_{ss}(x_0) \) is given by

\[
p_{ss}(x, x_0, \tau) = p_{ss}(x)p_{ss}(x_0) + \sum_{n=1}^{\infty} X_n(x_0)X_n(x) \exp \left[ -\lambda_n|\tau| \right]
\]

(7-72)

In particular, the correlation function of \( y(t) = F[x(t)] \) is given by

\[
R_y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x)F(x_0)p_{ss}(x, x_0, \tau) \, dx \, dx_0
\]

(7-73)

where \( F[\cdot] \) is any continuous function. Using (7-72) in (7-73), we have

\[
R_y(\tau) = \sum_{n=0}^{\infty} \sigma_n^2 \exp \left[ -\lambda_n|\tau| \right]
\]

(7-74)

where

\[
\sigma_n \triangleq \int_{-\infty}^{\infty} F(x)X_n(x) \, dx
\]

(7-75)

The spectral density of the \( y \) process is therefore

\[
S_y(\omega) = \sum_{n=1}^{\infty} \frac{2\lambda_n \sigma_n^2}{\omega^2 + \lambda_n^2}
\]

(7-76)

since \( \lambda_0 = 0 \). We point out that the method of separation of variables may not always work, for the "boundary conditions" (7-62) and (7-63) may be such that the solution does not exist or is not unique. In this case one must resort to other techniques.
7.7.1 Steady-State Solutions with Applications

For a first-order Markov process we have just shown that the transition p.d.f. depends on the steady-state p.d.f. We therefore look for solutions to (7-49) as \( t \) approaches infinity. Since \( \nabla \cdot J = 0 \) in the steady state, (7-43) becomes a linear differential equation in \( p_{ss}(x) = p(x) \),

\[
\frac{d}{dx} [K_2(x)p(x)] - 2K_1(x)p(x) = -2J 
\]  
(7-77)

and we have now dropped the redundant subscripts on \( p_{ss}(x) \). The general solution to (7-77) can be obtained by introducing the change of variables \( v(x) = K_2(x)p(x) \) to yield

\[
\frac{dv}{dx} - h(x)v(x) = -2J 
\]  
(7-78)

where we have introduced the restoring force, relative to \( K_2(x) \),

\[
h(x) \triangleq \frac{2K_1(x)}{K_2(x)} = -\nabla U(x) 
\]  
(7-79)

and the potential function

\[
U(x) \triangleq -\int^x h(y) \, dy = -\int^x \frac{2K_1(y)}{K_2(y)} \, dy 
\]  
(7-80)

Multiplying both sides of (7-78) by the factor \( \exp{[U(x)]} \), we can write

\[
\frac{d}{dx} \{v(x) \exp{[U(x)]}\} = -2J \exp{[U(x)]} 
\]  
(7-81)

when we integrate with respect to \( x \) and multiply the result by \( \exp{[-U(x)]} \), (7-81) becomes

\[
v(x) = \exp{[-U(x)]} \left[ C - 2J \int^x \exp{[U(y)]} \, dy \right] 
\]  
(7-82)

which says that

\[
p(x) = \frac{\exp{[-U(x)]}}{K_2(x)} \left[ C - 2J \int^x \exp{[U(y)]} \, dy \right] 
\]  
(7-83)

and \( C \) is an arbitrary constant of integration. The choice of the constant \( C \) is determined from the normalization condition, while the probability current
\( J \) is found from the boundary conditions of the particular problem. For a certain class of problems of practical interest, \( J(x) = 0 \) and (7-83) simplifies to

\[
p(x) = \frac{C}{K_2(x)} \exp \left[ -U(x) \right] \tag{7-84}
\]

Thus from a knowledge of the intensity coefficients \( K_1(x) \) and \( K_2(x) \), we can immediately write down the steady-state p.d.f. The p.d.f. \( p(x) \) is of course, valid over the region for which \( x \) varies. We now use a few examples to illustrate the effectiveness of stochastic methods based on the Fokker-Planck equation.

**Example 1.** Consider first the case where the external force \( K_1(x) \) is a linear restoring force \( K_1(x) = -bx \) and the potential function has the form

\[
U(x) = \frac{bx^2}{K_2}, \quad x \in [-\infty, \infty] \tag{7-85}
\]

with \( K_2 \) independent of \( x \). In this case the steady-state p.d.f., as determined from (7-79), (7-80), (7-84), and (7-85), is Gaussian,

\[
p(x) = \sqrt{\frac{b}{\pi K_2}} \exp \left( -\frac{bx^2}{K_2} \right) \tag{7-86}
\]

with \(|x| \leq \infty\).

**Example 2.** Suppose that the external force and potential function have the form \((K_2 = \text{constant})\)

\[
K_1(x) = -bx + \frac{K_2}{2x}, \quad U(x) = \frac{bx^2}{K_2} - ln(x) \tag{7-87}
\]

for \( x \geq 0 \). Then the p.d.f. of the random process \( \{x(t)\} \) is a Rayleigh process with steady-state p.d.f. given by

\[
p(x) = \frac{x}{\sigma^2} \exp \left( -\frac{x^2}{2\sigma^2} \right), \quad x \geq 0
\]

\[= 0 \text{ elsewhere} \tag{7-88}\]

and \( \sigma^2 = K_2/2b \).

**Example 3.** Consider the nonlinear external force,

\[
K_1(x) = -k_0 \sgn x = \begin{cases} -k_0 & x > 0 \\ k_0 & x < 0 \end{cases}, \quad x \in [-\infty, \infty] \tag{7-89}
\]
corresponding to the potential function

\[ U(x) = \frac{2k_0|x|}{K_2} \]  

(7-90)

In this case the steady-state p.d.f. is given by

\[ p(x) = \frac{k_0}{K_2} \exp \left[ -\frac{2k_0|x|}{K_2} \right] \]  

(7-91)

for \(|x| \leq \infty\).

### 7-7.2 Time Dependent Solutions for Some Special Cases

In this section we present explicit examples of transient solutions that correspond to the steady-state solutions presented in Section 7-7.1.

**Example 1.** The probability current defined in (7-43) is given by

\[ \mathcal{J}(x; t) = -\left[ (bx + \frac{K_2}{2} \frac{\partial}{\partial x}) p(x, t|x_0, t_0) \right] \]  

(7-92)

In the steady state (7-86) follows if \( \mathcal{J}(x) = 0 \). For this case (7-61) takes the form

\[ \sigma^2 \frac{d^2X}{dx^2} + \frac{d}{dx} (xX) + \frac{\lambda X}{b} = 0, \quad \sigma^2 = \frac{K^2}{2b} \]  

(7-93)

If we impose zero boundary conditions on \( p(x) \) and \( \mathcal{J}(x) \) at the boundaries \( x = \pm \infty \), then (7-93) has eigenvalues \( \lambda_n = nb(n = 0, 1, 2, \ldots) \) with corresponding eigenfunctions

\[ X_n(x) = \frac{1}{\sqrt{n!}} \frac{1}{\sigma} F^{(n+1)}(x/\sigma) \]  

(7-94)

where

\[ F^{(n+1)}(z) = \frac{1}{\sqrt{2\pi}} \frac{d^n}{dz^n} \exp \left[ -\frac{z^2}{2} \right] \]  

(7-95)

Thus the joint p.d.f. (7-72) is given by

\[ p(x, x_0, \tau) = \frac{1}{n! \sigma^2} \sum_{n=0}^{\infty} F^{(n+1)}(x/\sigma) F^{(n+1)}(x_0/\sigma) \exp \left[ -\lambda_n |\tau| \right] \]  

(7-96)

if the initial p.d.f. is given by \( p(x, t_0) = \delta(x - x_0) \).
Example 2. Consider the diffusion equation

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x} \left[ (bx - \frac{K_z}{2x}) p \right] + \frac{K_z}{2} \frac{\partial^2 p}{\partial x^2}$$  \hspace{1cm} (7-97)

which describes the Rayleigh process whose steady-state p.d.f. was given in (7-88). Here $p = p(x, t|x_0, t_0)$. The equation whose solution gives the eigenfunctions is given from (7-61) by

$$\sigma^2 \frac{d^2 X}{dx^2} + \frac{d}{dx} \left( x \frac{dX}{dx} \right) - \sigma^2 \frac{d}{dx} \left( \frac{X}{x} \right) + \frac{\lambda X}{b} = 0, \quad \sigma^2 = \frac{K_z}{2b}$$  \hspace{1cm} (7-98)

Imposing the boundary conditions such that $p(x, t|x_0, t_0) = 0$ at $x = 0$ and $x = \infty$, one can show that the eigenfunctions are given by

$$X_n(x) = \frac{1}{n!} \frac{x^n}{\sigma^n} \exp \left( -\frac{x^2}{2\sigma^2} \right) L_n \left( \frac{x^2}{2\sigma^2} \right)$$  \hspace{1cm} (7-99)

where the $L_n(z)$ are the zero-order Laguerre polynomials

$$L_n(z) \triangleq \exp(z) \frac{d^n}{dz^n} [z^n \exp(-z)]$$  \hspace{1cm} (7-100)

and the eigenvalues are $\lambda_n = 2nb$. The expansion (7-72) assumes the form

$$p(x, x_0, \tau) = \frac{x x_0}{\sigma^4} \exp \left( -\frac{x^2 + x_0^2}{2\sigma^2} \right) \sum_{n=0}^{\infty} \frac{1}{(n!)^2} L_n \left( \frac{x^2}{2\sigma^2} \right) L_n \left( \frac{x_0^2}{2\sigma^2} \right) \exp \left( -2nb|\tau| \right)$$  \hspace{1cm} (7-101)

This joint p.d.f. describes the statistics of the envelope of a narrowband Gaussian process.

Example 3. Consider the diffusion equation

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x} [(\text{sgn} \, x)p] + \frac{1}{2} \frac{\partial^2 p}{\partial x^2}$$  \hspace{1cm} (7-102)

where we have set $K_1(x) = -\text{sgn} \, x$, $K_2(x) = 1$. The steady-state solution corresponds to setting $k_0 = K_2$ in (7-91). Using (7-61) the eigenfunctions are found from the equations

$$\frac{d^2 X}{dx^2} - 2 \frac{dX}{dx} + 2\lambda X = 0, \quad x < 0$$  \hspace{1cm} (7-103)

$$\frac{d^2 X}{dx^2} + 2 \frac{dX}{dx} + 2\lambda X = 0, \quad x > 0$$  \hspace{1cm} (7-104)
if one assumes that $X(x)$ is twice differentiable and $X$ and $dX/dx$ are continuous at $x = 0$. The solutions of (7-103) and (7-104) that satisfy the continuity conditions are $\lambda_0 = 0$, $\lambda_n = (1 + n^2)/2$ with corresponding eigenfunctions

$$X_n(x) = \frac{\sin nx}{\sqrt{\pi}} \exp(-|x|)$$  \hspace{1cm}  (7-105)

for $n \geq 1$. The transition p.d.f. assumes the form

$$p(x, t|x_0, t_0) = \exp(-2|x|) + \frac{\exp(-|x_0|)}{\pi \exp(-|x|)} \sum_{n=1}^{\infty} \sin nx \sin n x_0 \exp[-\lambda_n(t - t_0)]$$  \hspace{1cm}  (7-106)

This example corresponds to a control system that has an integrator in the forward path and a $\text{sgn} \ x$ nonlinearity in the feedback path.

7-8 The First-Passage Time Problem in One Dimension

Assuming that $\{x(t)\}$ is a Markov process, we are frequently interested in the time $T_{fp}$ that it takes the sample function $x(t)$ to first reach the boundaries $x = b_1$ or $x = b_2$, given that at $t = t_0$, $x(t_0) = x_0$. This so-called first-passage time $T_{fp}$ takes place at times that are different from sample function to sample function of the process $\{x(t)\}$, so that $T_{fp}$ is actually a random variable dependent on $x_0$. Figure 7-4 depicts the situation for two typical sample functions,

![Diagram](image_url)

**Fig. 7-4.** Illustrating the First-Passage Time Problem.
while Fig. 7-5 illustrates the buildup of the transition p.d.f. \( p(x; t) = p(x, t|x_0, t_0) \) with the passage of time when absorbing boundaries are assumed. In particular, Fig. 7-5a illustrates the fact that at \( t = t_0 \) all the probability is concentrated at \( x = x_0 \); that is,

\[
\lim_{t \to t_0} p(x; t) \to \delta(x - x_0)
\]  

(7-107)

With the passage of time, various sample functions (particles) of the process \( \{x(t)\} \) are absorbed at the boundaries \( b_1 \) and \( b_2 \). Consequently, as indicated in Fig. 7-5b, there is a nonzero probability current at the boundaries that corresponds to a flow of representative points (sample functions) which are absorbed by the boundary. (For the discrete random walk problem, Section 7-4, this current was evaluated.) Thus the delta functions \( a_1 \delta(x - b_1) \) and \( a_2 \delta(x - b_2) \) in Fig. 7-5b represent the probability buildup as a result of the various sample functions being absorbed. In general, \( a_1 \neq a_2 \), since an external field bias may
exist. The function $Q(x; t) \triangleq Q(x, t|x_0, t_0)$ accounts for the remaining representative points that have not been absorbed. Since

$$\int_{b_1}^{b_2} p(x; t) \, dx = 1$$  \hspace{1cm} (7-108)$$

for all $t$, $Q(x; t)$ is not a transition p.d.f. because it does not have unit area; however,

$$\int_{b_1}^{b_2} Q(x; t) \, dx = 1 - (a_1 + a_2)$$  \hspace{1cm} (7-109)$$

for all $t$. As $t$ approaches infinity (see Fig. 7-5c), all random trajectories of $\{x(t)\}$ have been absorbed, so that if $a_1 = p$, then $a_2 = 1 - p$, or $a_1 + a_2 = 1$. So much for the qualitative discussion of the first-passage time problem. We now show how to calculate the moments of the r.v. $T_{fp}$ for the case where the intensity coefficients $K_1(x)$ and $K_2(x)$ are time independent.

Excluding from consideration any sample function of the random process $\{x(t)\}$ as soon as it reaches the boundaries for the \textit{first time}, then the integral

$$P(t|x_0) \triangleq \int_{b_1}^{b_2} Q(x; t) \, dx$$  \hspace{1cm} (7-110)$$
gives the probability that $x(t)$ has never reached the boundary during the time interval $[t_0, t]$ given that at $t = t_0$, $x = x_0$. Initially, when no sample function has managed to reach the boundary, the transition p.d.f. is concentrated at $x = x_0$, so that

$$\lim_{t \to t_0} P(t|x_0) = 1$$  \hspace{1cm} (7-111)$$

Sooner or later all trajectories arrive at the boundaries, and hence

$$\lim_{t \to \infty} P(t|x_0) = 0$$  \hspace{1cm} (7-112)$$

Inside the interval $[b_1, b_2]$, the behavior $Q(x; t)$ is described by the Fokker-Planck equation. Now $Q(x; t)$ describes the trajectories that have not been absorbed; thus the boundary conditions have to be altered in such a way that

$$Q(b_1; t) = Q(b_2; t) = 0$$  \hspace{1cm} (7-113)$$

for all $t > t_0$. Recall these facts from our discussion in Section 7-4, where we showed that boundary conditions like (7-113) are typical for a random walk with absorbing walls.
The initial and boundary conditions (7-111), (7-112), and (7-113) uniquely determine \( Q(x; t) \) as a solution to the Fokker-Planck equation. After calculating \( Q(x; t) \) we can find the probability

\[
P(t|x_0) - P(t|x_0) = 1 - P(t|x_0) \tag{7-114}
\]

that the boundary is first reached during the time interval \([t_0, t]\). Differentiating (7-114) with respect to \( t \), we obtain the p.d.f. of the first-passage time; that is,

\[
p(t|x_0) = -\frac{\partial P(t|x_0)}{\partial t} \tag{7-115}
\]

and the \( n \)th moment of the first-passage time becomes

\[
E[T_{f,p}^n(x_0)] = -\int_{t_0}^{\infty} t^n \frac{\partial P(t|x_0)}{\partial t} \, dt \tag{7-116}
\]

Integrating (7-116) by parts and using (7-111) and (7-112) gives

\[
E[T_{f,p}^n(x_0)] = n \int_{t_0}^{\infty} t^{n-1} P(t|x_0) \, dt + t_0^n \tag{7-117}
\]

and we have assumed that \( P(t|x_0) \) goes to zero faster than \( t^{-n} \) as \( t \) approaches infinity.

### 7-8.1 The \( n \)th Moment of the First-Passage Time and Its Relationship to the Probability Current

We now show how the p.d.f. of the first-passage time is related to the restricted probability current

\[
\mathcal{J}(x; t) = K_1(x)Q(x; t) - \frac{1}{2} \frac{\partial}{\partial x} [K_2(x)Q(x; t)]
\]

If we integrate the derivative of this equation between \( b_1 \) and \( b_2 \) and use (7-110), we immediately arrive at the expression for the p.d.f. of the first-passage time:

\[
\mathcal{J}(b_2; t) - \mathcal{J}(b_1; t) = -\frac{\partial}{\partial t} \int_{b_1}^{b_2} Q(x; t) \, dx = -\frac{\partial P(t|x_0)}{\partial t} \tag{7-118}
\]

From (7-115) and (7-118) we recognize that the p.d.f. of the first-passage time is given by

\[
p(t|x_0) = \mathcal{J}(b_2; t) + [-\mathcal{J}(b_1; t)] \tag{7-119}
\]
Equation (7-119) represents the sum of the number of particles moving to the right per unit time at \( b_2 \) and the number moving to the left per unit time at \( b_1 \). Stated yet another way, \( p(t|x_0) \) represents the density of the flow of particles per unit time that are leaving the probability space through the boundaries. Thus the \( n \)th moment of the first-passage time follows from (7-119).

\[
E[T_p^n(x_0)] = \int_{t_0}^{\infty} t^n \left[ \mathcal{J}(b_2; t) - \mathcal{J}(b_1; t) \right] dt. \tag{1-120}
\]

### 7-8.2 Development of the Recursive Formula for the \( n \)th Moment of the First-Passage Time

We now develop a general recursive formula for the \( n \)th moment of the first-passage time when \( t_0 = 0 \). This is the case of greatest interest in what follows. For convenience, we let \( \tau^n(x|x_0) \) denote the \( n \)th moment of the first passage to either \( x = b_2 \) or \( x = b_1 \), given that \( x = x_0 \) at \( t_0 = 0 \). From (7-110) and (7-117) we can then write

\[
\tau^n(x|x_0) = n \int_{0}^{b_1} t^{n-1} Q(x; t) dx \ dt \tag{7-121}
\]

If we define

\[
Q_n(x|x_0) \triangleq \int_{0}^{\infty} t^n Q(x; t) dt \tag{7-122}
\]

then using (7-122) in (7-121), we have

\[
\tau^n(x|x_0) = \int_{b_1}^{b_1} nQ_{n-1}(x|x_0) dx \tag{7-123}
\]

Since \( Q(x; t) \) satisfies the Fokker-Planck equation, our approach will be to find a solution for \( Q_{n-1}(x|x_0) \) from this equation. The first-passage time is then found from (7-123).

Using (7-43), (7-49), and (7-80), we can write

\[
\frac{\partial^2 Q(x; t)}{\partial x^2} - \frac{\partial}{\partial x} [h(x)Q(x; t)] = \frac{2}{K_2} \frac{\partial Q(x; t)}{\partial t} \tag{7-124}
\]

when we replace \( p(x; t) \) by \( Q(x; t) \). Multiplying both sides by \( t^n \) and integrating from zero to infinity produces, with the aid of (7-122),

\[
\frac{d^2 Q_n(x|x_0)}{dx^2} - \frac{d}{dx} [h(x)Q_n(x|x_0)] = \frac{2}{K_2} \int_{0}^{\infty} t^n \frac{\partial Q(x; t)}{\partial t} dt \tag{7-125}
\]
if we assume that $K_2$ is independent of $x$. Integrating by parts on the right-hand side of (7-125) and using (7-111) and (7-112) gives

$$
\frac{d^2 Q_a(x|x_0)}{dx^2} - \frac{d}{dx} \left[ h(x) Q_a(x|x_0) \right] = -\frac{2n}{K_2} \int_0^\infty t^{n-1} Q(x; t) \, dt \quad (7-126)
$$

and we have assumed that $Q(x; t)$ approaches zero faster than $t^{-n}$ as $t$ approaches infinity. Integrating on $x$ from $b_1$ to $x$ and using (7-122) yields

$$
\frac{dQ_a(x|x_0)}{dx} - h(x) Q_a(x|x_0) = -\frac{2}{K_2} \tau^n(x|x_0) + \frac{2}{K_2} C(n) \quad (7-127)
$$

where $C(n)$ is a constant and $\tau^n(x|x_0) = u(x - x_0)$ is the unit step function. Solving this equation for $Q_a(x|x_0)$ produces

$$
Q_a(x|x_0) = \exp \left[ -U(x) \right] \left\{ D' + \frac{2}{K_2} \int_{b_1}^{b_2} [C(n) - \tau^n(y|x_0)] \exp [U(y)] \, dy \right\} \quad (7-128)
$$

where $C(n)$ and $D'$ are constants to be determined. Since $Q_a(b_1|x_0) = 0$ from (7-113) and (7-122), we find that $D' = 0$; and since $Q_a(b_2|x_0) = 0$ from (7-113) and (122), we have that

$$
C(n) = \int_{b_1}^{b_2} \tau^n(x|x_0) \exp [U(x)] \, dx \quad (7-129)
$$

Making use of (7-123) and replacing $n$ by $n = 1$ in (7-128) gives the recursive formula

$$
\tau^n(x|x_0) = \frac{2n}{K_2} \int_{b_1}^{b_2} \int_{b_1}^{b_2} [C(n - 1) - \tau^{n-1}(y|x_0)] \exp [U(y) - U(x)] \, dy \, dx \quad (7-130)
$$

where $\tau^n(x|x_0) = u(x - x_0)$. In later chapters we shall make use of (7-130) several times.

**7-S.3 Evaluation of the Ratio of the Number of Particles That Get Absorbed at $x = b_2$ to Those at $x = b_1$**

In the study of the nonlinear behavior of SCRs in the presence of noise, one fundamental problem concerns determining the average number of cycles slipped to the right per second, say $N_+$, and to the left, say $N_-$. The sum $S =$
as well as the difference \( N = N_+ - N_- \), is also of interest. In this section we shall develop the theory required to make these evaluations. Although we treat the special case of a one-dimensional Markov process, it will turn out that our result applies to Markov processes of higher dimension.

More formally, let us consider the problem of determining the ratio of the number of particles (trajectories) that get absorbed at the barrier located at \( x = b_2 \) to the number of particles (trajectories) that get absorbed at the barrier located at \( x = b_1 \) in the steady state. Consider first the change in the net probability occurring in the interval \((x, x + \Delta x)\) during a time interval of \((t, t + \Delta t)\). Using (7-49) we can write

\[
\int_x^{x+\Delta x} \int_t^{t+\Delta t} \frac{\partial p(x; t)}{\partial t} \, dt \, dx = -\int_t^{t+\Delta t} \int_x^{x+\Delta x} \frac{\partial \mathcal{J}(x; t)}{\partial x} \, dx \, dt
\]

Integrating on \( t \) on the left-hand side, and on \( x \) on the right-hand side, produces

\[
\int_x^{x+\Delta x} [p(x; t + \Delta t) - p(x; t)] \, dx = \int_t^{t+\Delta t} [\mathcal{J}(x; t) - \mathcal{J}(x + \Delta x; t)] \, dt
\]

For sufficiently small \( \Delta t \) and \( \Delta x \), the foregoing equation can be written as

\[
\text{Pr} [x \leq X \leq x + \Delta x] = [\mathcal{J}(x; t) - \mathcal{J}(x + \Delta x; t)] \Delta t
\]

from which we can make the important observation that \( \mathcal{J}(x; t) \Delta t \) represents the amount of probability entering the interval \([x, x + \Delta x]\) in \( \Delta t \) seconds at \( x \), while \( \mathcal{J}(x + \Delta x; t) \Delta t \) represents the amount of probability leaving the interval \([x, x + \Delta x]\) in \( \Delta t \) seconds at \( x + \Delta x \). Instead of talking about the “amount of probability” entering and leaving the interval \((x, x + \Delta x)\) in this theory, we shall find it convenient to talk about the “number of trajectories” or the “number of particles” leaving or entering this interval in \( \Delta t \) seconds. Thus we conclude that

\[
N_x(T) = \int_0^T \mathcal{J}(x; t) \, dt
\]

represents the number of particles (trajectories) flowing to the right in \( T \) seconds \((t_0 = 0)\) past the point \( x \). The quantity \( N_x(T) \) also represents the amount of probability flowing to the right past the point \( x \) in \( T \) seconds. Allowing \( T \) to approach infinity and setting \( x = b_2 \), we note that

\[
\lim_{T \to \infty} N_{b_2}(T) = \int_0^{\infty} \mathcal{J}(b_2; t) \, dt
\]
represents the number of particles that are absorbed at the barrier located at \( x = b_k, \ k = 1, 2, \) in the steady state. Certainly we must have

\[
\lim_{T \to \infty} \{ N_{b_k}(T) - N_{b_k}(T) \} = 1 \quad (7-136)
\]

which is equivalent to saying that as \( T \) approaches infinity, all particles (trajectories) have been absorbed, either at \( x = b_1 \) or \( x = b_2 \), with probability one.

Suppose now that we ask for the relative number of particles, \( N_{b_k}(T)/T \), which are moving to the right and which are absorbed at \( b_k \) per \( T \) seconds; that is

\[
\frac{N_{b_k}(T)}{T} = \frac{1}{T} \int_0^T \mathcal{J}(b_k; t) \, dt \quad (7-137)
\]

Now let \( N_+ \) represent the relative number of particles (trajectories) moving to the right that are absorbed at \( x = b_2 \) in the steady state and \( N_- \) denote the relative number that are moving to the left and that are absorbed at \( x = b_1 \) in the steady state; that is,

\[
N_+ \triangleq \lim_{T \to \infty} \frac{N_{b_2}(T)}{T} \quad \quad N_- \triangleq \lim_{T \to \infty} -\frac{N_{b_1}(T)}{T} \quad (7-138)
\]

Using (7-137) and (7-138), we find that the ratio \( N_+/N_- \) is given by

\[
\frac{N_+}{N_-} = \frac{-\int_0^\infty \mathcal{J}(b_2; t) \, dt}{\int_0^\infty \mathcal{J}(b_1; t) \, dt} \quad (7-139)
\]

We now proceed to evaluate this ratio.

First consider the positive and negative parts of the probability current \( \mathcal{J}(x; t) \) by writing

\[
\mathcal{J}(x; t) = \mathcal{J}_+(x; t) - \mathcal{J}_-(x; t) \quad (7-140)
\]

This can be integrated to give

\[
N_+(T) = \int_0^T \mathcal{J}(x; t) \, dt = \int_0^T [\mathcal{J}_+(x; t) - \mathcal{J}_-(x; t)] \, dt \quad (7-141)
\]

which is the net number of particles that have flowed to the right past \( x \) in \( T \) seconds. Here \( \mathcal{J}_+(x; t) \) and \( \mathcal{J}_-(x; t) \) represent, respectively, the probability current flowing to the right and to the left per unit time through section \( x \) at
time \( t \). At the absorbing barriers \( b_1 \) and \( b_2 \), illustrated in Fig. 7-5, we further note that

\[
\mathcal{J}_+(b_1; t) = 0 \quad \mathcal{J}_-(b_2; t) = 0
\]  

(7-142)

for all \( t \). Thus from (7-139), (7-140), and (7-142), we have

\[
\frac{N_+}{N_-} = \frac{\int_0^\infty \mathcal{J}_+(b_2; t) \, dt}{\int_0^\infty \mathcal{J}_-(b_1; t) \, dt}
\]  

(7-143)

Now the probability current, \( \mathcal{J}(x; t) \), considered as a function of the restricted p.d.f. \( Q(x; t) \), can be integrated to give

\[
\int_0^\infty \mathcal{J}(x; t) \, dt = K_1(x)Q_0(x|x_0) - \frac{1}{2} \frac{d}{dx} [K_2(x)Q_0(x|x_0)]
\]  

(7-144)

so that from (7-113) we have

\[
\int_0^\infty \mathcal{J}_+(b_2; t) = -\frac{1}{2} \frac{d}{dx} [K_2(x)Q_0(x|x_0)]|_{b_2}
\]  

\[
\int_0^\infty \mathcal{J}_-(b_1; t) = \frac{1}{2} \frac{d}{dx} [K_2(x)Q_0(x|x_0)]|_{b_1}
\]  

(7-145)

and using (7-145) in (7-143) we have

\[
\frac{N_+}{N_-} = -\frac{(d/dx)[K_2(x)Q_0(x|x_0)]|_{b_2}}{(d/dx)[K_2(x)Q_0(x|x_0)]|_{b_1}}
\]  

(7-146)

Using (7-127) we can evaluate the required derivatives when \( K_2(x) \) is independent of \( x \). Thus

\[
\frac{dQ_0(b_2|x_0)}{dx} = \frac{2}{K_2} [C(0) - u(b_2 - x_0)]
\]  

(7-147)

and

\[
\frac{dQ_0(b_1|x_0)}{dx} = \frac{2}{K_2} [C(0) - u(b_1 - x_0)]
\]  

(7-148)

where \( C(0) \) is defined in (7-129). Substitution of these into (7-146) gives

\[
\frac{N_+}{N_-} = -\frac{[C(0) - u(b_2 - x_0)]}{[C(0) - u(b_1 - x_0)]}
\]  

(7-149)
which is the desired result. We shall have occasion to use (7-149) several times in the development of the nonlinear theory of SCSs. The constant $C(0)$, which depends upon $x_0$, plays a major role.

7-9 Vector Markov Process and Representations

To study $n$th order ($n > 1$) SCSs, one is lead to specifying the state of the system by a vector stochastic process. A vector stochastic process $\{x(t)\} = \{x_1(t), x_2(t), \ldots, x_N(t)\}$ is specified if the joint p.d.f.

$$p(x_1(t_1), x_2(t_2), \ldots, x_n(t_n)) = p[x_1(t_1), x_2(t_2), \ldots, x_n(t_n)]$$

is given for all possible (but arbitrary) times $t_1, t_2, \ldots, t_n$ and for all $n$. In (7-150)

$$x(t_1) = x_1 \triangleq [x_1(t_1), x_2(t_1), \ldots, x_N(t_1)]$$
$$x(t_2) = x_2 \triangleq [x_1(t_2), x_2(t_2), \ldots, x_N(t_2)]$$
$$\vdots$$
$$x(t_n) = x_n \triangleq [x_1(t_n), x_2(t_n), \ldots, x_N(t_n)]$$

The process $\{x(t)\}$ is vector Markov (first-order) if the conditional p.d.f. $p[x(t)|x(t_0), x(t_1), \ldots, x(t_n)]$ of its values at time $t$, given its values at arbitrary past times $t_0, t_1, t_2, \ldots, t_n$, with $t > t_0 > t_1 > \ldots > t_n$, depends only on its values $x(t_0)$ at the most recent time $t_0$. We then write for a vector Markov process the transition p.d.f.

$$p(x, t|x_0, t_0) = p[x(t)|x(t_0), x(t_1), \ldots, x(t_n)]$$

(7-152)

Thus to characterize a vector Markov process completely, one needs only the transition p.d.f.’s $p(x, t|x_0, t_0), p(x_0, t_0|x_1, t_1), \ldots, p(x_{n-1}, t_{n-1}|x_n, t_n)$, and the marginal p.d.f. $p(x_n, t_n)$, where $t > t_0 > t_1 > \ldots > t_n$; that is,

$$p(x, t, x_0, t_0, x_1, t_1, x_2, t_2, \ldots, x_n, t_n)$$
$$= p(x, t|x_0, t_0)p(x_0, t_0|x_1, t_1), \ldots, p(x_{n-1}, t_{n-1}|x_n, t_n)p(x_n, t_n)$$

(7-153)

for all $n$. The assertion (7-152) that the conditional p.d.f. of the variables $x$ at time $t$ depends only on the initial values $x_0$ of $x$ at some past time $t_0$ as expressed by the transition p.d.f. $p(x, t|x_0, t_0)$ is the statistical counterpart of the
statement that for a deterministic system the values of the variables \( x \) describing its state at time \( t \) depend only on the time \( t \) and on the initial values \( x(t_0) \) at an earlier time \( t_0 \).

For a stationary system in which the dynamical equations governing the response of the system do not contain time explicitly, the response depends only on the interval \( t - t_0 \). Hence the transition p.d.f. of a stationary vector Markov process depends only on the initial values \( x_0 \).

### 7.9.1 The Smoluchowski Equation for a Vector Markov Process

The main reason for presenting the Smoluchowski equation is its historical significance; that is, the derivation of the Fokker-Planck equation was first derived using the Smoluchowski equation. It turns out, however, that a generalized Fokker-Planck equation can also be derived from the statement of conservation of probability for an arbitrary continuous random process, the Markov assumption not being required (Ref. 7).

We now begin to study how a Markov process evolves in time by deriving an integral equation for the transition p.d.f. If we assume successive times \( t, t_1, \) and \( t_0, \) with \( t > t_1 > t_0 \), it follows that the joint p.d.f. \( p(x, t, x_0, t_0) \) must satisfy the compatibility relation

\[
p(x, t, x_0, t_0) = \int \cdots \int p(x, t, x_1, t_1, x_0, t_0) \, dx_1 \quad (7-154)
\]

where \( dx_1 \) is the incremental volume element in the \( N \)-dimensional probability space of components \( [x_1(t_1), x_2(t_1), \ldots, x_N(t_1)] \) and the integration is taken over all of that space. The components of a vector Markov process are frequently referred to as the projections of \( x \). If we write (7-154) in terms of conditional p.d.f.'s, then

\[
p(x, t, x_0, t_0) = \int \cdots \int p(x, t|x_1, t_1, x_0, t_0) p(x_1, t_1, x_0, t_0) \, dx_1 \quad (7-155)
\]

But because \( x(t) \) is a vector Markov process, we have from (7-152)

\[
p(x, t|x_1, t_1) = p(x, t|x_1, t_1, x_0, t_0), \quad t > t_1 > t_0 \quad (7-156)
\]

and (7-155) becomes

\[
p(x, t, x_0, t_0) = \int \cdots \int p(x, t|x_1, t_1) p(x_1, t_1|x_0, t_0) p(x_0, t_0) \, dx_1 \quad (7-157)
\]
Dividing both sides of (7-157) by the p.d.f. \( p(x_0, t_0) \) yields the \textit{Smoluchowski equation} (or the \textit{Chapman-Kolmogorov equation}) for the transition p.d.f.

\[
p(x, t|x_0, t_0) = \int \cdots \int p(x, t|x_1, t_1)p(x_1, t_1|x_0, t_0) \, dx_1 \tag{7-158}
\]

Equations (7-157) and (7-158) represent a statement of how the probability unfolds with the passage of time. In effect, they show that the transition p.d.f. can be regarded as an integral operator that transforms the p.d.f. of the state of the system \( x_1 \) at one time \( t_1 \) into the p.d.f. of \( x \) at a later time \( t = t_1 + \tau \).

If the vector Markov process is stationary, we can replace \( t - t_1 \) by \( \tau \); then (7-158) can be written as

\[
p(x, t_1 + \tau|x_0, t_0) = \int \cdots \int p(x, \tau|x_1, 0)p(x_1, t_1|x_0, t_0) \, dx_1 \tag{7-159}
\]

If we think of \( \tau \) as very small and imagine repeating the operation in (7-158) a large number of times \( t_1, t_1 + \tau, t_1 + 2\tau \) and so on up to the present time \( t \), then the p.d.f. \( p(x, t|x_0, t_0) \) can be regarded as being generated from \( p(x_1, t_1|x_0, t_0) \) by a succession of infinitesimal transformations in much the same way as the state of a dynamical system at any time \( t \) is the result of a large number of infinitesimal contact transformations of the state at an earlier time \( t_1 \).

A physical interpretation of the Smoluchowski equation is possible by reverting back to the smoke puff discussed at the end of Chapter 6. Consider a small volume \( dV = dx_1 \) surrounding the point \( x_1 \) at time \( t_1 \) (see Fig. 7-6). If we adopt the point of view that the representative point \( x(t) \) is the position of a particle in \( N \) space at time \( t \), then the fraction of the number of particles in the smoke puff that pass through the incremental volume \( dV = dx_1 \), starting

![Fig. 7-6. Representative Points that Lead from \( x_0 \) to \( x \), which Pass through the Volume \( dx \) via \( x_1 \) at Time \( t_1 \).](image-url)
from \(x_0\) at \(t_0\), is given by \(p(x_1, t_1|x_0, t_0) \, dx_1\). On the other hand, the fraction of the number of representative points or particles that pass through \(dV = dx_1\), starting from \(x_0\) at time \(t_0\) and arriving at \(x\) at time \(t\), is given by

\[
p(x, t|x_1, t_1)p(x_1, t_1|x_0, t_0) \, dx_1.
\]

If we now sum over all the possible paths of the particles that lead from \(x_0\) at \(t_0\) and arrive at \(x\) at time \(t\), we have the Smoluchowski equation. The computation has a direct connection with the random walk problem where we computed the probability of arriving at a point by counting the number of paths (sequences) that could lead to a point in the \((x, t)\)-plane given \(N\) steps.

Returning to (7-157), we can write, by integrating over \(x_0\)

\[
p(x, t) = \int \cdots \int_{2N\text{-fold}} p(x, t|x_1, t_1)p(x_1, t_1|x_0, t_0)p(x_0, t_0) \, dx_1 \, dx_0 \tag{7-160}
\]

Integrating on \(x_0\) on the right hand side

\[
p(x, t) = \int \cdots \int_{N\text{-fold}} p(x, t|x_1, t_1)p(x_1, t_1) \, dx_1 \tag{7-161}
\]

and if \([x(t)]\) is *temporally homogeneous* (homogenous with respect to time), then

\[
p(x, t|x_1, t_1) \triangleq p(x, t - t_1|x_1, 0) \tag{7-162}
\]

so that (7-151) becomes

\[
p(x, t) = \int \cdots \int_{N\text{-fold}} p(x, t - t_1|x_1, 0)p(x_1, t_1) \, dx_1 \tag{7-163}
\]

If the marginal p.d.f. satisfies the equation

\[
p(x, t) = p(x, t_1) = p_\ast(x) \tag{7-164}
\]

for all \(t_1\) and \(t\), then the vector Markov process \([x(t)]\) is *stationary* of order one, and we write

\[
p_\ast(x) = \int \cdots \int_{N\text{-fold}} p(x, t - t'|x', 0)p_\ast(x') \, dx' \tag{7-165}
\]

since \(x_1\) is a dummy vector variable. The steady-state p.d.f. is defined by (when the limit exists)
\[ p_{x_{t_0}}(x) \triangleq \lim_{t \to \infty} p(x, t|x_0, t_0) \]  

(7-166)

and in most cases of practical interest the stationary p.d.f. equals the steady-state p.d.f.; that is,

\[ p_{x_{t_0}}(x) = p_{x_{t_1}}(x) \]  

(7-167)

Notice that \( p_{x_{t_0}}(x) \) is independent of time. Figure 7-7 depicts the set of all first-order Markov processes in terms of the various subsets just defined.

![Figure 7-7](image)

**Fig. 7-7. Illustrating the Set of All Vector Markov Processes.**

As the time \( t \) is decreased to the initial time, the transition p.d.f. becomes more and more concentrated near \( x_0 \), and in the limit it becomes the vector delta function

\[ \lim_{t \to t_0} p(x, t|x_0, t_0) = \delta(x - x_0) \]  

(7-168)

with \( x_{t_0} = x_{t}(t_0) \). On the other hand, as the time interval \( t - t_0 \) increases, the transition p.d.f.'s for the vector process depend less and less on the initial conditions \( x_0 \) at \( t_0 \) as the steady-state p.d.f. is approached.

In this way one could proceed to attempt to characterize the next order Markov process by assuming that the future values of a vector stochastic process depend on the present and one step into the past—namely, at time \( t_1 \); however, in the practical applications, few examples of such higher-order processes are studied because the equations cannot be solved.

### 7-9.2 Representation of Markov Process by the Method of Projections

Very often when a process is not a Markov process one can sometimes extend the process into projections of a more complicated Markov process.
Whether it is always possible to find the appropriate extension in terms of the desired projections so as to complete the observed process to a Markov process will, of course, depend on the physical causes generating the process. In most cases where a dynamical system is forced by white noise (or a delta-correlated Gaussian process), such an extension is usually possible. For example, any scalar Gaussian process \( \{x(t)\} \) with a rational spectrum approaching zero for high frequencies can be represented by the stochastic differential equation

\[
\sum_{k=0}^{m} a_k \frac{d^{m-k}x(t)}{dt^{m-k}} = \sum_{k=1}^{m} \lambda_k \frac{d^{m-k}n(t)}{dt^{m-k}}
\]  

(7-169)

where \( a_0 = 1, a_1, \ldots, a_m \) and \( \lambda_1, \ldots, \lambda_m \) are constants and \( \{n(t)\} \) is formally a white Gaussian process. In fact, \( \{x(t)\} \) can be realized by passing \( \{n(t)\} \) through the filter

\[
H(i\omega) = \frac{\sum_{k=1}^{m} \lambda_k (i\omega)^{m-k}}{\sum_{k=0}^{m} a_k (i\omega)^{m-k}}
\]

(7-170)

A particular state representation in terms of a vector Markov process \( \{x(t)\} \) has been given by Zadeh and Desoer (Ref. 8). The projections of \( \{x(t)\} \) are defined through

\[
\begin{align*}
\frac{dx_1}{dt} &= -a_1 x_1 + x_2 + \lambda_1 n \\
\frac{dx_2}{dt} &= -a_2 x_1 + x_3 + \lambda_2 n \\
&\vdots \\
\frac{dx_{m-1}}{dt} &= -a_{m-1} x_1 + x_m + \lambda_{m-1} n \\
\frac{dx_m}{dt} &= -a_m x_1 + \lambda_m n
\end{align*}
\]

(7-171)

where \( x(t) \triangleq x_1(t) \) and \( x = (x_1, x_2, \ldots, x_m) \). In matrix notation we write (7-171) as

\[
\frac{dx}{dt} = Fx + \lambda n
\]

(7-172)

where
\[ F \triangleq \begin{bmatrix}
-a_1 & 1 & 0 & 0 & \cdots & 0 \\
-a_2 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
-a_{m-1} & 0 & \cdots & 0 & 1 \\
-a_m & 0 & \cdots & 0 & 0 \\
\end{bmatrix} \quad \lambda \triangleq \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_m \\
\end{bmatrix} \quad (7-173) \]

A nonstationary Gaussian random process can be generated by (7-169) or (7-171) with time-varying coefficients \( a_0 = 1, a_1(t), \ldots, a_m(t), \) and \( \lambda_1(t), \ldots, \lambda_m(t). \) When we study the nonlinear theory of coherent analog demodulation, we will make use of (7-171).

**7-10 The Multidimensional Fokker-Planck Equation**

The Smoluchowski equation can now be used to derive the \textit{Fokker-Planck} equation. The Fokker-Planck equation will provide us with the most powerful weapon for use in studying the theory of synchronization, tracking, and coherent demodulation by means of SCs. The principles learned from the study of the random walk problem, the law of diffusive flow, and the equations of stochastic field theory derived in Chapter 6 render powerful methods for dissecting the Fokker-Planck equation to a point where insight can be gained relative to the design and analysis of dynamical systems. In the language of a mathematician, the Fokker-Planck equation is a partial differential equation of the parabolic type (Refs. 9, 10, and 11). The equation arose initially in studying the problem of Brownian motion—that is, the study of the erratic motion of smoke particles or of powder in a liquid as they are battered about by the molecules of the medium in which they are suspended. To some extent this random motion was discussed in Chapter 6 for the free-particle case as well as for the case where an external force, of the spring type, acted on the particle. It is rather striking that the same general methods of analysis may be applied to the study of problems arising in communication theory as in the fields of plasma physics, fluid dynamics, electromagnetic field theory, heat flow, stellar dynamics, colloid chemistry, the generation of coherent sources of light, and the list could continue at some length.

If the future p.d.f. of the state vector \( x \) can be generated by a succession of infinitesimal transformations like that manifested in (7-158), we can expect it to be the solution to a differential equation of the first order in time. This conjecture can be made on the basis of the fact that a dynamical system is described by Hamilton’s equations (7-29) in much the same way as the electromagnetic fields are related through Maxwell’s equations or the state of a quantum-mechanical system is governed by Schrödinger’s wave equation. Such a differential equation will exist if the moments of the transition p.d.f. \( p(x, t + \tau | x_0, t_0) \) obey
certain conditions. We have previously derived the form of the Fokker-Planck equation in Chapter 6 from the point of view of stochastic field theory. Both approaches to the derivation possess merit. In fact, the stochastic-field theoretic methods of Chapter 6 yields much physical insight into its meaning, whereas the method to follow yields mathematical insight.

To derive the conditions imposed on the transition p.d.f., we use the Smoluchowski equation (7-158) to work out the change in the transition p.d.f. \( p(x, t|x_0, t_0) \) over a small interval \((t, t + \tau)\) as a function of the moments of the random increment \((x - x_1)\). Thus we write from (7-158)

\[
\begin{align*}
p(x, t + \tau|x_0, t_0) - p(x, t|x_0, t_0) & \quad = \int \cdots \int p(x, t + \tau|x_1, t)p(x_1, t|x_0, t_0) \, dx_1 - p(x, t|x_0, t_0) \\
& \quad = \frac{1}{N} \int \cdots \int R(x)[p(x, t + \tau|x_0, t_0) - p(x, t|x_0, t_0)] \, dx
\end{align*}
\tag{7-174}
\]

We wish to express this change in terms of the moments of the random increment \((x - x_1)\) for small \(\tau\).

If we multiply both sides of (7-174) by the function \(R(x)\) and integrate over \(N\) space of the vector \(x\), we obtain, after dividing by \(\tau\),

\[
\begin{align*}
\frac{1}{\tau} \int \cdots \int R(x)[p(x, t + \tau|x_0, t_0) - p(x, t|x_0, t_0)] \, dx & \quad = \frac{1}{\tau} \int \cdots \int R(x)p(x, t + \tau|x_1, t)p(x_1, t|x_0, t_0) \, dx \, dx_1 \\
& \quad - \int \cdots \int R(x)p(x, t|x_0, t_0) \, dx
\end{align*}
\tag{7-175}
\]

The function \(R(x)\) is selected so that it and all its derivatives are continuous everywhere and vanish at infinity; otherwise \(R(x)\) is arbitrary. Notice that the integrals in (7-175) are actually volume integrals. The standard technique is now used to obtain the required differential equation; that is, we expand \(R(x)\) is a Taylor series about the point \(x = x_1\). Thus

\[
R(x) = R(x_1) + \sum_{i=1}^{N} (x_i - x_{1i}) \frac{\partial R(x)}{\partial x_{1i}} \bigg|_{x=x_1}
\]

\[
+ \frac{1}{2!} \sum_{i=1}^{N} \sum_{j=1}^{N} (x_i - x_{1i})(x_j - x_{1j}) \frac{\partial^2 R(x)}{\partial x_{1i} \partial x_{1j}} \bigg|_{x=x_1}
\]

\[
+ \frac{1}{3!} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} (x_i - x_{1i})(x_j - x_{1j})(x_k - x_{1k}) \frac{\partial^3 R(x)}{\partial x_{1i} \partial x_{1j} \partial x_{1k}} \bigg|_{x=x_1} + \cdots
\]

\tag{7-176}
The term \( R(x_i) \) will cancel with the last term in (7-175) after we apply the conservation of probability condition to the transition probability density.

\[
\int \cdots \int p(x, t + \tau | x_1, t) \, dx = 1 \quad (7-177)
\]

If we now define the following intensity coefficients

\[
K^{(1)}_{k}(x_1, t) \triangleq \lim_{\tau \to 0} \frac{1}{\tau} \int \cdots \int (x_k - x_{1k}) p(x, t + \tau | x_1, t) \, dx \\
K^{(2)}_{k\beta}(x_1, t) \triangleq \lim_{\tau \to 0} \frac{1}{\tau} \int \cdots \int (x_k - x_{1k})(x_{\beta} - x_{1\beta}) p(x, t + \tau | x_1, t) \, dx \\
K^{(3)}_{k\beta\gamma}(x_1, t) = \lim_{\tau \to 0} \frac{1}{\tau} \int \cdots \int (x_k - x_{1k})(x_{\beta} - x_{1\beta})(x_{\gamma} - x_{1\gamma}) p(x, t + \tau | x_1, t) \, dx
\]

(7-178)

and assume the limits exist, then (7-175) becomes, using (7-176) and (7-178),

\[
\oint_{\Omega} R(x) \frac{\partial p}{\partial t} \, dx = \oint_{\Omega} \frac{\partial R(x_1)}{\partial x_{1z}} K^{(1)}_{z}(x_1, t)p(x_1, t | x_0, t_0) \, dx_1 \\
+ \frac{1}{2!} \oint_{\Omega} \frac{\partial^2 R(x_1)}{\partial x_{1z} \partial x_{1\beta}} K^{(2)}_{z\beta}(x_1, t)p(x_1, t | x_0, t_0) \, dx_1 \\
+ \frac{1}{3!} \oint_{\Omega} \frac{\partial^3 R(x_1)}{\partial x_{1z} \partial x_{1\beta} \partial x_{1\gamma}} K^{(3)}_{z\beta\gamma}(x_1, t)p(x_1, t | x_0, t_0) \, dx_1 + \cdots
\]

(7-179)

where \( p = p(x, t | x_0, t_0) \) and we have adopted the summation convention whereby Greek subscripts, which denote the components of the vector \( x \), are summed from 1 to \( N \). Letting

\[
u = K^{(1)}_{z}(x_1, t)p(x_1, t | x_0, t_0) \quad dv = \frac{\partial R(x_1)}{\partial x_{1z}} \\
u = R(x_1) \quad du = \frac{\partial}{\partial x_{1z}} \left[K^{(1)}_{z}(x_1, t)p(x_1, t | x_0, t_0)\right]
\]

(7-180)

in (7-179), integrating each term by parts a sufficient number of times, and making use of the fact that \( R(x) \) and its derivatives vanish at the limits of integration yields
\[ \oint_{\Omega} R(x) \frac{\partial p}{\partial t} \, dx = \oint_{\Omega} R(x) \left\{ -\frac{\partial}{\partial x_a} [K_a^{(1)}(x, t)p] + \frac{1}{2!} \frac{\partial^2}{\partial x_a \partial x_b} [K_a^{(2)}(x, t)p] \right. \\
\left. - \frac{1}{3!} \frac{\partial^3}{\partial x_a \partial x_b \partial x_c} [K_a^{(3)}(x, t)p] \right\} \, dx + \cdots \]  

(7-181)

Since the equation is valid for arbitrary \( R(x) \), the integrands must be equal; consequently, we have the following partial differential equation for the transition p.d.f. at a point

\[ \frac{\partial p(x, t|x_0, t_0)}{\partial t} = -\sum_{k=1}^{\infty} \frac{\partial}{\partial x_k} \left[ K_k^{(1)}(x, t)p(x, t|x_0, t_0) \right] + \frac{1}{2!} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{\partial^2}{\partial x_k \partial x_j} \left[ K_k^{(2)}(x, t)p(x, t|x_0, t_0) \right] - \frac{1}{3!} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \frac{\partial^3}{\partial x_k \partial x_j \partial x_l} \left[ K_k^{(3)}(x, t)p(x, t|x_0, t_0) \right] + \cdots 
\]

(7-182)

For an important class of Markov processes, the quantities \( K_k^{(i)}(x, t) \) and so on vanish; that is, the third-, fourth-, and higher-moments of the components of the change in the increment \( (x - x_i) \) over a small interval \( \tau \) are proportional to powers of \( \tau \) greater than one. The partial differential equation for the transition probability p.d.f. is then of first order in time and second order in the space coordinates of \( x \), and it is called the multidimensional Fokker-Planck equation (or diffusion equation).

\[ \frac{\partial p(x, t|x_0, t_0)}{\partial t} = -\sum_{k=1}^{\infty} \frac{\partial}{\partial x_k} \left[ K_k^{(1)}(x, t) - \frac{1}{2} \sum_{j=1}^{\infty} \frac{\partial}{\partial x_j} K_k^{(2)}(x, t) \right] p(x, t|x_0, t_0) \]

(7-183)

Sometimes a Markov process is said to be continuous if its higher-order (i.e., greater than two) intensity coefficients are zero.

Introducing the probability current density

\[ \mathcal{J}_k(x; t) \triangleq \left[ K_k^{(1)}(x, t) - \frac{1}{2} \sum_{j=1}^{\infty} \frac{\partial}{\partial x_j} K_k^{(2)}(x, t) \right] p(x, t|x_0, t_0) \]

(7-184)

for all \( k = 1, 2, \ldots, N \), we can write the Fokker-Planck equation in the form of an equation of flow or continuity:

\[ \nabla \cdot \mathcal{J}(x; t) + \frac{\partial p(x, t|x_0, t_0)}{\partial t} = 0 \]

(7-185)
where $\mathcal{J}(x; t) \equiv [\mathcal{J}_1(x; t), \mathcal{J}_2(x; t), \ldots, \mathcal{J}_N(x; t)]$. Recall from Chapter 6 that $\mathcal{J}(x; t) \equiv \mathbb{V}p(x, t|x_0, t_0)$, where $v$ is the average velocity, and that velocity times a point mass is momentum; then it is clear that (7-185) indicates the flow of probability per unit time.

**7-10.1 Physical Interpretation for the Intensity Coefficients**

When $\tau$ is small, the intensity coefficient

$$K_{ik}^{(1)}(x, t) = \lim_{\tau \to 0} \frac{E[x_k - x_{1k}]}{\tau}$$  \hspace{1cm} (7-186)

is roughly the average change in the coordinate $x_k$ in a small interval $\tau$ in time. If we think of the transition p.d.f. $\rho(x, t|x_0, 0)$ as the p.d.f. in $N$ space of particles (a smoke puff) that started at time $t = 0$ at the point $x_0$, the quantity $K_{ik}^{(1)}(x, t)$ is the average velocity with which the particles move through an incremental volume surrounding the point $x$ in the direction on the $k$th coordinate vector $e_k$ (see Fig. 7-8) at time $t$.

Fig. 7-8. Diffusion in Velocity Space after $\tau$ Seconds Have Elapsed.

The intensity coefficients

$$K_{ij}^{(2)}(x, t) = \lim_{\tau \to 0} \frac{E[(x_k - x_{1k})(x_j - x_{1j})]}{\tau}$$  \hspace{1cm} (7-187)

are the variances and covariances measuring the uncertainty about where a particle is $\tau$ seconds after it is observed at time $t$. The meaning of these “coefficient of velocity” are illustrated in Fig. 7-8. These terms are due to thermal noise. Similar interpretations can be given to the higher moments $K_{ijkl}^{(3)}$ and so on.
For a temporarily homogeneous process, the quantities $K_k^{(1)}(x, t)$, $K_k^{(2)}(x, t)$ and so on are independent of time. Equation (7-185) is to be solved with the initial condition (7-168) and, in the steady state, we have $\nabla \cdot \mathcal{J} = 0$; that is, the total probability flow from a point $x$ surrounded by an incremental volume is zero, after a sufficiently long time period has elapsed.

### 7.10.2 The Multidimensional Backward Equation

The transition p.d.f. $p(x, t|x_0, t_0)$ is also a function of the initial values $x_0$ and $t_0$, and how it depends on them is described by the adjoint of the partial differential equations (7-182).

\[
\frac{\partial p(x, t|x_0, t_0)}{\partial t_0} + K_k^{(1)}(x_0, t_0) \frac{\partial}{\partial x} p(x, t|x_0, t_0) + \frac{1}{2} K_x^{(2)}(x_0, t_0) \frac{\partial^2}{\partial x_0^2} p(x, t|x_0, t_0) + \cdots = 0 \quad (7-188)
\]

This equation is sometimes referred to as the backward equation. It can be derived by using the Smoluchowski equation and writing for two successive times $t_0$ and $t_0 - \tau$, $\tau > 0$,

\[
\frac{1}{\tau} \left[ p(x, t|x_0, t_0) - p(x, t|x_0, t_0 - \tau) \right]
\]

\[=
\frac{1}{\tau} \int \left[ p(x, t|x_0, t_0) - p(x, t|x_0, t_0) \right] p(\xi_0, t_0|x_0, t_0 - \tau) d\xi_0 \quad (7-189)
\]

If the transition p.d.f. $p(x, t|x_0, t_0)$ is expanded in a Taylor series about the point $x_0$, and one uses the definitions in (7-178), we obtain (7-188), after passing to the limit $\tau \to 0$. The assumptions about the vanishing of the higher moments $K_k^{(j)}$, etc., needed to derive the Fokker-Planck equation (7-183) and its adjoint, can be replaced by the so-called Lindeberg-Lévy (Ref. 12) conditions for the Fokker-Planck equation; that is, if for any $\delta > 0$,

\[
\lim_{\tau \to 0} \int_{|x - x_0| > \delta} p(x, t + \tau|x_0, t_0) dx = 0 \quad (7-190)
\]

the higher-order coefficients are zero. For a derivation of this equation, see Bharucha-Reid (Ref. 13). Equation (7-190) is, in essence, a statement of the fact that the probability flow from an incremental volume surrounding the point $x$ does not change instantaneously; that is, the transportation of probability throughout the probability space cannot change instantaneously. This statement is analogous to the statement that momentum cannot suffer an abrupt change in magnitude.
Finally, we relate the solution to the Fokker-Planck equation; hence Markov process theory, to Gauss’s theorem, the divergence theorem, and Stoke’s theorem for stochastic fields through the current emerging through the arbitrary surface \( S \); that is,

\[
i(t) = \oint_V \nabla \cdot \mathbf{J} \, dV = \oint_S \mathbf{J} \cdot dS = -\frac{\partial}{\partial t} \oint_V p(x, t|x_0, t_0) \, dV \tag{7-191}
\]

and

\[
\oint_{\text{cap}} \nabla \times \mathbf{J} \cdot dS = \oint_{\text{cap}} \mathbf{J} \cdot d\mathbf{r} \tag{7-192}
\]

and if the surface \( S \) encloses the probability space \( \Omega \), then \( i(t) = 0 \).

7.11 Physical Interpretation of the Probability Current Density

The coefficients of the Fokker-Planck equation can be given the following physical interpretation. Starting with an ensemble of test particles, all having the same velocity \( v_0 \) at \( t_0 \), it will be noted after a short time that the average velocity has changed and that a spread of velocities has appeared (see Fig. 7-8). The coefficients \( K_\alpha(x, t) \) and \( K_{\alpha\beta}(x, t) \) are thus the transition moments of the Fokker-Planck equation and describe, respectively, the change in velocity and the spread in velocity of the test particle for short times (see Fig. 7-8). We have dropped the redundant superscripts on the \( K \)’s.

If we represent the probability current density in terms of its component breakdown, then the drift component

\[
\mathbf{J}_\alpha(x; t) \triangleq K_\alpha(x, t)p(x, t|x_0, t_0) \tag{7-193}
\]

represents the probability current density due to the signal component (field force) in the \( \alpha \) direction, while

\[
\mathbf{J}_{\alpha\beta}(x; t) \triangleq -\frac{\partial}{\partial x_\beta} \left[ K_{\alpha\beta}(x, t)p(x, t|x_0, t_0) \right] = -\nabla_\beta pK_{\alpha\beta}(x, t) \tag{7-194}
\]

represents that component of the probability current density produced in the \( \alpha \) direction due to diffusion taking place in the \( \beta \) direction. In communication systems, this component is due to the additive noise effects. Thus the projection of \( \mathbf{J}(x, t) \) in the \( \alpha \) direction of the probability space is given by

\[
\mathbf{J}_\alpha(x; t) = \mathbf{e}_\alpha \cdot \mathbf{J}(x; t) = \mathbf{J}_{\alpha\alpha}(x; t) + \mathbf{J}_{\alpha d}(x; t) \tag{7-195}
\]

Drift component due to signal  
Due to noise that causes diffusion in each direction of \( N \) space
The Potential Case

where

$$\mathbf{J}_\alpha(x; t) = \sum_{\beta=1}^{N} \mathbf{J}_{\alpha\beta}(x, t)$$  \hfill (7-196)

In terms of the intensity coefficients, we can write

$$\mathbf{J}_\alpha(x; t) = K_\alpha(x, t)p(x, t|x_0, t_0) - \frac{1}{2} \nabla_\beta pK_{\alpha\beta}(x, t)$$  \hfill (7-197)

where $\nabla_\beta$ denotes the directional derivative (see Chapter 6) and $p = p(x, t|x_0, t_0)$. Therefore the presence of a signal produces a drift component of probability current density in the $\alpha$ direction in the amount of $K_\alpha$, while the presence of a concentration gradient in each direction of $N$ space leads to the appearance of a diffusive current density flow in the $\alpha$ direction, $\alpha = 1, 2, \ldots, N$; namely, the sum over $\beta$ of the components $-\nabla_\beta pK_{\alpha\beta}$. The minus sign is, of course, due to the fact that the particles are tending to diffuse to a region of lower probability. In statistical mechanics, the intensity coefficients $K_\alpha(x, t)$ and $K_{\alpha\beta}(x, t)$ are frequently called, respectively, the coefficient of dynamical friction due to an external force and the diffusion-of-velocity coefficient due to random disturbances. Subject to the diffusion plus the frictional effects, the transition p.d.f. tends toward the equilibrium p.d.f. $p_{ss}(x)$.

Two approaches can be used in determining the transition p.d.f. On the one hand, one can attempt to solve the differential equations for which the solution must satisfy certain conditions—the initial conditions and the boundary conditions. This is the so-called boundary value problem. On the other hand, we can attempt to find solutions for $p(x, t|x_0, t_0)$ by a determination of the eigenvalues and eigenfunctions by means of integral equation techniques—transform theory, numerical integration, iteration, variation iteration, and so on. As we shall see later, synchronization, tracking, and coherent demodulation by means of a SCS involves the system of equations just derived and discussed; therefore certain communication theoretic studies merge into this mathematical framework quite naturally.

7-12 The Potential Case

Here we consider steady-state conditions whereby

$$\lim_{t \to \infty} \nabla \cdot \mathbf{J}(x; t) = \nabla \cdot \mathbf{J}(x) = 0$$  \hfill (7-198)

The potential case is defined by the conditions $\mathbf{J}_k(x) = 0$ for all $k = 1, 2, \ldots, N$ and all $x$ in the probability space $\Omega$. Even if $\mathbf{J}(x) = 0$ is zero on the surface $S$ of the boundaries of $\Omega$, it does not have to vanish inside $\Omega$ because rotational
probability flows can occur. To avoid this we require that

$$\nabla \times \mathcal{H} = \mathcal{J}(x) = 0$$  \hspace{1cm} (7-199)$$

in the steady state for all \( x \in \Omega \). The fact that \( \mathcal{J}_k(x) \) is identically zero for all \( x \in \Omega \) yields the conditions

$$K_k(x)p_{s_1}(x) = \frac{1}{2} \sum_{j=1}^{N} \frac{\partial}{\partial x_j} K_{kj}(x)p_{s_2}(x), \quad k = 1, 2, \ldots, N$$  \hspace{1cm} (7-200)$$

Assuming a solution of the form

$$p_{s_1}(x) = C \exp \left[ -U(x) \right]$$  \hspace{1cm} (7-201)$$

where \( U(x) \) is an integrating factor (the potential function), we find from (7-200) that

$$\sum_{j=1}^{N} K_{kj}(x) \frac{\partial U(x)}{\partial x_j} = -2K_k(x) + \sum_{j=1}^{N} \frac{\partial K_{kj}(x)}{\partial x_j}$$  \hspace{1cm} (7-202)$$

for all \( k = 1, 2, \ldots, N \). This set of equations can be written in matrix form

$$\mathbf{K}(x)(\nabla'U(x)) = -2\mathbf{W}(x) + \nabla' \cdot \mathbf{K}(x)$$  \hspace{1cm} (7-203)$$

where the prime denotes the transpose and

$$\mathbf{K}(x) = \begin{bmatrix} K_{11}(x) & \cdots & K_{1N}(x) \\ \vdots & \ddots & \vdots \\ K_{N1}(x) & \cdots & K_{NN}(x) \end{bmatrix}$$  \hspace{1cm} (7-204)$$

and \( \mathbf{W}(x) = [K_1(x), \ldots, K_N(x)]' \). Multiplying both sides of (7-203) by the inverse \( [\mathbf{A}(x)] = [\mathbf{K}(x)]^{-1} \), we obtain

$$\nabla'U(x) = -2\mathbf{A}(x)\mathbf{W}(x) + \mathbf{A}(x)\nabla' \cdot \mathbf{K}(x)$$  \hspace{1cm} (7-205)$$

which can be written as

$$\nabla_{x_1} U(x) = \frac{\partial U(x)}{\partial x_1} = \sum_{k=1}^{N} \sum_{j=1}^{N} A_{kj}(x) \frac{\partial K_{jk}(x)}{\partial x_j} - 2 \sum_{k=1}^{N} A_{ik}(x)K_k(x)$$  \hspace{1cm} (7-206)$$

for all \( t = 1, 2, \ldots, N \). This says that the right-hand side of (7-206) are the components of the gradient of a certain function \( U(x) \). Define the \( l \)th component of the vector \( \mathbf{E}(x) = [E_1(x), \ldots, E_N(x)] \) by
\[ E_i(x) = -\nabla_x U(x) = -\frac{\partial U(x)}{\partial x_i} \]  \hspace{1cm} (7-207)

so that

\[ \mathbf{E}(x) \triangleq -\nabla U(x) \]  \hspace{1cm} (7-208)

Thus the potential function \( U(x) \) can be written in terms of the line integral

\[ U(x) = -\int \mathbf{E}(x) \cdot dr \]  \hspace{1cm} (7-209)

which is the integrating factor needed in (7-201). Thus to find \( p_{z\pi}(x) \) we need only determine the potential function \( U(x) \). The components of the gradient of \( U(x) \) must satisfy the potential conditions

\[
\sum_{k=1}^{N} \sum_{j=1}^{N} \frac{\partial}{\partial x_m} \left[ A_{lj}(x) \frac{\partial K_{j\pi}(x)}{\partial x_j} \right] = 2 \sum_{k=1}^{N} \frac{\partial}{\partial x_m} \left[ A_{l\pi}(x) K_k(x) \right] - 2 \sum_{k=1}^{N} \frac{\partial}{\partial x_l} \left[ A_{m\pi}(x) K_k(x) \right] \]  \hspace{1cm} (7-210)

since

\[
\frac{\partial^2 U(x)}{\partial x_m \partial x_l} = \frac{\partial^2 U(x)}{\partial x_l \partial x_m} \]  \hspace{1cm} (7-211)

for all \( l \) and \( m = 1, 2, \ldots, N \). The constant \( C \) in (7-201) is found from the normalization condition of \( p_{z\pi}(x) \).

We now consider the special case of isotropic fluctuations defined by

\[ [K_{k\pi}(x)] = K(x) \delta_{k\pi} \]  \hspace{1cm} (7-212)

where \( \delta_{k\pi} = 1 \) for \( k = \pi \) and \( \delta_{k\pi} = 0 \) for \( k \neq \pi \). For this situation (7-205) becomes

\[ \nabla_x U(x) = -2 \frac{K_{k}(x)}{K(x)} + \frac{1}{K(x)} \frac{\partial K(x)}{\partial x_k} \]  \hspace{1cm} (7-213)

In this case the components of \( \mathbf{E} \) are given by

\[ E_k(x) = -\nabla_{x_k} U(x) = -\frac{1}{K(x)} \frac{\partial K(x)}{\partial x_k} + \frac{2}{K(x)} \frac{K_k(x)}{K(x)} \]  \hspace{1cm} (7-214)
where \( E \) and \( U(x) \) are defined in (7-207) and (7-208). For this case the potential conditions (7-210) become

\[
\frac{\partial}{\partial x_m} \left[ \frac{1}{K(x)} \frac{\partial K(x)}{\partial x_j} \right] - 2 \frac{\partial}{\partial x_m} \left[ \frac{K_k(x)}{K(x)} \right] = \frac{\partial}{\partial x_i} \left[ \frac{1}{K(x)} \frac{\partial K(x)}{\partial x_j} \right] - 2 \frac{\partial}{\partial x_i} \left[ \frac{K_k(x)}{K(x)} \right]
\]

(7-215)

If \( K(x) = K \) is a constant, then the potential conditions become

\[
\frac{\partial K_k(x)}{\partial x_m} = \frac{\partial K_k(x)}{\partial x_j}, \quad j \neq m = 1, 2, \ldots, N
\]

(7-216)

which says that only in this simple case can one regard the components \( K_k(x) \) as components of a force which can be derived from a potential function. For \( N = 1 \), the solution \( p_{ss}(x) \) is given by (7-81).

Regardless of whether the potential conditions are met, the steady-state solution to the Fokker-Planck equation can be simplified in the cases where \( K_1(x), \ldots, K_N(x) \) are linear functions of the arguments \( x_1, \ldots, x_N \) and the functions \( K_k(x) \) are independent of \( x \).

Within the class of Markov processes, the simplest are those described by the Fokker-Planck equation (7-185), where the coefficients \( K_k(x) \) constants and the coefficients \( K_k(x) \) are the linear functions

\[
K_k(x) = -\sum_{n=1}^{N} \beta_{km} x_n + b_k, \quad k = 1, 2, \ldots, N
\]

(7-217)

This situation corresponds to the case of a multidimensional, Gaussian Markov process for which it can be shown that each component \( x_k(t) \) of the process \( \{x(t)\} \) has a rational spectral density. Although we do not prove it here, the converse assertion is also true: every Gaussian process with a rational spectral density can be represented as a component of a multidimensional, Gaussian Markov process.

### 7.13 The Relationship Between the Fokker-Planck Equation, Stochastic Field Theory, and Continuous Markov Processes

This section serves as a summary of most of the material of this chapter. It also provides us a summary of most of the material given in Chapter 6. The equations, which we refer to as the equations for stochastic field theory, Chapter 6, are four partial differential equations between five vector quantities, \( \mathcal{E}, \mathcal{F}, \mathcal{Q}, \mathcal{B}, \) and \( \mathcal{H} \), together with the scalar concept of the probability density function \( p(x, t|x_0, t_0) \). As previously noted, these equations are analogous to Maxwell’s equation arising in electromagnetic theory if \( p(x, t|x_0, t_0) \) is replaced.
by the volume density of electric charge. A theorem of vector analysis, due to Helmholtz, states that a vector is specified when its divergence and curl have been specified. We therefore have the differential and integral formulation

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \oint \mathbf{E} \cdot d\mathbf{r} = -\frac{\partial}{\partial t} \oint \mathbf{B} \cdot dS
\]

\[
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \oint \mathbf{H} \cdot d\mathbf{r} = \oint (\mathbf{J} + \mathbf{D}) \cdot dS
\]

\[
\nabla \cdot \mathbf{D} = p(x, t|x_0, t_0) \quad \oint \mathbf{D} \cdot d\mathbf{S} = \oint_v p(x, t|x_0, t_0) \, dx
\]

\[
\nabla \cdot \mathbf{B} = 0 \quad \oint \mathbf{B} \cdot d\mathbf{S} = 0
\]

(7-218)

(7-219)

The vector quantities are related at each point through

\[
\mathbf{D} = \varepsilon \mathbf{E} \quad \mathbf{B} = \mu \mathbf{H}
\]

\[
\mathbf{J} = -\mu \mathbf{E} - \nu p(x, t|x_0, t_0)
\]

(7-220)

with

\[
\mathbf{J}_k(x; t) = \nu_k p(x, t|x_0, t_0)
\]

\[
= \left\{ \left[ K_k(x, t) - \frac{1}{2} \nabla' \cdot K(x, t) \right] p(x, t|x_0, t_0) \right\}
\]

(7-221)

for all \( k = 1, 2, \ldots, N \). Whatever the form of the intensity coefficients \( K_x(x, t) \) and \( K_{x\beta}(x, t) \) (the notation of using subscripts from the Greek alphabet implies that \( x \) and \( \beta \) take on all possible values), the general Fokker-Planck equation is a general diffusion equation in position space.

For the potential case, Poisson's equation becomes, from (6-69) and (7-201),

\[
\nabla^2 V(x) = -\exp \left[ \frac{-U(x)}{\epsilon} \right]
\]

(7-222)

Thus the potential of \( p_{x_i}(x) \) is related to the potential function \( U(x) \) of the dynamical system through Poisson's equation.

7.14 Dynamical Systems Described by a Set of Stochastic Differential Equations and the Diffusion Approximation

In later work we shall be dealing with a dynamical system (a SCS) for which the system response \( p(y; t) \) is governed by the set of stochastic differential equations
\[ \begin{align*}
\dot{y}_0(t) &= F_0[y, n_0(t)] \\
\dot{y}_1(t) &= F_1[y, n_0(t)] \\
&\vdots \\
\dot{y}_N(t) &= F_N[y, n_0(t)]
\end{align*} \] (7-223)

which define the vector random process \( \{y(t)\} \) when the excitation process \( \{n_0(t)\} \) is arbitrary. For fixed values \( y_1(t), \ldots, y_N(t) \), the expressions \( F_1[y, n_0(t)], \ldots, F_N[y, n_0(t)] \) are determined by the dynamics of the system (which we assume known); in fact, they are random functions of time. Selecting the initial values \( y_k(t_0) = y_{k0}; k = 0, 1, \ldots, N \), a question that immediately arises is, for an arbitrary random process \( \{n_0(t)\} \), when can one approximate or replace the vector process \( \{y(t)\} \) by a vector Markov process \( \{x(t)\} \)? Although beyond the scope of this book, it can be shown (Ref. 14) that if the correlation time of \( \{n_0(t)\} \) is much less than system correlation time (which is inversely related to the system bandwidth), then system response \( p(y; t) = p(y, t|y_0, t_0) \) can be obtained from the Fokker-Planck equation

\[ \nabla \cdot \mathcal{J}(y; t) + \frac{\partial p(y; t)}{\partial t} = 0 \] (7-224)

where

\[ \mathcal{J}(y; t) \triangleq \left\{ \left[ K_k(y, t) - \frac{1}{2} \sum_{l=1}^N \frac{\partial}{\partial y_l} K_{lk}(y, t) \right] p(y; t) \right\} \] (7-225)

and

\[ K_k(y, t) \triangleq E[F_k[y, n_0(t)]] \] (7-226)

\[ K_{lk}(y, t) \triangleq \int_{t_0 - t}^0 \frac{1}{2} \text{Cov} \{F_l[y, n_0(t)], F_k[y, n_0(t + \tau)]\} \, \text{d}\tau \]

Notice that \( y \) is held fixed in calculating the averages \( E[F_k] \) and the covariance functions in the integrand. For \( t - t_0 \) much greater than the correlation time of \( \{n_0(t)\} \), the lower limit \( t_0 - t \) can be replaced by minus infinity. The replacement of a non-Markov process by an equivalent Markov process is called the diffusion approximation and is equivalent to neglecting the higher-order intensity coefficients.

As an example, consider the stochastic differential equation, with restoring force \( h(y) \), given by

\[ \dot{y}(t) = F[y(t), n(t)] = h(y) + g(y)n_0(t) \] (7-227)
Here \( h(y) \) and \( g(y) \) are assumed known, while \( \{ n_0(t) \} \) is a stationary, zero-mean process with intensity coefficient and correlation function given by

\[
K_{n_0} = \int_{-\infty}^\infty R_{n_0}(\tau) \, d\tau \quad R_{n_0}(\tau) = K_{n_0} \delta(\tau)
\]

(7-228)

that is, we have replaced \( \{ n_0(t) \} \) by a delta-correlated random process. From (7-227) we find that

\[
K_1(y) = h(y) \quad K_{00}(y) = K_{n_0} g_0^2(y)
\]

(7-229)

so that the steady-state p.d.f. can be easily determined from (7-83).

We can also solve the inverse problem of finding the stochastic differential equation that gives rise to a given Fokker-Planck equation. In general, this problem does not have a unique solution; however, the solution will be unique if \( \{ n_0(t) \} \) is a Gaussian, delta-correlated, random process, say \( \{ n(t) \} \), with zero mean and correlation function \( R_n(\tau) = K_n \delta(\tau) \). Then setting

\[
h(y) = K_1(y) \quad g(y) = \sqrt{\frac{K_{00}(y)}{K_n}}
\]

(7-230)

we have that the arbitrary Fokker-Planck equation (7-49) corresponds to the stochastic differential equation

\[
\dot{x}(t) = K_1(x) + \sqrt{\frac{K_{00}(x)}{K_n}} n(t)
\]

(7-231)

In general, then, we can associate the more "complicated" equation (7-227) with the "simpler" equation (7-231). The reason, of course, in that (7-227) and (7-231) have the same Fokker-Planck equation. When this is true, the two dynamical systems are said to be \textit{stochastically equivalent}. This method of replacing or identifying a complicated dynamical system with a stochastically equivalent, simpler dynamical system turns out to be useful later.

\section*{7.15 The Escape of Particles over a Potential Wall}

As a final illustration of the application of the principles of Brownian motion, we consider the problem of the escape of particles over potential walls. The solution to this problem has important applications in a variety of physical, chemical, and astronomical problems (Ref. 6). Our concern in SCS theory will be to determine the average number of phase-jumps (cycles slipped) per unit of time that occur in a SCS. The general problem concerns the rate at which
particles will escape over the potential barrier as a consequence of Brownian motion. In the most general form, however, its solution is most likely to have considerable difficulties.

Limiting ourselves for simplicity to a one-dimensional problem, we consider a particle in a potential field $U(x)$ of the type illustrated in Fig. 7-9. We consider an ensemble—cloud of smoke particles, for instance—moving in the potential field $U(x)$ without any mutual interference, and we assumed that the particles are initially caught in the potential well at point $x = x_a$. The general problem concerns the rate at which particles will escape over the potential wall as a consequence of the Brownian motion and eventually wind up in the potential well at $x = x_p$. We know from quantum mechanics that an orbital electron of an atom (analogous to a particle here) can exist only in discrete energy levels, say $U_n (n = 0, 1, 2, \ldots)$. The atom can absorb (or emit) pulses of electromagnetic energy (photons) in discrete amounts given by the difference in energy levels. If a large number of atoms (particles) is in thermal equilibrium and $U(x_a) - U(x_p) \gg 1$, the ratio of the population of particles of the two energy levels is given by the Boltzmann factor

$$\frac{n_a}{n_b} = \exp \left\{ - [U(x_a) - U(x_p)] \right\} \tag{7-232}$$

In an analogous way, this ratio may be used to represent the ratio of the number of phase-jumps “to the right” to the number of phase-jumps “to the left” in a tracking system in much the same way as it is used to determine the current flow across a transition region ($p - n$ junctions) in semiconductors. The
“deeper” the wells—that is, larger the potential at $x_c$—the less net current flow one will have from $a$ to $b$.

7-16 Further Studies

An advanced mathematical treatise on the subject of random walk can be found in the book by Spitzer (Ref. 1). Chandrasekhar (Ref. 6) and Kac (Ref. 6) present fundamental discussions of the random walk problem; the references cited in each of these papers form a working bibliography of the literature up to about 1940–1945. Spitzer’s work (Ref. 1) and the books by Doob (Ref. 2), Parzen (Ref. 3), and Feller (Ref. 4) describe more recent work and contain supplementary references to the earlier investigations.

In 1801 Robert Brown sailed to study plant life off the coast of Australia. Although mathematicians remember him as the discoverer of Brownian motion, his biography in the *Encyclopedica Britannica* makes no mention of this discovery. In fact, Brown did not discover Brownian motion, for in his 1828 paper (Ref. 15) he mentions one precursor and ten more in his 1829 paper (Ref. 16).

As mentioned earlier, Bachelier (Ref. 17) unknowingly was the first to study the theory of Brownian Motion around 1900 in connection with the stock market; however, it was not until 1905 that Einstein (Ref. 18) advanced an acceptable theory. Einstein’s Brownian-motion model was subsequently used by Perrin to obtain an accurate determination of Avogadro’s number, for which Perrin received the Nobel Prize. Smoluchowski, Fürth, and others (Ref. 19) generalized much of Einstein’s earlier work. Ornstein and Uhlenbeck (Ref. 20) and Wang and Uhlenbeck (Ref. 6) present a summary with references relative to the theory of Brownian motion accomplished prior to 1945. More recently, Breiman (Ref. 21) and Nelsen (Ref. 22) give separate accounts of the theory of Brownian motion. The early work of Fokker (Ref. 23), Planck (Ref. 24), and Blan-Lapierre, Foret, (Ref. 25) is excellent for purposes of further study. Fundamental material relative to principles of statistical mechanics can be found in books by Tolman (Ref. 26) and Landau and Lifshitz (Ref. 27). References 2 and 28 to 31 represent books on Markov processes. Recently Fisher and Stear (Ref. 32) have obtained some interesting integropartial differential equations of flow for the case of jump processes. Finally, we should mention that the early work on the first-passage time problem was accomplished by Siegert (Ref. 33) and Darling and Siegert (Ref. 34). For an excellent survey of papers pertaining to applications of the FP equation to the study of nonlinear vibrations and control systems, as well as more rigorous discussions pertaining to stability and existence of the stationary solutions, see the tutorial paper by Fuller (Ref. 35).
Problems*

7-1 Find the steady-state p.d.f. of the random walk problem if the distribution
$P(n|N)$ is reduced modulo $m$, $m$ any integer greater than two.

7-2 Consider a one-dimensional system of particles undergoing diffusion in the
presence of the linear restoring force $K_1(x) = -\beta x$ for $|x| \leq \infty$. Assume that
the process is continuous and that $K_2(x, t) = D$.
(a) Find the probability current $J(x; t)$ if $p(x; t_0) = \delta(x - x_0)$.
(b) Find the correlation function $R_x(\tau)$.
(c) Find the spectral density $S_x(\omega)$.

7-3 A phase-tracking system is designed such that the restoring force (coefficient
of dynamical friction) is given by

$$K_1(x, t) = \Lambda_0 - AK \sin x,$$

and the diffusion-of-velocity coefficient is given by

$$K_2(x, t) = \frac{N_0K^2}{4}$$

Find
(a) The probability current density due to the signal in terms of $p(x; t)$.
(b) The probability current density due to the noise in terms of $p(x; t)$.
(c) The system Fokker-Planck equation.
(d) The p.d.f. of the steady-state phase error if $p(\pi) = p(-\pi)$.
(e) The probability current density due to the signal in the steady state.
(f) The probability current density due to the noise in the steady state.
(g) The steady-state probability current density.

7-4 Establish the validity of (7-97), (7-101), and (7-106).

7-5 Find the ratio, $N_+/N_-$, of the probability currents that flow in the steady state
of Prob. 7-3.

7-6 Assume that the intensity coefficients $K_k(x)$ are characterized by (7-217) and
that the $K_k(x)$'s are independent of $x$. Show that the steady-state multidimen-
sional p.d.f. is Gaussian.

7-7 Develop an expression for the moments of the first-passage time for the Markov
process described in Prob. 7-3.

References

7-1 Spitzer, F., Principles of Random Walk. Van Nostrand, Princeton, N. J.
1964.

*Problems that exercise the theory in this chapter are reserved for Chapter 8 after
stochastic differential equations for dynamical systems are further discussed.
References


7-23 Fokker, A. D., Die mittlere Energie rotierender elektrischer Dipole im Strahlungsfeld, Ann. Physik, 43 (1914), 810-820.


8

STOCHASTIC DIFFERENTIAL EQUATIONS AND THE FOKKER-PLANCK EQUATION

8-1 Introduction

This chapter discusses the modeling of a large and important class of dynamical systems by means of stochastic differential equations. Such equations are frequently referred to in the literature as stochastic state-space equations. Our main purpose here is to present background material and to introduce analytical methods that unite the classical theory of Markov processes given in Chapter 7 to certain elements of the vast body of work on the subject of nonlinear filter theory and Itô stochastic integration theory. Stochastic differential equations that represent models for dynamical systems are considered; and the idea that such equations, which are the stochastic analog of deterministic state equations (Ref. 1), describe certain Markov processes is developed.

The chapter begins with formal remarks concerning the difficulties associated with the classical definition of white noise and explains why white Gaussian noise is best treated as the formal derivative of Brownian motion (the Wiener process). Next, the stochastic or Itô integral is defined and Itô's differential rule is presented. Use of this rule is demonstrated by deriving the multidimensional FP equation. This derivation has the powerful advantage of
showing how the coefficients that characterize the stochastic differential equation of system operation enter into the FP equation in a natural way. Using the Itō calculus, the intensity coefficients, for both the multidimensional FP equation given in Chapter 7, Section 7-10, and Pawula’s reduced, generalized FP equation (Ref. 2) in one dimension, are derived. The result of this derivation is that two methods of nonlinear system analysis—the sequence method and the conditional expectation method—are motivated. These two methods for solving the FP equation will be discussed in later chapters. It is necessary to assume that the reader is familiar with matrix algebra. Appendix I defines the matrix notation needed in what follows.

8-2 State-Space Representation of a Dynamical System

Snyder (Ref. 3) showed that state-space representation of a dynamical system is extremely useful for a large class of communication system problems in which a stochastic message is transmitted by either a linear (AM) or a nonlinear modulation (FM or PM) scheme through a randomly time-varying channel. He wrote stochastic state-space equations for the message, the modulator, and the channel and used nonlinear filtering theory to produce approximately optimum demodulators.

The primary interest in what follows is in analysis, not synthesis. State-space representation is just as useful for analysis at it is for synthesis. For the case of analysis or performance evaluation, a state-space model can be written for the entire system including the receiver demodulator. The linear portions of the system can be represented in state-space form by the methods of Zadeh and Desoer (Ref. 1) and the nonlinear elements can be included to produce the overall state-space equations

$$\frac{dx(t)}{dt} = f[t, x(t)] + B(t)u(t) \quad z(t) = Cx(t) \quad (8-1)$$

where $x(t)$ and $f[t, x(t)]$ are column vectors, $C$ is a row vector, $B(t)$ a matrix, $u(t)$ a column vector of white noise, and $z(t)$ a scalar. The quantity $z(t)$ is, in general, a linear combination of the state variables and is the system output for the purpose of analysis. It can represent either an estimate of the message (the desired output signal) or an error signal. For example, when the case of coherent demodulation is considered, $z(t)$ will represent either the frequency tracking error or the phase error.

It is worth noting that the state-space equations in (8-1) can be written in many different levels of generality. Doing so allows for the inclusion of different classes of dynamical systems. However, for most communication system problems, (8-1) is sufficiently general. It allows essentially any model
made up of blocks consisting of memoryless, nonlinear functions and linear, lumped parameter filters with white noise or deterministic signal inputs. For most problems in communications, however, the system model can be simplified by removing the carrier frequency to produce an equivalent baseband model. This simplification usually has the effect of removing the additional time dependence from $f[t, x(t)]$ and $B(t)$ to produce a stationary model. Although it is rare in communications, still other problems may require that the model be generalized further so that $B(t)$ depends also on $\{x(t)\}$. In such cases it will be written as $B[t, x(t)]$. Even though the emphasis in what follows is on the stationary model, the model is generalized whenever this can be done without sacrificing clarity.

Equation (8-1) will be discussed in more detail in the next section; however, it is important to note that it can be shown (Ref. 4) that for a reasonable function $f[t, x(t)]$, (8-1) produces a vector Markov process. It is shown in many references (Refs. 5, 6, 7, 8) that if $\{x(t)\}$ is a Markov process satisfying certain conditions, then its p.d.f. satisfies the FP equation (7-183) given in Chapter 7. It should also be noted that the solution $p(x; t)$ to (7-183) is actually the transition p.d.f. $p(x, r|x_0, t_0)$ based on the system starting at time $t_0$ from the initial state $x_0$. However, if the starting conditions are fixed and are implied in the definition of the probability measure, $p(x; t)$ will be a total p.d.f. As noted earlier, the FP equation is a multidimensional, parabolic partial differential equation that cannot be solved exactly except in a few isolated cases. It is from this form that the sequence method given in a later chapter will be derived. Pawula (Ref. 2) shows that the FP equation is not restricted to Markov processes but can be applied to a much larger class. In particular, he shows that the process $\{z(t)\}$ satisfies a one-dimensional version of (7-183) with $x$ replaced by $z$ and that the intensity coefficients are still defined by (7-186) and (7-187). The only basic difference is apparent when the intensity coefficients, evaluated in Section 8-4, are found to require knowledge of certain conditional expectations. It is from this one-dimensional equation that the conditional expectation method given in a later chapter is developed. This method is essentially a generalization of the approach developed by the author and does not require integrating out the state variables to arrive at a reduced FP equation.

8-3 Basic Rules of Itô Calculus

Equation (8-1) represents a mathematical model of a dynamical system (e.g., a synchronous control system) forced by white Gaussian noise. White Gaussian noise, like the Dirac $\delta$-function, does not have any rigorous meaning by itself; however, both are of great value in analysis and both can be made mathematically rigorous only in terms of an integral.
The appropriate integral that gives white noise meaning is the stochastic or Itô integral. Although analysis of many problems can be accomplished using only engineering approximations and intuitive feelings for white noise, we shall see that some of the basic rules of Itô calculus often make even the analysis more simple. For example, evaluation of the intensity coefficients for the FP equation can be accomplished with ease by using a few rules of Itô calculus. It is the purpose of this section to present some of these rules without rigorous proofs and, in order to reach a larger audience, without relying on the deeper concepts of measure theory. The proofs are readily available in the literature (see Refs. 4, 5, 6, 7, 8).

8-3.1 White Gaussian Noise (WGN) and Brownian Motion (the Wiener Process)

The underlying idea is the description of the white Gaussian noise (WGN) process \( n(t) \) as the formal derivative of Brownian motion; that is,

\[
n(t) = \frac{db(t)}{dt} \quad \text{or} \quad b(t) = \int_{t_0}^{t} n(\tau) \, d\tau
\]  

(8-2)

where \( \{b(t)\} \) is a scalar Brownian motion (Wiener) process and \( b(t_0) = 0 \). In Chapter 1 it was shown that \( \{b(t)\} \) is a Gaussian Markov process with zero-mean, orthogonal increments and a variance that increases linearly with time.

\[
E[b(t)] = 0
\]  

(8-3)

\[
E[(b(t_2) - b(t_1))^2] = (t_2 - t_1) \frac{N_0}{2}
\]  

(8-4)

and

\[
E[(b(t_3) - b(t_1))(b(t_4) - b(t_1))] = 0
\]  

(8-5)

where \( t_1 \leq t_2 \leq t_3 \leq t_4 \). Actually (8-5) is the definition of orthogonal increments.

As noted earlier, (8-2) does not have meaning any more than does WGN; however, we presently show that (8-2) represents a limiting quantity that fits with the intuitive concepts of WGN. First we define an approximation \( n_h(t) \) to \( n(t) \) by

\[
n_h(t) \triangleq \frac{b(t + h) - b(t)}{h}
\]  

(8-6)

where \( h > 0 \). Now \( n(t) \) is the limit of \( n_h(t) \) as \( h \) goes to zero. Also, we note that the mean value of \( \{n_h(t)\} \) is zero, while its variance is given by
which goes to infinity as \( h \) approaches zero. On the other hand, the covariance of \( \{n_h(t)\} \) is given by

\[
E[n_h(t) n_h(t)] = \frac{1}{h^2} E[(b(t + \tau + h) - b(t + \tau)) [b(t + h) - b(t)]]
\]

so that the right-hand side of (8-8) can be evaluated by adding and subtracting terms in order to apply (8-4) and (8-5). Thus the covariance function of the process \( \{n_h(t)\} \) becomes

\[
E[n_h(t + \tau) n_h(t)] = \begin{cases} 
\frac{N_0}{2} \left( \frac{h - |\tau|}{h^2} \right) & \text{if } |\tau| \leq h \\
0 & \text{if } |\tau| > h 
\end{cases}
\]

The spectral density is given (see Chapter 1) by the Fourier transform of (8-9); that is,

\[
S_h(\omega) = \frac{N_0}{2h^2} \int_{-h}^{h} (h - |\tau|) e^{-i\omega \tau} \, d\tau = \frac{N_0}{\omega^2 h} [1 - \cos(\omega h)]
\]

If \( h \) is small, the spectral density in (8-10) actually approaches \( N_0/2 \). Therefore WGN can be thought of as a limiting process that approaches the derivative of Brownian motion just as the Dirac \( \delta \)-function can be thought of as a limiting process that approaches the derivative of a unit step.

According to Doob (Ref. 5), white noise can be given a formal definition only in terms of an integral. This integral can take any of the forms

\[
\int_{t_0}^{t} g(\tau) n(\tau) \, d\tau = \int_{t_0}^{t} g(\tau) \left[ \frac{db(\tau)}{d\tau} \right] \, d\tau = \int_{t_0}^{t} g(\tau) \, db(\tau)
\]

where \( g(\tau) \) is an arbitrary function, which can, in its most general form, be a random process. It must, however, be completely determined by the past history of \( n(t) \) or, equivalently, of \( b(t) \). None of the integrals in (8-11) is defined by ordinary methods. The last integral on the right in (8-11) is called a stochastic integral and is interpreted as the Itô integral (Refs. 4, 8). It is not an ordinary Stieltjes integral because \( b(t) \) is not of bounded variation; however, it can be defined in a similar manner to the Stieltjes integral by stochastic methods following Doob (Ref. 5).
Before the stochastic integral is defined more precisely, it is useful to
generalize the discussion to include vector processes. If we let \( \mathbf{b}(t) \) represent a
column vector of Brownian motion—that is, a vector process each of whose
components is Brownian motion—then it follows that

\[
E[(\mathbf{b}(t_2) - \mathbf{b}(t_1))\mathbf{b}(s)] = 0 \quad (8-12)
\]
\[
E[(\mathbf{b}(t_2) - \mathbf{b}(t_1))\mathbf{b}(t_2) - \mathbf{b}(t_1)\mathbf{b}(s)] = (t_2 - t_1)Q \quad (8-13)
\]
\[
E[(\mathbf{b}(t_2) - \mathbf{b}(t_1))\mathbf{b}(t_2) - \mathbf{b}(t_1)]' = 0 \quad (8-14)
\]

where \( s \leq t_1 \leq t_2 \leq t_3 \leq t_4 \) and \( Q \) is a symmetric, positive-definite matrix
that is taken without loss of generality in many of the later sections to be the
identity matrix multiplied by \( N_0/2 \). Notice that the expectations in (8-12),
(8-13), and (8-14) are conditioned on \( \mathbf{b}(s) \), where \( s \) represents any past value
of time. The orthogonal increments in (8-14) are sufficient to show that an
increment of Brownian motion is independent of all past values of the process
(see Chapter 1). The condition on \( \mathbf{b}(s) \) is, therefore, not necessary; however,
it is included for clarity later.

The Brownian motion process is usually assumed to start at zero at
some time \( \tau \) seconds in the past, perhaps at \( \tau = -\infty \). In order to show indepen-
dence of past values, assume that \( \tau \leq t_1 \leq t_2 \leq t_3 \); then from (8-14) we see that

\[
E[(\mathbf{b}(t_3) - \mathbf{b}(t_2))\mathbf{b}'(t_1)] = E[(\mathbf{b}(t_3) - \mathbf{b}(t_2))\mathbf{b}(t_1) - \mathbf{b}(\tau)]' = 0 \quad (8-15)
\]

Since the increments themselves each have zero mean, then

\[
E[(\mathbf{b}(t_3) - \mathbf{b}(t_2))\mathbf{b}'(t_1)] = E[\mathbf{b}(t_3) - \mathbf{b}(t_2)]E[\mathbf{b}'(t_1)] \quad (8-16)
\]

The process \( \{\mathbf{b}(t)\} \) is Gaussian, so that (8-16) implies that an increment of
Brownian motion is independent of all past values of the process.

### 8-3.2 The Stochastic or Itô Integral

First we define \( \{G(t)\} \) as a matrix random process (corresponding in the
scalar case to \( g(t) \)) that is completely determined by values of \( \mathbf{b}(\tau) \) for \( \tau \leq t \).
The stochastic or Itô integral is now defined by first defining it for a step
function \( G_n(t) \) (a function with only a finite number of values) subject to these
same restrictions as \( \{G(t)\} \). The definition is then extended via a limiting argu-
ment to include a larger class of functions. We begin by selecting \( G_n(t) \) to be the
\( N \)th function in a sequence that converges to \( G(t) \), where \( G_n(t) \) is right-
continuous and takes on only a finite number of values. Its points of discon-

-continuity within the interval \( t_0 \leq \tau \leq t = t_n \) are assumed to be contained in the
The stochastic integral of \( G_N(t) \) is defined in accordance with Itô (Ref. 4) and Doob (Ref. 5) by

\[
\int_{t_0}^{t} G_N(\tau) \, d\mathbf{\beta}(\tau) \triangleq \sum_{n=0}^{N-1} G_N(t_n)[\mathbf{\beta}(t_{n+1}) - \mathbf{\beta}(t_n)]
\]  

(8-17)

In order to determine the properties of (8-17), we must establish that \( \{G_N(t_n)\} \) is independent of \( [\mathbf{\beta}(t_{n+1}) - \mathbf{\beta}(t_n)] \) for each \( n \). We recall that \( \{G(t)\} \), and therefore \( \{G_N(t)\} \), is restricted to be a matrix random process whose value is completely determined by the past history of \( \{\mathbf{\beta}(t)\} \). It is already established that \( [\mathbf{\beta}(t_{n+1}) - \mathbf{\beta}(t_n)] \) is independent of all past values of the \( \{\mathbf{\beta}(t)\} \) process; therefore it is independent of \( \{G_N(t_n)\} \). Using this independence, the following important properties of the stochastic integral are easily established directly via (8-17). These properties are written in terms of \( G(t) \) and are given by

\[
E\left[ \int_{t_0}^{t} G(\tau) \, d\mathbf{\beta}(\tau) \big| \mathbf{\beta}(s) \right] = E\left[ \int_{t_0}^{t} G(\tau) \, d\mathbf{\beta}(\tau) \right] = 0 
\]  

(8-18)

\[
E\left[ \int_{t_0}^{t} G(\tau) \, d\mathbf{\beta}(\tau) \bigg| \int_{t_0}^{t} G(\tau) \, d\mathbf{\beta}(\tau) \bigg| \mathbf{\beta}(s) \right] = \int_{t_0}^{t} E[G(\tau)QG'(\tau)|\mathbf{\beta}(s)] \, d\tau
\]  

(8-19)

for any \( s < t_0 \). Since they hold given \( \mathbf{\beta}(s) \) for any \( s \leq t_0 \), they also hold given any other random variable that is generated by the process \( \{\mathbf{\beta}(s)\} \) for \( s \leq t_0 \). This fact will be used when the intensity coefficients are evaluated for the FP equation in the next section.

If \( \{G(t)\} \) takes on an infinite set of values, the definition of the stochastic integral must be extended. This is accomplished by forming a sequence of functions \( G_n(t) \), each of which has only a finite number of values and whose limit is \( \{G(t)\} \). The stochastic integral of \( G(t) \) is then defined by

\[
\int_{t_0}^{t} G(\tau) \, d\mathbf{\beta}(\tau) \triangleq \lim_{N \to \infty} \int_{t_0}^{t} G_N(\tau) \, d\mathbf{\beta}(\tau)
\]  

(8-20)

The rigorous details of this extension are omitted; however, they are readily available in the literature (see Doob, Ref. 5). It can be seen that the properties (8-18) and (8-19) hold for this extended integral as well.

The stochastic differential equation (8-1) can now be given a formal meaning. In its most general form, both \( f \) and \( B \) can be allowed to depend on both \( x(t) \) and \( t \) so that (8-1) becomes, in the notation of Itô calculus,

\[
dx(t) = f[t, x(t)] \, dt + B[t, x(t)] \, d\mathbf{\beta}(t)
\]  

(8-21)

This notation is simply a shorthand notation for the integral equation
\[ x(t + \tau) - x(t) = \int_t^{t+\tau} f[\sigma, x(\sigma)] \, d\sigma + \int_t^{t+\tau} B[\sigma, x(\sigma)] \, dB(\sigma) \quad (8-22) \]

where we have made use of the stochastic integral in (8-20) to give meaning to the white-noise input. Notice that the difference on the left-hand side of (8-22) is defined in computing the intensity coefficients. Suppose that \( f(t, x) \) and \( B(t, x) \) are both globally Lipschitz in \( x \). This means that

\[ ||f(t, x_2) - f(t, x_1)|| \leq k ||x_2 - x_1|| \quad (8-23) \]

and

\[ ||B(t, x_2) - B(t, x_1)|| \leq k ||x_2 - x_1|| \quad (8-24) \]

for some constant \( k \) and for all vectors \( x_1 \) and \( x_2 \). Itô (Ref. 4) proves that given this condition, (8-21) has a unique global solution \( \{x(t)\} \) that is a vector Markov process of the diffusion type (see Chapter 7) with continuous sample functions (almost everywhere) and that \( \{x(t)\} \) is independent of \( \{B(t)\} \) for any \( \tau \leq t \).

This is a significant result in that it ensures that the state vector of almost any dynamical system (linear or nonlinear) driven by white noise is a vector Markov process. This does not imply that the individual states are Markov, only the entire state vector. The existence and uniqueness of solutions to (8-21) are discussed in Ref. 7.

### 8-3.3 The Itô Differential Rule

One of the most important rules for analysis using (8-21) to model a dynamical system can be summarized in the following: For simplicity, the quantities in (8-21) are, for the moment, taken as one dimensional. Suppose that a limiting solution to some problem is to be obtained by expanding a nonlinear function \( F[\Delta x(t)] \) in a Taylor series in \( \Delta x(t) \triangleq x(t + \Delta t) - x(t) \) by eliminating all terms that go to zero faster than \( \Delta t \). Therefore the function \( F \) can be expanded as

\[ F(\Delta x) = F(0) + \frac{dF(0)}{d(\Delta x)} \Delta x + \frac{1}{2} \frac{d^2F(0)}{d(\Delta x)^2} (\Delta x)^2 + \cdots \quad (8-25) \]

The natural tendency is to retain only the first two terms; however, the third term must also be retained, as we now show. We see from (8-21) that \( [\Delta x(t)]^2 \) contains \( [\Delta \beta(t)]^2 \) and from (8-13) that \( E[|\Delta \beta(t)|^2] = Q \Delta t \) so that the expected value of this third term approaches zero with \( \Delta t \), not \( (\Delta t)^2 \). If this term is eliminated, an error will result. Itô's lemma (Refs. 4, 7, 8) solves this problem. Essentially it states that a Taylor series expansion such as (8-25) is valid for a stochastic process; however, if terms of order \( \Delta t \) are to be retained, the third
term must be kept. In matrix notation this implies that $E[\Delta \mathbf{B}(t) \Delta \mathbf{B}'(t)]$ must be replaced by $\mathbf{Q} \Delta t$.

More formally, Itô's lemma (differential rule) is stated as follows: Let $F[t, x(t)]$ be a nonlinear function of $t$ and $x(t)$ that is differentiable in $t$ and twice differentiable in $x$. If we assume that

$$\int_0^t B_k^j \{ \tau, x(\tau) \} \, d\tau < \infty \quad (\text{all } k \text{ and } j)$$

(8-26)

and

$$\int_0^t |f_k \{ \tau, x(\tau) \} | \, d\tau < \infty \quad (\text{all } k)$$

(8-27)

for any finite $t$, then $F[t, x(t)]$ satisfies the Itô differential rule

$$dF[t, x(t)] = \frac{\partial F[t, x(t)]}{\partial t} \, dt + \sum_k \frac{\partial F[t, x(t)]}{\partial x_k} \, f_k[t, x(t)] \, dt$$

$$+ \frac{1}{2} \sum_{k,j} \{ \mathbf{B}[t, x(t)] \mathbf{Q} \mathbf{B}'[t, x(t)] \}_{kj} \frac{\partial^2 F[t, x(t)]}{\partial x_k \partial x_j} \, dt$$

$$+ \sum_{k,j} \frac{\partial F[t, x(t)]}{\partial x_k} B_k^j[t, x(t)] \, d\beta_j(t)$$

(8-28)

Equation (8-28) is a shorthand notation that is formally interpreted as an integral equation similar to (8-22). The third term on the right-hand side in (8-28) is derived from the third term in the Taylor series expansion (8-25). This term is often overlooked in nonlinear system analysis, thus leading to questionable results. We now demonstrate the use of (8-28) by presenting an alternate derivation of the FP equation. Other examples of its use are given in Probs. 8-1 to 8-7. Consider the function

$$F[t, x(t)] = \exp \{ iu' [x(t) - x(t_0)] \}$$

(8-29)

and introduce the shortened notation

$$f = f[t, x(t)] \quad B_{kj}[t, x(t)] = [\mathbf{B}[t, x(t)]]_{kj}$$

$$\mathbf{M}[t, x(t)] = \mathbf{B}[t, x(t)] \mathbf{Q} \mathbf{B}'[t, x(t)]$$

(8-30)

in (8-28). Thus (8-28) can be written as

$$dF[t, x(t)] = \left\{ \sum_k \left( iu_k f_k - \frac{1}{2} \sum_j \{ \mathbf{M}[t, x(t)] \}_{kj} u_k u_j \right) F[t, x(t)] \, dt \right\}$$

$$+ \sum_k \sum_j iu_k B_{kj}[t, x(t)] F[t, x(t)] \, d\beta_j(t)$$

(8-31)
Integrating from $t_0$ to $t$ with $F[t_0, x(t_0)] = 1$ and taking expectations on both sides of (8-31) conditioned on $x_0 = x(t_0)$, we have

$$E[F[t, x(t)]|x_0] = 1 + \int_{t_0}^{t} E\left[ \sum_k (\cdot) F[t, x(t)]|x_0 \right] dt$$  \hspace{1cm} (8-32)

where the term in the large parenthesis is the same as in (8-31). Now the conditional characteristic function of $\{x(t)\}$ is defined by

$$C_x(u|x_0) \triangleq E[\exp(iu'x)|x_0]$$  \hspace{1cm} (8-33)

Differentiating (8-32) with respect to time and using (8-33), we can write

$$\frac{\partial C_x(u|x_0)}{\partial t} = E\left[ \sum_k (\cdot) \exp(iu'x)|x_0 \right]$$  \hspace{1cm} (8-34)

where we have suppressed the time variable. This conditional characteristic function is just the Fourier transform of the transition p.d.f. $p(x; t) = p(x, t|x_0, t_0)$. If we now take the inverse Fourier transform of (8-34) and remove the expectation operator by integrating over $p(x; t)$, using the dummy variable $y$, then

$$\frac{\partial p(x; t)}{\partial t} = \frac{1}{2\pi} \sum_k \frac{\partial}{\partial x_k} \left\{ \int_{-\infty}^{\infty} \left( -f_k[t, y(t)] + \frac{1}{2} \sum_j [M[t, y(t)]_{kj} \frac{\partial}{\partial y_j}] \right) \times p(y, t|x_0, t_0) \exp[iu'(y - x)] dy du \right\}$$  \hspace{1cm} (8-35)

The integration on $u$ becomes $2\pi \delta(y - x)$, so that the resulting integration over $y$ yields

$$\frac{\partial p(x; t)}{\partial t} = -\sum_k \left[ \frac{\partial}{\partial x_k} (f_k[t, x(t)] - \frac{1}{2} \sum_j \frac{\partial}{\partial x_j} [M[t, x(t)]_{kj}] p(x; t) \right]$$  \hspace{1cm} (8-36)

which is recognized as the FP equation (7-183) wherein the $k$th projection of the probability current density, $J_k(x; t)$ in (7-184), is given by the terms in the large brackets. This is a striking and powerful result! In essence, the intensity coefficients, (7-186) and (7-187), which characterize the FP equation (7-183) are found by inspection from the stochastic differential equation (8-21) of system operation; that is,

$$K_k(x, t) = f_k[t, x(t)]$$

$$K_k(x, t) = [B[t, x(t)]]QB'[t, x(t)]_{kj}$$  \hspace{1cm} (8-37)
where we have dropped the superscript on the coefficients without loss in
generality. We now evaluate these coefficients using (8-22).

8-4 Evaluation of the FP Intensity Coefficients Using Itô Calculus

Itô calculus, as described in the previous section, provides a simple
method of relating the intensity coefficients in the FP equation to the stochastic
differential equation of the dynamical system involved. Even though a simpler
model is used in the later sections, the intensity coefficients will first be evalu-
ated for the system model defined by

\[
dx(t) = f(t, x(t)) \, dt + B(t, x(t)) \, dB(t) \quad (8-38)
\]

\[
z(t) = Cx(t) \quad (8-39)
\]

Then by making use of Pawula’s results (Ref. 2), the fact that the FP equa-
tion holds for non-Markov processes as well as for Markov processes, the
intensity coefficients are evaluated for the non-Markov process \{z(t)\}.

8-4.1 Evaluation of the FP Intensity Coefficients for the
Process \{x(t)\}

The intensity coefficients of \{x(t)\} were defined [see Chapter 7, (7-186)
and (7-187)], by

\[
\{K_k(x, t)\} \triangleq \lim_{\tau \to 0} \frac{1}{\tau} E[x(t + \tau) - x(t)|x(t)] \quad (8-40)
\]

\[
\{K_{k,j}(x, t)\} \triangleq \lim_{\tau \to 0} \frac{1}{\tau} E[\{x(t + \tau) - x(t)\}[x(t + \tau) - x(t)]^j|x(t)] \quad (8-41)
\]

where, in general, \(A = \{A_{k,j}\}\) denotes a matrix whose \(k, j\)th element is \(A_{k,j}\). It
was pointed out in Section 8-3.1 that (8-21) is simply a shorthand notation for
the integral equation given in (8-22). In order to determine the intensity coef-
ficient causing drift in the transition p.d.f., we simply take the expected value
of (8-22) conditioned on \(x(t)\). Thus

\[
E[x(t + \tau) - x(t)|x(t)] = \int_{t}^{t+\tau} E[f(\sigma, x(\sigma))|x(t)] \, d\sigma
\]

\[
+ E \left[ \int_{t}^{t+\tau} B(\sigma, x(\sigma)) \, dB(\sigma)|x(t) \right] \quad (8-42)
\]

when we make use of (8-22). Since \(x(t)\) is determined only by past values of
\(B(t)\), the last term in (8-42) is zero by property (8-18) so that
\[ E[x(t + \tau) - x(t)|x(t)] = \int_{t}^{t+\tau} E[f[\sigma, x(\sigma)]|x(t)] \, d\sigma \quad (8-43) \]

Since the integrand is no longer a random function, the right-hand side of (8-43) is simply an ordinary integral. If (8-43) is inserted into (8-40), then

\[
[K_k(x, t)] = \lim_{\tau \to 0} \frac{1}{\tau} \int_{t}^{t+\tau} E[f[\sigma, x(\sigma)]|x(t)] \, d\sigma
= E[f[t, x(t)]|x(t)] = f[t, x(t)]
\]

(8-44)

and this evaluates the intensity coefficients that cause \( p(x; t) \) to drift.

In order to evaluate the second-order intensity coefficient, we notice that

\[
\int_{t}^{t+\tau} f[\sigma, x(\sigma)] \, d\sigma
\]

(8-45)

is nothing more than an ordinary integral and

\[
\lim_{\tau \to 0} \frac{1}{\tau} \int_{t}^{t+\tau} f[\sigma, x(\sigma)] \, d\sigma = f[t, x(t)]
\]

(8-46)

If terms that go to zero faster than \( \tau \) are eliminated, then

\[
\int_{t}^{t+\tau} f[\sigma, x(\sigma)] \, d\sigma \approx f[t, x(t)]\tau
\]

(8-47)

Using (8-47), the expectation in (8-41) can be expanded as

\[
E[(x(t + \tau) - x(t))[x(t + \tau) - x(t)]|x(t)]
= f[t, x(t)]f'[t, x(t)]\tau^2
+ f[t, x(t)]\left( E\left\{ \int_{t}^{t+\tau} B[\sigma, x(\sigma)] \, dB(\sigma)|x(t) \right\} \right)' \tau
+ E\left\{ \int_{t}^{t+\tau} B[\sigma, x(\sigma)] \, dB(\sigma)|x(t) \right\} f'[t, x(t)]\tau
+ E\left( \int_{t}^{t+\tau} B[\sigma, x(\sigma)] \, dB(\sigma) \bigg| \int_{t}^{t+\tau} B[\sigma, x(\sigma)] \, dB(\sigma) \right)\right. \left. \int_{t}^{t+\tau} B[\sigma, x(\sigma)] \, dB(\sigma)|x(t) \right) \quad (8-48)
\]

If (8-48) is inserted into (8-41) and the properties of the stochastic (Itô) integral (8-18) and (8-19) are used, then

\[
[K_{k_2}(x, t)] = \lim_{\tau \to 0} \frac{1}{\tau} \int_{t}^{t+\tau} E[B[\sigma, x(\sigma)]QB'[\sigma, x(\sigma)]|x(t)] \, d\sigma
\]

(8-49)
Therefore the intensity coefficients, \( K_{k}(x, t) \) of (7-187), are given by the matrix

\[
[K_{k}(x, t)] = B[t, x(t)]QB'[t, x(t)]
\]  
(8-50)

A similar procedure can be used to show that all higher-order intensity coefficients defined in Chapter 7 are zero.

8-4.2 Evaluation of the FP Intensity Coefficients for the Process \([z(t)]\)

The intensity coefficients for the scalar process \([z(t)]\) are now determined from

\[
K_{0}(z, t) \triangleq \lim_{\tau \to 0} \frac{1}{\tau} E[z(t + \tau) - z(t) | z(t)]
\]  
(8-51)

\[
K_{00}(z, t) \triangleq \lim_{\tau \to 0} \frac{1}{\tau} E[[z(t + \tau) - z(t)]^{2} | z(t)]
\]  
(8-52)

using (8-39) and (8-42). Since \( z(t) = Cx(t) \), we notice from (8-51) that

\[
K_{0}(z, t) = C \lim_{\tau \to 0} \frac{1}{\tau} E[x(t + \tau) - x(t) | z(t)]
\]

\[
= CE \left[ \lim_{\tau \to 0} \frac{1}{\tau} E[x(t + \tau) - x(t) | x(t)] | z(t) \right]
\]  
(8-53)

which leads to the result

\[
K_{0}(z, t) = CE [[K_{0}(x, t)] | z(t)] = CE [f[t, x(t)] | z(t)]
\]  
(8-54)

when we use (8-42).

The second-order intensity coefficient \( K_{00}(z, t) \) defined in (8-52) can be broken down in a similar manner with the result that

\[
K_{00}(z, t) = CE [[K_{00}(x, t)] | z(t)] C'
\]

\[
= CE [B[t, x(t)]QB'[t, x(t)] | z(t)] C'
\]  
(8-55)

Therefore (8-45) and (8-46) can be used to characterize the generalized FP equation suggested by Pawula (Ref. 2)—that is, (7-42) and (7-43) with \( K_{0}(x) \) replaced by \( K_{0}(z; t) \), \( K_{0}(x) \) replaced by \( K_{00}(z, t) \) and \( x \) replaced by \( z \).

A comparison of (8-44) and (8-50) with (8-54) and (8-55) reveals important differences in the FP equations for the vector Markov process \([x(t)]\) and for the non-Markov scalar process \([z(t)]\). First the intensity coefficients for the process \([x(t)]\) are easily determined from the parameters of the state-space...
equations in (8-38) and (8-39). The solution, however, to the resulting multi-dimensional FP equation is not available in general and must be approximated. (Later we present a sequence method that is applicable in this case.) On the other hand, it was shown in Chapter 7, Section 7-7.1, that the steady-state solution to the one-dimensional FP equation for the scalar process \( z(t) \) is readily available; however, the intensity coefficients (8-53) and (8-55) are seen to contain conditional expectation. Generally these are not known a priori and must therefore be approximated.

For the purpose of evaluating the intensity coefficients, the stochastic state-space equations (8-38) and (8-39) are quite general. It is rare in communication systems that \( B \) depends on \( \{x(t)\} \). If \( B \) does not depend on \( \{x(t)\} \), the conditional expectation can be eliminated from \( K_{00}(z, t) \) in (8-55) so that only those appearing in \( K_0(z, t) \) must be approximated. The covariance matrix \( Q \) for the white Gaussian noise process is symmetric and positive definite; therefore there exists a matrix \( T \) such that \( Q = TT' \). If the matrix \( B \) in (8-38) is redefined as \( BT \), then \( Q \) can, without loss of generality, be taken to be the identity matrix. Using (8-53) and (8-55), reduced FP equations can be written down by inspection of the equations of state. Thus the method of integrating out the state variables to obtain reduced FP equations is not required. This also alleviates the problem of defining numerous boundary conditions. The preceding concepts will be clarified when we develop the nonlinear theory of SCSs, using the results in this chapter.

8-5 Further Studies

For further information demonstrating how state variable concepts can be applied to a variety of problems in communication theory, the reader is referred to Refs. 3 and 9. Additional reference material can be found in the papers by Van Trees (Ref. 10), Stratonovich (Ref. 11), Kailath (Ref. 12), and Fleming (Ref. 13).

The more advanced reader will notice that we have not touched on the subject of Stratonovich calculus (Ref. 14), for it lies outside the scope of this text. Wong and Zakai (Ref. 15) and Kailath and Frost (Ref. 16) have investigated the relationship between Itô and Stratonovich calculus. Skorokhod (Ref. 7, pp. 14–29) gives a detailed and readable proof of the Itô differential rule for the scalar case. In some recent work McShane (Ref. 17) introduces another definition of the stochastic integral. On the other hand, Fisk (Ref. 18) has introduced a stochastic integral definition, which, like Stratonovich’s, can be evaluated according to ordinary calculus. According to Kailath and Frost (Ref. 16), there is no right definition in the sense that all others are wrong. The real question has to do with which definition is most appropriate for a given purpose. Needless to say, the definition of a stochastic integral has generated some controversy.
As with ordinary differential equations, the existence of a solution to stochastic differential equations when \( f(t, x(t)) \) and \( B(t, x(t)) \) are not globally Lipschitz cannot easily be determined. Recently Stroock and Varadhan (Refs. 19, 20) have given somewhat more general conditions derived on the basis of a martingale approach to the whole theory. Other studies pertaining to the convergence of the transition p.d.f. are given by Khasminskii (Ref. 21).

It is worth noting that Langevin (Ref 22) was among the first to consider "stochastic" differential equations in studying Brownian motion. Later Pontryagin et al. (Ref. 23) proposed to model a randomly disturbed dynamical system by a stochastic differential equation; and, more recently, in the context of control theory, Chuang and Kazda (Ref. 24), Barrett (Ref. 25), and Khazen (Ref. 26). Wonham (Ref. 27) considers various questions pertaining to control systems represented by stochastic differential equations. Wonham (Ref. 27) also gives an extensive and excellent bibliography concerning stochastic differential equations with brief indications of how these equations arise as models for randomly perturbed dynamical systems. The reader is also referred to the recent books by Sage and Melsa (Ref. 28) and Nahi (Ref. 29) on estimation theory for further applications of stochastic differential equations to problems in communications and control. Also see the recent book by Aström (Ref. 30).
APPENDIX I

MATRIX NOTATION

In general, all boldface variables are either column vectors, row vectors, or matrices.

\[ \Omega \] Multidimensional probability space.
\[ \Omega_i \] Component of \( \Omega \) in the \( i \)th direction.
\[ \Omega_i^* \] Multidimensional probability space formed by deleting the \( i \)th component of \( \Omega \).
\[ x(t) \] The state vector.
\[ x_i(t) \] The \( i \)th component \( x(t) \).
\[ x_i^*(t) \] The vector formed from \( x(t) \) by deleting its \( i \)th component.
\[ f(t, x(t)) \] The column vector whose components are functions of \( t \) and \( x(t) \).
\[ f_x(t, 0) \] The matrix whose \( i, j \)th element is \( \frac{\partial f_i(t, x)}{\partial x_j} \) evaluated at \( x = 0 \).
\[ 0 \] The zero vector or matrix.
\[ A_{ij} \] The \( i, j \)th element of the matrix \( A \).
\[ \{A_{ij}\} \] The matrix whose \( i, j \)th element is \( A_{ij} \).
\[ ||x(t)||^2 \] The sum of the squares of the components of the vector \( x(t) \).
\[ ||x(t)||_A^2 \] The quadratic form associated with the matrix \( A \).
\[ E \] Expected value operator.
\[ \tilde{E} \] Expected value operator based on a linear model of the system. (to be used in later chapters)
\[ \text{tr } [A] \] Trace of the matrix \( A \) defined as the sum of its diagonal elements.
\[ A' \] Transpose of the matrix \( A \).
\[ I \] Identity matrix.
Problems

8-1 Let \( \{x(t)\} \) be the Itô process defined by

\[
    dx(t) = f(t) \, dt + \mathcal{B}(t) \, d\beta(t)
\]

with \( x(0) = 0 \). Assuming that \( F[t, x(t)] = x^2(t) \), show that

\[
    dx^2(t) = 2x(t) [f(t) \, dt + \mathcal{B}(t) \, d\beta(t)] + \mathcal{B}^2(t) \, dt
\]

\[
    = 2x(t) \, dx(t) + \mathcal{B}^2(t) \, dt
\]

using Itô's differential rule with \( Q = 1 \).

8-2 In Prob. 8-1 let \( f(t) = 0 \) and \( \mathcal{B}(t) = 1 \) and show that

\[
    2 \int_0^t \beta(\tau) \, d\beta(\tau) = \beta^2(t) - t
\]

which demonstrates the difference from the usual integration rule.

8-3 Suppose that \( F[t, x(t)] = \exp[\alpha x(t)] \) with \( dx(t) \) defined in Prob. 8-2. By using Itô's differential rule show that

\[
    \exp[\alpha x(t)] = 1 + \int_0^t \exp[\alpha x(\tau)] \left( \alpha f(\tau) + \frac{\alpha^2 \mathcal{B}^2(\tau)}{2} \right) \, d\tau
\]

\[
    + \int_0^t \alpha \exp[\alpha x(\tau)] B(\tau) \, d\beta(\tau)
\]

if \( Q = 1 \) and \( x(0) = 0 \).

8-4 If \( dx_i(t) = f_i(t) \, dt + \mathcal{B}_i(t) \, dt, i = 1, 2 \), represents an Itô process, find the product \( d[x_1(t)x_2(t)] \), using Itô's differential rule. When \( x_1(t) = x_2(t) = x(t) \), show that your answer reduces to the result given in Prob. 8-1.

8-5 Consider the function \( \varphi(t) = \exp[i\omega(x(t) - x(t_0))] \), where \( x(t) \) is a Markov process represented by

\[
    x(t) = \int_{t_0}^t f[\tau, x(\tau)] \, d\tau + \int_{t_0}^t B[\tau, x(\tau)] \, d\beta(\tau)
\]

(a) Apply Itô's differential rule to \( \varphi(t) \) (see Prob. 8-3) and evaluate the conditional characteristic function of \( x(t) \) given \( x(t_0) \)—that is, \( C[i\omega|x(t_0)] \). This approach complements the development given in Section 7-6. Assume that \( F[t_0, x(t_0)] = 1 \).

(b) Write (a) in differential form and inverse transform the result to obtain the Fokker-Planck equation.

8-6 Write Itô's differential rule 8-28 in matrix form.
8-7 In a manner analogous to that used in Prob. 8-5, to derive the one-dimensional FP equation, use Itô's differential rule, in matrix form, to derive the multidimensional FP equation. Assume that the vector Markov process is generated by the stochastic differential equation (8-29).

8-8 A zero-mean white Gaussian noise \([n(t)]\) of single-sided spectral density \(N_0\) watts/Hz drives the feedback circuit illustrated in Fig. P8-8. Assume that \(g(x)\) is a memoryless, single-valued feedback element.

(a) Determine the stochastic differential equation of operation. Is \([y(t)]\) a Markov process? If so, why?
(b) Determine the intensity coefficients \(K_1(y)\) and \(K_2(y)\).
(c) Determine the probability current \(J(y; t)\).
(d) Establish the FP equation for this system.
(e) Describe suitable initial conditions and boundary conditions for determining its solution.

8-9 For the feedback circuit illustrated in the preceding problem, find the steady-state p.d.f. when

(a) \(g(x) = k_0 x + \xi x^{2n+1}, \) \(n\) any integer and \(|x| \leq \infty\).
(b) \(g(x) = \left(\frac{2k_0 d}{\pi m}\right) \tan \frac{\pi x}{2d}, \) \(|x| < d\).
(c) \(g(x) = \frac{k_0}{mb} \tanh bx, \) \(|x| \leq \infty\).
(d) \(g(x) = A_1 \sin x, \) \(|x| \leq \pi\).
(e) For the restoring forces given in (a), (b), (c), and (d), find the mean value of the first-passage time when the absorbing barriers are set at \(a\) and \(b\).

8-10 A zero-mean, white Gaussian noise \((N_0/2)\) acts as the input to the circuit illustrated below. The circuit receives a bias \(B^+\) voltage through the resistance \(R_2(x)\), which depends on the output \(x(t)\). (In general, it could even depend on \(t\).)

(a) Write the stochastic differential that relates the input to its output. Is \([x(t)]\) Markov? If so, why?
(b) Show that \(x(t + \tau) - x(t)\) is given by

\[
\int_t^{t + \tau} \dot{x}(\lambda) \, d\lambda = \int_t^{t + \tau} \left[ g(\lambda) + \frac{n(\lambda)}{R_1 C} \right] \, d\lambda
\]
where
\[
g(x) = \left( \frac{CR_2(x)}{B} \right)^{-1} - \frac{1}{C} \left[ \frac{1}{R_1} + \frac{1}{R_2(x)} \right]
\]

is the system restoring force.

(c) Using the results in (b), show that
\[
K_1(x) = h(x) \quad \text{and} \quad K_2(x) = \frac{N_0}{2(R_1 C)^2}
\]

8-11 Assume that
\[
R_2(x) = R_+ u(x - B) + R_- u(B - x)
\]
in Prob. 8-10. Here \( u(x) \) represents the unit step function.

(a) Find the steady-state p.d.f. \( p(x) \). Why doesn’t the solution \( p(x) \) contain a delta function at \( x = B \)?

(b) What does the solution in (d) reduce to when an ideal ideal diode is considered—that is, \( R_+ \to 0 \) and \( R_- \to \infty \)?

(c) Under what conditions would a delta function in \( p(x) \) be observed?

8-12 Consider the general first-order system shown in Fig. P8-12. Here \( \{n(t)\} \) is a zero-mean, white Gaussian noise process.

(a) Write down the stochastic differential equation that relates the output process \( \{x(t)\} \) to the input process.

(b) Is \( \{x(t)\} \) a first-order Markov process if \( F[x(t), t] \) and \( B[x(x), t] \) are memoryless devices? If so, state why.

(c) If \( F[x(0), 0] = F_0, \quad B[x(0), 0] = B_0, \quad \partial F/\partial x|_{t=0} = F'_0, \quad \partial B/\partial x|_{t=0} = B'_0, \quad \partial F/\partial t|_{t=0} = F'_0 \) and \( \partial B/\partial t|_{t=0} = B'_0 \) show that the intensity coefficients are given by
\[
K_1(x) = F_0 + \frac{B'_0 B_0 N_0}{4}
\]
\[
K_2(x) = \frac{B'_0 N_0}{4}
\]
by integrating the stochastic differential equation of operation and using the definitions for $K_1(x)$ and $K_2(x)$.

(d) Assuming the boundary conditions

\[ J(\infty; t) = J(-\infty; t) = J \]

find the steady-state p.d.f.

(e) If $J(\infty; t) = J(-\infty; t) = 0$ as $t$ approaches infinity, find the corresponding steady-state solution.

(f) If absorbing barriers are placed at $b_1$ and $b_2$, develop an expression for the $n$th moment of the first-passage time.

8-13 When subjected to the random loading $\{n(t)\}$, the equation of motion for a single-mode oscillator with nonlinear spring force $g(x)$ is given by

\[ \ddot{x} + b\dot{x} + g(x) = n(t) \]

where the excitation $\{n(t)\}$ is a stationary white Gaussian process with zero mean and correlation function $E[n(t_1)n(t_2)] = N_0 \delta(t_1 - t_2)/2$.

(a) Is $\{x(t)\}$ Markov?

(b) Letting $y_1 = x$ and $y_2 = \dot{x} = \dot{y}_1$, convert the second-order equation into two first-order equations. Does $(x, \dot{x})$ form a Markov vector? If so, why?

(c) Using the results in (b), find the intensity coefficients that characterize the two-dimensional FP equation.

8-14 Using the results obtained in Prob. 8-13, show that the FP equation is given by

\[ \frac{N_0}{4} \frac{\partial^2 p}{\partial y_2^2} - \frac{\partial}{\partial y_1} (y_2 p) + \frac{\partial}{\partial y_2} [IG(y_1) + b y_2] p = \frac{\partial p}{\partial t} \]

where $p = p(y_1, y_2; t) = p(x, \dot{x}; t)$ [We note that this is the form of the FP equation that arises in a SCS with nonlinearity $g(\phi)$ and loop filter $F(s) = 1/(1 + ts)$.

(b) Show that the steady-state solution
\[ p_{x_2}(y_1, y_2) = C \exp \left\{ -\frac{4b}{N_0} \int_0^{\gamma_0} g(\lambda) d\lambda + \frac{y_2^2}{2} \right\} \]

solves this equation.

(c) Discuss this solution in regard to the statistics of \( y_1 = x \) and \( y_2 = \dot{x} \).

(d) Let \( y_1 = \varphi \) and \( g(\varphi) = \sin \varphi, |\varphi| \leq \pi \) in (b) and determine the solution.

8-15 Find explicit solutions to the two-dimensional FP equations of Probs. 8-13 and 8-14 when

(a) \( g(x) = k_0 x + \epsilon x^2, k_0 \) is the initial linear stiffness and \( \epsilon \) is a constant coefficient. What happens as \( \epsilon \to 0 \)?

(b) \( g(x) = (2k_0 d/\pi m) \tan (\pi x/2d), |x| < d \), in which \( m \) and \( d \) are constants. Discuss the p.d.f. \( p(x) \) as \( d \) approaches zero and infinity. Setting \( n = 32b k_0 d/\pi m N_0 \), then \( p(x) \) is known as the cosine-power p.d.f.

(c) \( g(x) = (k_0/mb)[\tanh (bx)]; \) notice here that the restoring force \( h(x) \) developed during the oscillations (vibrations) is bounded between \( \pm k_0/bm. \) This can be used to model the elastic-perfect-plastic behavior in various physical situations. Here \( p(x) \) is frequently referred to as the sech-power p.d.f.

(Hint: Use the general solution given in Prob. 8-14.)

8-16 A narrowband noise process is generated by the stochastic differential equation

\[ \ddot{x} + \epsilon \dot{x} + x = \epsilon n(t) \]

where \([n(t)]\) is white Gaussian noise.

(a) Find the Fokker-Planck equation for the transition p.d.f. \( p(x, \dot{x}; t) \).

(b) Determine the solution to this equation.

8-17 The stochastic differential equation of operation for a Van der Pol oscillator system driven by white noise \([n(t)]\) is given by

\[ \ddot{x} + \tau \dot{x}(1 - x^2) + x(t) = n(t) \]

(a) Find the FP equation whose solution yields the transition p.d.f. \( p(x, \dot{x}, t|x_0, \dot{x}_0, t_0) \).

(b) State appropriate boundary and initial conditions.

8-18 In the measurement of the rotation of the plane of polarization of the transmitted Pioneer VI spacecraft signal, a polarization tracker was built whose dynamical equations for polarization angle \( \theta \) and carrier phase error \( \varphi \) are well approximated by

\[ \dot{\theta} = -(A_1 \cos \theta) \sin \theta + n_1 \]
\[ \dot{\varphi} = -(A_2 \cos \theta) \sin \varphi + n_2 \]

where \([n_1(t)]\) and \([n_2(t)]\) are statistically independent white Gaussian noise processes with single-sided spectral densities of \( N_{01} \) and \( N_{02} \) respectively.
(a) Find the intensity coefficients that characterize the FP equation.
(b) Find the probability current densities \( \mathcal{J}_0(\varphi, \theta; t) \) and \( \mathcal{J}_1(\varphi, \theta; t) \) in terms of \( p(\varphi, \theta, t; \varphi_0, \theta_0, t_0) \).
(c) Determine the two-dimensional FP equation.

8-19 Consider a class of \( n \)-degree-of-freedom, nonlinear dynamical systems whose motions are defined by the following system of stochastic differential equations:

\[
\dot{x}_i + a_i \dot{x}_i [1 + \epsilon_i D_i(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)] \\
+ b_i x_i [1 + \mu_i S_i(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)] = \eta_i(t), \quad i = 1, 2, \ldots, n
\]

where \( a_i \) and \( b_i \) are linear damping and stiffness coefficients, respectively, \( \epsilon_i \) and \( \mu_i \) are nonlinear parameters, \( D_i \) and \( S_i \) are nonlinear functions, and the \( [\eta_i(t)] \) are random processes.

(a) Transform the motion in \( n \) space into \( 2n \)-dimensional space. Notice this converts \( n \) second-order equations in \( 2n \) first-order equations. (Hint: Let \( y_i = x_i \) and \( y_{i+1} = \dot{x}_i \).)

(b) Assuming that \( E[\eta_i(0)] = 0 \) and \( E[\eta_i(t)\eta_j(t_2)] = S_{ij}\delta(t_1 - t_2) \) for constant \( S_{ij} \), find the intensity coefficients that characterize the FP equation.

8-20 Write expressions for the probability components of the probability current density vector \( \mathcal{J}(y_1, \ldots, y_{2n}; t) \) in terms of the transition p.d.f. \( p(y_1, y_2, \ldots, y_{2n}; t) \), using the results of the previous problem.

8-21 If each component of \( \mathcal{J} \) in Prob. 8-19 vanishes everywhere in \( \Omega \), find the solution of the potential form; that is, write an expression for the potential function \( U(y_1, \ldots, y_{2n}) \) in the steady state. (See Chapter 7.)

8-22 For the isotropic case—the case where the matrix

\[
[K_{ij}] = K(y)[\delta_{ij}]
\]

verify by direct substitution into

\[
\sum_{i=1}^{2n} \frac{\partial \mathcal{J}_i}{\partial y_i} = 0
\]

that \( p_{ss}(y) = C \exp \{-U(y)\} \) solves the FP equation.

8-23 Assume that the multidimensional FP equation

\[
\sum_{k=1}^{N} \frac{\partial \mathcal{J}_k(x)}{\partial x_k} = 0
\]

can be put in the form

\[
\sum_{k=1}^{N} L_k \left[ g_k(x_k)p + h_k(x_k) \frac{\partial p}{\partial x_k} \right] = 0
\]
where \( L_k \) are arbitrary first-order partial differential operators, and assume that there exists a \( p(x) \) satisfying

\[
g_k(x_k) + h_k(x_k) \frac{\partial p}{\partial x_k} = 0
\]

for all \( k = 1, 2, \ldots, N \). Show that the steady-state solution has the form

\[
p_{ss}(x) = C \prod_{k=1}^{N} \exp \left[ -\int_0^{x_k} \frac{g_k(\lambda)}{h_k(\lambda)} \, d\lambda \right]
\]

8-24 Two first-order PLL loops are connected in cascade for use in a two-way Doppler and phase-tracking system. The dynamical equations of operation are given by

\[
\begin{align*}
\dot{\varphi}_1 &= \Omega_1 - A_1 K_1 \sin \varphi_1 - K_1 n_1 \\
\dot{\varphi}_2 &= -\dot{\varphi}_1 - \Omega_2 - A_2 K_2 \sin \varphi_2 - K_2 n_2
\end{align*}
\]

where \( \{n_1(t)\} \) and \( \{n_2(t)\} \) are statistically independent, white Gaussian noise sources.

(a) Is \( \{\varphi_i(t)\} \) a Markov process?
(b) Characterize the FP equation whose solution is \( p(\varphi_1, \varphi_2; t) \).
(c) If \( \varphi_1 = \theta_0 - \hat{\theta}_1 \) and \( \varphi_2 = \hat{\theta}_1 - \hat{\theta}_2 \) characterize the FP equation whose solution is \( p(\hat{\theta}_1, \hat{\theta}_2; t) \).

8-25 Let \( y(t) \) denote the output of an RC filter driven by the random telegraph signal \( x(t) \). Setting \( t_0 = 0 \), the input-output transformation law is given by

\[
y(t) = y_0 \exp (-\beta t) + \int_0^t \beta \exp [-\beta(t - \tau)] x(\tau) \, d\tau
\]

where \( \beta^{-1} \) is the time constant of the RC filter. With equal probability, the random telegraph signal, \( x(t) \), is either \( +1 \) or \( -1 \) and the number of transitions between \( +1 \) or \( -1 \), \( N(t, t') \), in the time interval \( T = |t' - t| \) is a Poisson-distributed r.v. characterized by (1-138).

(a) Show that the intensity coefficients in the FP equation are given by

\[
K_1(y) = -\beta f y \quad K_2 = \beta^2 (1 - y^2)
\]

(b) Write an expression for the FP equation. Show that the steady-state p.d.f. is given by

\[
p(y) = \begin{cases} 
(1 - y^2)^{(f/\beta)-1} & |y| \leq 1 \\
B(f/\beta, 1/2) & |y| > 1
\end{cases}
\]

where \( B(x, y) \) denotes the beta function.
Often in communication and tracking-system analysis, the problem under consideration can be modeled by using the theory of vector Markov processes. For example, the vector Markov process \( x = (x_1, x_2, x_3, x_4) \) is characterized by the set of stochastic differential equations

\[
\begin{align*}
\dot{x}_1(t) &= f_1[x_1(t), x_2(t)] + C_1 n_1(t) \\
\dot{x}_2(t) &= f_2[x_1(t), x_2(t)] + C_2 n_2(t) \\
\dot{x}_3(t) &= f_3[x_1(t), x_2(t), x_3(t), x_4(t)] + C_3 n_3(t) - C_1 n_1(t) \\
\dot{x}_4(t) &= f_4[x_3(t), x_4(t)] + C_4 n_2(t)
\end{align*}
\]

where \( f_1(\cdot), f_2(\cdot), f_3(\cdot), \) and \( f_4(\cdot) \) are all real-valued nonlinear functions; \( n_1(t) \) and \( n_2(t) \) are independent white Gaussian noise process with zero mean and one-sided spectral densities \( N_{01} \) and \( N_{02} \) respectively; \( C_i, i = 1, 2, 3, 4 \) are constants.

(a) Why is \( x \) a vector Markov process?
(b) Is \( x_i, i = 1, 2, 3, 4 \), a Markov process? Why or why not?
(c) Is the vector \((x_1, x_2)\) a vector Markov process? Why?
   If not, then under what condition would it be?
(d) Is the vector \((x_3, x_4)\) a vector Markov process? Why?
   If not, then under what condition would it be?

In a two-way, phase-coherent communication system used in deep-space applications, the functions \( f_i(\cdot), i = 1, 2, 3, 4 \), in Prob. 8-26 are characterized by

\[
\begin{align*}
f_1[x_1(t), x_2(t)] &= \Omega_{01} - F_1 A_1 K_1 \sin x_1(t) \\
f_2[x_1(t), x_2(t)] &= -\frac{x_2(t)}{\tau_1} - \frac{(1 - F_1)[A_1 K_1 \sin x_1(t)]}{\tau_1} \\
f_3[x_1(t), x_2(t), x_3(t), x_4(t)] &= \Omega_{02} + F_1 A_1 K_1 [x_1(t) - x_2(t)] \\
&\quad - F_2 A_2 K_2 \sin [x_2(t) + x_4(t)] \\
f_4[x_3(t), x_4(t)] &= -\frac{x_4(t)}{\tau_2} - \frac{(1 - F_3)[A_3 K_3 \sin x_4(t)]}{\tau_2}
\end{align*}
\]

(a) Find the FP equation whose solution is the transition p.d.f. \( p(x(t) | x(t_0)) \),
where \( x(t_0) \) is the value of \( x(t) \) at the initial time \( t_0 \).
(b) Which of the transition p.d.f.'s

\[
\begin{align*}
(1) & \quad p(x_1(t), x_2(t) | x_1(t_0), x_2(t_0)) \\
(2) & \quad p(x_3(t), x_4(t) | x_3(t_0), x_4(t_0)) \\
(3) & \quad p(x_1(t) | x_1(t_0)) \\
(4) & \quad p(x_2(t) | x_4(t_0)) \\
(5) & \quad p(x_3(t) | x_3(t_0)) \\
(6) & \quad p(x_4(t) | x_4(t_0))
\end{align*}
\]
satisfies a FP equation? For those which satisfy a FP equation in (b), find the corresponding FP equation.

References


PART THREE
NONLINEAR THEORY OF FIRST-ORDER SYNCHRONOUS CONTROL SYSTEMS

9-1 Introduction

In this chapter we begin our study of the nonlinear theory of synchronous control systems. A synchronous control system (SCS) is a generic term that will be used to refer to any electronic, mechanical, or electromechanical system which derives a coherent reference signal in a closed-loop mechanization by using the phase-lock or entrainment principle discussed in Chapters 3 and 5. For example, a sinusoidal PLL represents a special case of a SCS.

The theory presented in the previous four chapters provides us with the mathematical tools necessary to study the nonlinear behavior of synchronous control systems employing the phase-lock or entrainment principle. Even though the first-order loop is not of great practical importance, it remains, for the most part, the only loop for which one can obtain exact answers to questions requiring the use of the nonlinear theory. Moreover, as we shall see, the first-order loop theory gives considerable insight into the nonlinear behavior of higher-order loops. Aside from this, the locking properties (in the absence of
noise) of a first-order sinusoidal PLL are identical with those of the RF injection-locked oscillator (Ref. 1). This point is particularly interesting in view of the fact that the practical limitations of the two techniques are quite different. For example, an injection-locked tunnel diode oscillator has been locked (Ref. 2) to an input signal at a frequency of 160 MHz—a feat not presently possible with a sinusoidal PLL. On the other hand, a laser in a sinusoidal PLL has been locked over a 100-MHz range, whereas the measured range for an injection-locked helium-neon laser is less than 1 MHz.

It is important to note at the outset that a first-order SCS operates in two distinct modes—the signal acquisition mode and the synchronous or tracking mode. Therefore, in the design of a SCS, the engineer must account for and characterize the performance of these modes of operation. In the acquisition mode, measures of system performance include the characterization of the acquisition behavior—for example, signal acquisition range and synchronization stability. In the synchronous mode of operation, system performance is characterized by the steady-state p.d.f. of the phase error, moments of the phase error, moments of the time to first loss of synchronization, average number of cycles-slipped per unit time, probability of sync failure (losing lock), probability of $k$ cycle-slips per unit time, and average number of cycles-slipped to the right and left.

The purpose of this chapter is to characterize the performance of a first-order SCS. We begin by discussing the nonlinear behavior of a first-order SCS in the absence of noise by use of the phase-plane method. Analytical formulas that specify the signal acquisition range and time are then developed. Next we derive the steady-state p.d.f. of the phase error process when noise is present and when the loop is used as a tracking filter. Moments of the phase error, phase error rate, and the average number of cycles-slipped per unit time are derived, followed by the evaluation of the moments of the mean time to first slip and time to acquire the signal. The probability of acquisition is then developed. Since SCSs frequently must operate in the presence of impulse and Gaussian noise, we next develop the theory that specifies the steady-state p.d.f. of the phase error in the presence of non-Gaussian noise. Various approximate theories that have evolved to specify the phase error variance are also discussed. We shall study the transient behavior of the phase error process, its spectral density, and correlation function in Chapter 16.

9-2 Acquisition Behavior and Synchronization Stability in the Absence of Noise

The primary function of a SCS designed to operate as a tracking receiver or a synchronization system is to deliver to a communications terminal the frequency and timing information contained in the received signal. To per-
form this function, however, the receiver must first acquire the received signal. This is the most difficult part of its task and accounts for a substantial part of the mechanization complexity. Once frequency and phase have been acquired, the loop continuously attempts to maintain both phase and frequency alignment.

Although a SCS always operates in the presence of noise, it is instructive to consider the nonlinear behavior of an ideal system in the absence of noise, oscillator instabilities, and random modulation. When \( \theta(t) = \Omega_0 t + \theta_0 \), we have from (3-7) that

\[
\Omega_0 = \phi + AKg(\phi) + K_v e
\]  

(9-1)

when \( F(p) = 1 \). As in the case of the Van der Pol oscillator the behavior of the nonlinear differential equation for various initial conditions is best understood by making use of phase-plane plots. Furthermore, a study of these plots will shed considerable light on the nonlinear behavior of higher-order SCSs in the presence of noise.

In accordance with the foregoing assumptions, the equation of operation of a generalized, first-order SCS can be rewritten, using (9-1), as

\[
\frac{d\phi}{d\tau} = v(\phi) = \gamma - g(\phi)
\]

(9-2)

where \( \gamma \triangleq \Lambda_0/AK \) and we have normalized time by setting \( \tau = AKt \) and \( \Lambda_0 = \Omega_0 - K_v e \). In the phase-plane \((v, \phi)\), the equation of the energy curve (9-2) represents the phase-detector characteristic \( g(\phi) \) with sign reversed and raised or lowered above or below the horizontal axis by the value of the effective normalized detuning \( \gamma \triangleq \Lambda_0/AK \).

The single phase-plane trajectory is illustrated in Fig. 9-1 for the case of a PLL; that is, \( g(\phi) = \sin \phi \). If we envision the value of \( \phi(t) \) as the position of a particle at time \( t \) and \( v(\phi) \) its velocity at time \( t \), then whenever \( v(\phi) \) is positive, \( \phi \) tends to increase; whenever \( v(\phi) \) is negative, \( \phi \) tends to decrease. The vertical energy curves indicate instantaneous "jumps" of the representative points on to the energy curve (9-2) from any point in the phase-plane. Notice that when \( \Lambda_0 < AK \), there are singularities occurring when \( \dot{\phi} = AKv = 0 \). Stable points of phase lock or synchronization occur when

\[
\phi_{ik} = 2n\pi + \sin^{-1} \gamma
\]

(9-3)

and unstable lock points occur at

*The subscript "ik" is used to denote points where potential lock points occur. It should not be considered as an indexing integer or variable. The position of the points are indexed via the integer \( n \).
Fig. 9-1. Acquisition Behavior of a First-Order PLL.
\[ \varphi_{ik} = (2n - 1)\pi - \sin^{-1} \gamma \]  

(9-4)

where \( n \) is any integer. Once the system reaches a stable point, it remains there until further disturbed by external conditions. If \( \Lambda_0 > AK \), then \( v \neq 0 \) for any \( \varphi \) so that phase-lock is not possible; the loop phase error cycles through multiples of \( 2\pi \) radians (Fig. 9-1b) and synchronization stability cannot be achieved. This behavior is analogous with the rotations through multiples of \( 2\pi \) radians of the frictionless pendulum discussed in Chapter 5. In a PLL this continuous increase or decrease of the phase error by multiples of \( 2\pi \) radians is defined as cycle-slipping in a first-order loop. It is analogous to the “slip-a-pole” phenomenon occurring in the operation of alternating current motors and the skipping of a heartbeat in biological organisms. When the initial frequency detuning \( \Lambda_0 < AK \), the loop will phase-lock when \( \varphi \) changes from its initial value by \( 2\pi \) radians at most. Notice, also, from (9-3) that in the presence of VCO detuning, the steady-state phase error can be made zero by setting \( \gamma = 0 \), that is, \( e = \Omega_0 / K_\nu \).

For a first-order loop, the synchronization band (analogous with that of the Van der Pol oscillator) is defined by the condition

\[ |\gamma| = \frac{|\Lambda_o|}{AK} = \frac{|\Lambda_o|}{2W_L} = 1 \]  

(9-5)

If \( |\Lambda_o| < AK \), the incoming Doppler signal—that is, the frequency offset \( d(t) = \theta_0 + \Omega_0 t \)—will ultimately be tracked, and phase-lock is possible with no frequency error. A constant lagging phase error given by (9-3) does occur. In the absence of noise, the loop never skips a cycle when \( |\Lambda_o| < AK \). Thus for a first-order system the synchronization band is defined by

\[ B_s = AK = 4B_L = \Lambda_0 \]  

(9-6)

Sometimes \( B_s \) is referred to as the acquisition range, since \( \Lambda_0 = AK = B_s \) represents the largest frequency offset for which the loop will phase-lock regardless of the value of \( \theta_0 \).

### 9-2.1 Acquisition Time and Transient Performance in the Absence of Noise

Now the multiples of \( 2\pi \) in (9-3) are usually understood to occur so that \( \sin \varphi_{ik} = \gamma \). In general, the amount of time required for the phase error to change from \( \varphi_0 \) to \( \varphi \) can be determined from (9-2). If at \( t = t_0 = 0 \), \( \varphi = \varphi_0 \), then the phase error at time \( t \) can be determined from (9-2).

\[
\int_{0}^{Aki} d\tau = AK \int_{0}^{t} dt = \int_{\varphi_0}^{\varphi} \frac{dx}{\sin \varphi_{ik} - g(x)}
\]  

(9-7)

This represents the development of \( \varphi \) as a function of \( t \).
In practice, the acquisition time, defined as the time $T_{\text{acq}}$ in seconds required for the SCS to achieve frequency and phase lock, is an important parameter. Frequency acquisition (lock) time is defined as the time $T_f$ in seconds required for a SCS to reach the state such that there exists no further cycle-slipping—that is, changes of the VCO phase by multiples of $2\pi$ relative to the incoming signal phase. Phase acquisition time is the time $T_p$ in seconds, once frequency acquisition has occurred, for phase-lock to occur; that is, phase lock is achieved when $\dot{\phi} = 0$. Without loss in generality, we now explore the acquisition properties for the sinusoidal PLL in the absence of noise.

Consider first the case $|\gamma| < 1$. Within the acquisition range, the phase synchronization process stabilizes according to the integral given in (9-7) when $t$, in the upper limit, is replaced by $T_p$. Carrying out the integration with $g(\phi) = \sin \phi$ gives

$$
(\Lambda K \cos \varphi_{ik})T_p = \ln \left[ \frac{\lambda - \lambda_{ik}^{-1}}{\lambda - \lambda_{0}^{-1}} \frac{\lambda_{0} - \lambda_{ik}}{\lambda_{0} - \lambda_{ik}^{-1}} \right]
$$

(9-8)

where $\lambda = \tan (\varphi/2)$, $\lambda_{ik} = \tan (\varphi_{ik}/2)$, and $\lambda_{0} = \tan (\varphi_{0}/2)$. Typical phase error transients are sketched in Fig. 9-1c, where the change in the phase error is plotted relative to $\varphi_{ik}$. The starting point in this sketch is determined by $\varphi = \varphi_{0} - \varphi_{ik}$. For high-loop gains, $\varphi_{ik} < 90^\circ$, we see that for most practical purposes $\Lambda K T_p < 10$; however, the point $\varphi_{ik}$ is reached only after an infinite amount of time has elapsed.

Outside the acquisition range, $|\gamma| > 1$, the phase error, as a function of time, is defined by

$$
AKt = \frac{2}{\sqrt{\gamma^2 - 1}} \tan^{-1} \left[ \frac{\gamma \tan \varphi/2 - 1}{\sqrt{\gamma^2 - 1}} \right]_{\varphi_0}
$$

(9-9)

which is the integral of (9-7). This represents a periodic oscillation characterized by its waveform and its fundamental frequency. Figure 9-1d illustrates examples of this periodic variation between the beat note, $\sin \varphi$, versus $t$ for $\gamma = 1.05$ and $\gamma = 3$. Notice that this waveform is unsymmetrical and as such possesses a dc value. The time scale is normalized to the beat note period $T_{bn} = 1/f_{bn}$. The period, $T_{bn}$, of the beat note, is such that as $t$ increases by $T_{bn}$, then $\varphi$ increases $2\pi$ radians; therefore from (9-7) we have

$$
T_{bn} = \frac{2\pi}{AK \sqrt{\gamma^2 - 1}} = \frac{2\pi}{\sqrt{\Lambda_{0}^2 - (AK)^2}}
$$

(9-10)

The average dc control voltage developed at the input to the VCO may now be determined using the preceding relationships. Integrating (9-1) over a cycle, we obtain the average generated frequency shift
\[ AK \langle \sin \varphi \rangle_a \triangleq \frac{AK}{T_{bn}} \int_{t_a}^{t_a + T_{bn}} \sin \varphi \, dt = \Lambda_0 - \frac{2\pi}{T_{bn}} \] (9-11)

or

\[ \langle \sin \varphi \rangle_a = \begin{cases} \gamma - \sqrt{\gamma^2 - 1} & |\gamma| \geq 1 \\ \gamma & |\gamma| < 1 \end{cases} \]

This average pull-in potential is sketched in Fig. 9-1e and represents the magnitude of the dc voltage developed in the loop as a function of \( \gamma \). In fact, \( \langle \sin \varphi \rangle_a \) is maximum when \( \gamma = 1 \) and decreases beyond that point. When \( |\gamma| < 1 \), the phase error does not shift \( 2\pi \) radians in a finite time. Enough dc voltage is developed, however, to shift the VCO frequency by \( \Lambda_0 \) radians. This voltage is represented by the straight-line portion in Fig. 9-1e. Finally, when \( \sin \varphi_{lk} = \gamma = 1 \), we find from (9-7) that

\[ AKt = \tan \left( \frac{\varphi + \pi/2}{2} \right) - \tan \left( \frac{\varphi_0 + \pi/2}{2} \right) \] (9-12)

9-3 Characterization of the Statistical Behavior in the Presence of Noise*

In this section we begin our study of the nonlinear behavior of a SCS in the presence of noise. The presence of the additive noise \( n(t) \) leads, however, to certain deviations from the fully synchronized regime (stable lock points) defined by (9-3) in the absence of noise when \( \Lambda_0 < AK \). These deviations consist of the following: (1) Small phase error fluctuations occur away from the synchronized value observed in the absence of noise. (2) Sometimes the phase error deviations assume values comparable with \( \pi/2 \). (3) Still more infrequently, there occur cycle-slips by a whole number of periods. Such phase error changes are not compensated but are accumulated. This result leads to the mean frequency of the VCO not coinciding with the frequency of the synchronizing signal (the applied signal), as well as to nonreversible phase diffusion, and, consequently, diffusion of the VCO frequency stability (see the Prelude).

The first effect (small fluctuations in \( \varphi \)) can be accounted for by the linear PLL theory; however, the linearization method ceases to be applicable if the amplitude of the synchronizing signal becomes small or if the initial detuning approaches the boundary of the synchronization band. In order to study

*It should be noted that the reader who is not interested in the details of the nonlinear theory of first-order loops may prefer to go directly to Chapter 11 where the more general treatment is given. However, in the author’s opinion, anyone who elects to do so requires a fairly advanced knowledge of the nonlinear behavior in order to be successful.
the second and third effects, we must use the nonlinear theory. Moreover, in the linear approximation, nothing at all can be said about the cycle-slipping problem. This effect can only be understood in the context of the nonlinear theory. The method of approach is to apply the Fokker-Planck equation. This approach has the advantage of giving directly the steady-state p.d.f. of the phase error in the nonlinear region of loop operation.

First we shall discuss the qualitative aspects of the statistical behavior of the phase error process. In the presence of noise we showed in Chapter 3 that the stochastic differential equation of loop operation is given by

\[ \dot{\phi} = \Lambda_0 - K[Ag(\phi) + N(t, \phi)] \]

\[ \Lambda_0 = \Omega_0 - K_v e \]  \hspace{1cm} (9-13)

where \( N(t, \phi) \) is defined in (3-23). Here we have assumed that \( \Delta \psi = M = 0 \) and \( \theta(t) = d(t) = \Omega_0 t + \theta_0 \). This equation is analogous to that describing the motion, along the \( \phi \) axis, of a particle in a field of irregular external forces that depend nonlinearly on the position. The particle becomes trapped in the vicinity of one of the stable lock points given in (9-3); however, its motion about this point is random. From our discussion in Chapter 5, the restoring force that tends to hold the particle in the vicinity of the stable point is recognized from (9-13) to be \( \Lambda_0 - AK \sin \phi \), where, for the moment, we set \( g(\phi) = \sin \phi \) for convenience. The potential at any point along the \( \phi \) axis due to this force is just

\[ V(\phi) = -\int^\phi (\Lambda_0 - AK \sin x) \, dx \]

\[ = -\Lambda_0 \phi - AK \cos \phi \]  \hspace{1cm} (9-14)

Thus the presence of an input signal causes the formation of a potential having the same form as the potential of a particle that moves along a wavy inclined surface depicted in Fig. 9-2. If \( \Lambda_0 > 0 \) and \( \Lambda_0 \leq AK \), this surface has "well bottoms" in the vicinity of \( \phi_{ik} = 2n\pi + \sin^{-1} \gamma \) which are stable points of the particle in the absence of random disturbances. Suppose that at \( t = t_0 \) the particle is in the well bottom \( B_0 \) of Fig. 9-2. First, the noise disturbance moves the particle in the vicinity of the well bottom in a random manner, thereby causing small fluctuations in the instantaneous frequency and phase of the VCO. Secondly, the noise dislodges the particle more or less frequently from well bottom \( B_0 \) and causes it to gradually slip down (probability flow to the right) to the next lower well bottom \( B_1 \). Less frequently, the random impacts of the noise throw the particle into well bottom \( B_{-1} \), which is situated on a higher level (probability flow to the left). This random downward motion of the particle from well bottom to well bottom corresponds to an increase in the phase error by multiples of \( 2\pi \) radians. This phenomenon is known as cycle-
slipping. When $\Lambda_0 \neq 0$, the particle generally slips down the inclined surface more rapidly; the greater the detuning (slope) and the smaller the signal amplitude A (well depths), the more often the phase error increases (particle slides) by multiples of $2\pi$ radians. If the input noise is small, then the changes of the particle from $B_0$ to $B_{-1}$ and so on is not a highly probable event and, in practice, cycle-slipping can be ignored (linear theory applies).

When $\Lambda_0 > AK$, there will be no well bottoms and the particle slips down the incline incessantly; the synchronized state is impossible because the loop continuously slips cycles. This is analogous to the phase-plane plot of Fig. 9-1b and accounts for a continuous probability flow down the incline in the presence of noise. In the presence of noise, the inclined surface of Fig. 9-2 experiences random amplitude fluctuations, which are distributed according to the statistics of $N(t, \varphi)$.

Figure 9-3a illustrates a typical trajectory of the phase-error process, while Fig. 9-3b illustrates a plot of the phase-error trajectory $\varphi(t)$ reduced modulo $2\pi$ about axes $\varphi = (2n - 1)\pi$ and $\varphi = (2n + 1)\pi$. We shall denote the phase error process reduced modulo $2\pi$ by $[\varphi(t)]$. Note the phase-jumps of $2\pi$ in $[\varphi(t)]$.*

The oscillograms presented in Fig. 9-4 illustrate the phase error variations of $\varphi(t)$ when it is affected by external noise. Oscillogram a is for a low noise level. A single phase-jump of $2\pi$ is seen in oscillogram c. Oscillograms b and d illustrate phase-jumps when detunings $\Lambda_0$ of different signs are introduced into the system.

The buildup of the steady-state p.d.f. of $\varphi(t)$ from its initial p.d.f. at $t = t_0$ is illustrated in Fig. 9-5. For the moment we shall assume that the initial p.d.f. is given by

$$\lim_{t \to t_0} P(\varphi, t|\varphi_0, t_0) = \delta(\varphi - \varphi_0) \quad (9-15)$$

*The modulo-$2\pi$ process is explicitly defined by $[\varphi(t)] = [\varphi(t) + \pi \text{ mod } 2\pi - (2n - 1)\pi]$. 

---

**Fig. 9-2.** Motion of the Particle from Well to Well.
Fig. 9-3. Top Figure Illustrates Typical Trajectories of the Phase Error while the Bottom Illustrates These Trajectories Reduced Modulo $2\pi$ about the Axes $(2n + 1)\pi$ and $(2n - 1)\pi$. Note the Phase-Jumps of $2\pi$.

Fig. 9-4. Oscillograms of the Phase Error Variations in the Presence of Noise.
Characterization of the Statistical Behavior in the Presence of Noise

\[ p(\varphi, t | \varphi_0, t_0) \]

(a) Initially all the probability is concentrated at \( \varphi = \varphi_0, t = t_0 \).

\[ \delta(\varphi - \varphi_0) \]

(b) Diffusion of the phase-error, \( t = t_1 > t_0 \).

\[ \varphi \]

(c) Diffusion of the phase-error and probability flow to the right, \( t = t_2 > t_1 \).

\[ (2n-4)\pi \quad (2n-2)\pi \quad 2n\pi \quad (2n+2)\pi \quad (2n+4)\pi \]

(d) Diffusion and probability flow to the left and to the right, \( t = t_3 > t_2 \), due to cycle slipping.

Fig. 9-5. Evolution of \( P(\varphi, t | \varphi_0, t_0) \) with the Passage of Time.

The position \( \varphi = \varphi_0 \) is taken to be that position along the \( \varphi \) axis which corresponds to the well bottom \( B_0 \) of Fig. 9-2. With the passage of time, the initial p.d.f. given in (9-13) begins to diffuse in the vicinity of \( \varphi = \varphi_0 \) (see Fig. 9-5). Sooner or later the loop slips a cycle, probability flows to the right and begins to build up in the new well bottom, say \( B_1 \), as the phase error (particle) wanders about. This buildup (Fig. 9-5) of the probability about the various stable lock points continues until, in the steady state,

\[
\lim_{t \to \infty} P(\varphi, t | \varphi_0, t_0) = \lim_{t \to \infty} P(\varphi + 2n\pi, t | \varphi_0, t_0) = 0 \quad (9-16)
\]

for all \( |\varphi| \leq \infty \), \( n \) any integer. This says that the transition p.d.f. \( P(\varphi, t | \varphi_0, t_0) \) has completely diffused and that it is equally probable to find the particle anywhere along the \( \varphi \) axis. Thus in the steady state \( P(\varphi, t | \varphi_0, t_0) \) possesses an unbounded variance. This behavior is reminiscent of the random walk problem discussed in Chapter 7 and, of course, is due to the fact that the loop slips
cycles at random times. The cycle-slips are "analogous" to the steps in the random walk model.

Since the steady-state behavior of the loop is of most practical interest and since (9-16) is true, we must determine the transition p.d.f. of an equivalent phase-error process that retains all the information about the statistical behavior of \( \varphi \) yet yields a meaningful mathematical solution in the steady state. This is accomplished by considering the periodic extension

\[
\bar{p}(\varphi, t|\varphi_0, t_0) \triangleq \sum_{n=-\infty}^{\infty} P(\varphi + 2n\pi, t|\varphi_0, t_0)
\]

(9-17)

of the transition p.d.f. \( P(\varphi, t|\varphi_0, t_0) \) for all \( \varphi \). It is easy to see that (9-17) contains all the probability represented by \( P(\varphi, t|\varphi_0, t_0) \), provided we restrict \( \varphi \) to any interval \( I(n) \triangleq [(2n - 1)\pi, (2n + 1)\pi] \) and dictate that \( \varphi_0 \) is a point in this interval. The reason, of course, is that all the probability has been trapped in a region of the width 2\( \pi \) radians. From (9-15) and (9-17) we note that \( \bar{p} \) satisfies the initial condition

\[
\lim_{t \to t_0} \bar{p}(\varphi, t|\varphi_0, t_0) = \sum_{n=-\infty}^{\infty} \delta(\varphi + 2n\pi - \varphi_0)
\]

(9-18)

and that \( \bar{p} \) is periodic in \( \varphi \). However, the conservation of probability requires that

\[
\int_{-\infty}^{\infty} P(\varphi, t|\varphi_0, t_0) \, d\varphi = 1
\]

(9-19)

so that \( \bar{p} \) is not a transition p.d.f. because (1) it is an infinite sum of p.d.f.'s, each with unit area and (2) its initial condition (9-18) is not a delta function. Consequently, other special considerations are required.

In accordance with Figs. 9-3 and 9-4 we note that the phase error trajectories \( \varphi(t) \) can be written as \( \{\varphi(t)\} = [\hat{\varphi}(t) + 2\pi J(t)] \), where \( [\hat{\varphi}(t)] \) represents the modulo 2\( \pi \) reduced version of \( \{\varphi(t)\} \) about the axis \( \varphi = (2n - 1)\pi \) and \( \varphi = (2n + 1)\pi \). In essence, \( [\hat{\varphi}(t)] \) represents the phase error process measured by a phase meter.* The process \( \{J(t)\} \) accounts for the net number of phase-jumps (not cycle slips) that have occurred since \( t = t_0 \). At any instant of time, \( J(t) \) is a discrete r. v. that takes on integer values \( j = 0, \pm 1, \pm 2, \pm 3, \ldots \). Notice that in the limit as \( t \) approaches \( t_0 \), \( J(t_0) = 0, \varphi(t_0) = \hat{\varphi}(t_0) = \varphi_0 = \phi_0 \), and, from (9-17), that \( \bar{p}(\phi \pm 2n\pi, t|\phi_0, t_0) = \bar{p}(\phi, t|\phi_0, t_0) \). If we define the conditional (upon \( n \)) transition p.d.f.

\[
p(\phi, t|\phi_0, t_0, n) \triangleq \begin{cases} \bar{p}(\phi + 2\pi n, t|\phi_0, t_0); & \phi \in I(n) \\ 0 & \text{elsewhere} \end{cases}
\]

(9-20)

*The modulo-2\( \pi \) process in explicitly defined by \( [\hat{\varphi}(t) = [\varphi(t) + \pi] \mod 2\pi - (2n-1)\pi] \).
we see that the diffusion of probability from $\phi = \phi_0$ is trapped in a region along the $\phi$ axis, $\phi \in I(n)$ for all $t$; $n$ any fixed integer. In what follows, we shall characterize the steady-state conditioned p.d.f. $p(\phi|n)$. Time-dependent solutions for $p(\phi, t|\phi_0, t_0, n)$ will be studied in Chapter 16.

9.3.1 The Fokker-Planck Equation for the First-Order SCS

The p.d.f. $p(\phi, t|\phi_0, t_0, n)$ can be determined from the Fokker-Planck equation provided that we can justify the condition that $\phi$ is Markov. It is obvious from (9-13) that $\phi$ would be Markov if $N(t, \phi) = N(t, \phi)$ were a white Gaussian process.* The reason, of course, is that if $N(t, \phi)$ were a white Gaussian process, then $\dot{\phi}$ would depend only on the present value of $\phi$ and a white Gaussian noise process—that is, its future value would depend only on the present.

When $\theta(t) = \Omega_o t + \theta_0$, then, using (3-18) and (3-23), the correlation function of $N(t, \phi)$ becomes

$$R_N(\tau) \approx r(\tau) \left[ \cos \Omega_o \tau \cos (\phi - \phi_0) - \sin \Omega_o \tau \sin (\phi - \phi_0) \right]$$

(9-21)

and we have assumed that the phase error at time $t$ is independent of channel noise at time $t$. This is approximately true when the loop bandwidth is small in comparison with the bandwidth of the noise process $\{n(t)\}$. Introducing such an approximation is equivalent to the condition that, for values of $\tau$ greater than the correlation time $\tau_n$ of the additive noise, then $R_N(\tau) \approx 0$ if $\tau > \tau_n$, since $\tau > \tau_n, r(\tau) \approx 0$. For $\tau < \tau_n$, $\cos (\phi - \phi_0) \approx 1$ and $\sin (\phi - \phi_0) \approx 0$, so that (9-22) reduces to $R_N(\tau) \approx r(\tau) \cos \Omega_o \tau$. We previously arrived at this result via another route in Chapter 3, Section 3-4. Assuming further that $r(\tau)$ is of the form given in (3-34), then the correlation function of $N(t, \phi)$ is well approximated by

$$R_N(\tau) \approx \left( \frac{N_0}{2} B_i \text{sinc} \pi B_i \tau \right) \cos \Omega_o \tau$$

(9-22)

so that for large $B_i$ we have

$$\lim_{B_i \to \infty} R_N(\tau) \to \frac{N_0}{2} \delta(\tau)$$

(9-23)

Thus when $B_i$ is large in comparison to $B_L$, the phase-noise process $\{N(t, \phi)\}$ seen by the loop is approximately white. Consequently, the Fokker-Planck equation and the diffusion approximation of Chapter 7 apply.

*It is worth noting that $\sin \phi = \sin \phi$ and $\cos \phi = \cos \phi$; however, $\phi = \phi$ only in the linear region of loop operation. Thus from (3-23) we have that $N(t, \phi) = N(t, \phi)$.
Under the preceding assumptions, the intensity coefficients that characterize the Fokker-Planck (FP) equation can be determined from (7-41) and (9-13) with \( \varphi \) replaced by \( \phi \). These turn out to be given by

\[
K_1(\phi) = \Lambda_0 - AKg(\phi) \tag{9-24}
\]

\[
K_2(\phi) = \frac{N_0K^2}{2} \tag{9-25}
\]

so that the probability current density given in (7-43) through the section \( \phi \) due to signal and noise becomes

\[
\mathcal{J}(\phi; t) = [\Lambda_0 - AKg(\phi)]p(\phi; t) - \frac{K^2N_0}{4} \frac{\partial p(\phi; t)}{\partial \phi} \tag{9-26}
\]

while the equation of probability flow, (7-42), becomes

\[
\frac{\partial}{\partial \phi}[[\beta - \alpha g(\phi)]p] - \frac{\partial^2 p}{\partial \phi^2} + \frac{4}{K^2N_0} \frac{\partial p}{\partial t} = 0 \tag{9-27}
\]

where \( p \triangleq p(\phi; t) = p(\phi, t|\phi_0, t_0, n) \) and

\[
\alpha \triangleq \frac{4A}{N_0K} = \frac{2A^2}{N_0W_L} = \rho \quad \beta \triangleq \frac{4(\Lambda_0)}{N_0K^2} = \gamma \alpha \tag{9-28}
\]

are the parameters that characterize loop performance. In (9-28) we have made use of the parameters defined by the linear theory for first-order loops, see Chapter 4, Section 4-3. From (9-26) and (9-27) it is clear that in the steady state the probability current density is constant.

### 9-3.2 Boundary Conditions and Initial Conditions

In order to obtain solutions to the FP equation, we have to supplement it with initial conditions and boundary conditions. From (9-18) and (9-20) we see that the initial transition p.d.f. is given by

\[
\lim_{t \to -\infty} p(\phi, t|\phi_0, t_0, n) = \delta(\phi - \phi_0) \tag{9-29}
\]

where \( \phi_0 \) belongs to the interval \( I(n), n \) any integer. The subsequent evolution of this p.d.f. is found from (9-27). Since \( \bar{p} \) of (9-17) is periodic, we have from (9-20) that

*Actually, these coefficients can be determined by inspection using the theory given in Chapter 8, in particular, (8-44), (8-50) and comparison of (8-1) with (9-13).*
Characterization of the Statistical Behavior in the Presence of Noise

\[ p[(2n - 1)\pi; t] = p[(2n + 1)\pi; t] \quad (9-30) \]

and from (9-17), (9-20), and (9-26) we have that

\[ J[(2n - 1)\pi; t] = J[(2n + 1)\pi; t] \quad (9-31) \]

for all \( t \geq t_0 \). Finally, the conservation of probability requires that

\[ \int_{(2n-1)\pi}^{(2n+1)\pi} p(\phi; t) \, d\phi = 1 \quad (9-32) \]

for all \( t \). Boundary conditions (9-30) and (9-32) are required when finding the steady-state solution to (9-27).

**9-3.3 Steady-State Probability Density of the Phase Error**

We are interested in the steady-state p.d.f. of the phase error which, from (9-27), satisfies the equation

\[ \frac{d^2 p(\phi|n)}{d\phi^2} - \frac{d}{d\phi} \left\{ [\beta - \alpha g(\phi)] p(\phi|n) \right\} = 0 \quad (9-33) \]

where we have made use of the fact that as \( t \) approaches infinity, \( p(\phi; t) \) approaches \( p(\phi|n) \). Using (7-83), the steady-state solution to (9-27) or (9-33) can be written as

\[ p(\phi|n) = \frac{2 \exp \left[ -U_0(\phi) \right]}{N_0 K^2} \left\{ C - 2 \int_{(2n-1)\pi}^{\phi} \exp \left[ U_0(y) \right] \, dy \right\} \quad (9-34) \]

where

\[ U_0(\phi) = -\int^{\phi} h_0(y) \, dy \quad h_0(\phi) = \frac{2K_1(y)}{K_2(y)} \quad (9-35) \]

\[ = -\beta \phi + \alpha \int^{\phi} g(x) \, dx \]

is the potential relative to the noise and \( J \) is the steady-state probability current density. Using (9-30) and (9-34), it is easy to show (see Appendix I) that

\[ p(\phi|n) = C'_0 \exp \left[ -U_0(\phi) \right] \int_{\phi}^{\phi+2\pi} \exp \left[ U_0(y) \right] \, dy \quad (9-36) \]

where \( C'_0 \) is the normalization constant and \( \phi \in I(n) \). Since (9-36) depends upon \( n \) only through the condition \( \phi \in I(n) \) we set \( p(\phi) = p(\phi|n) \) for convenience.
9-3.4 Phase Error Density for the Sinusoidal PLL

With \( g(\phi) = \sin \phi \) in (9-35) and (9-36), the constant \( C'_0 \) is found from the normalization condition (9-32). This is given by (see Appendix I)

\[
C'_0 = [4\pi^2 \exp (-\pi \beta |I_{j\beta}(\alpha)|^2)]^{-1}
\]

so that the steady-state p.d.f.

\[
p(\phi) = \frac{\exp (\beta \phi + \alpha \cos \phi)}{4\pi^2 \exp (-\pi \beta |I_{j\beta}(\alpha)|^2)} \int_{\phi}^{\phi+2\pi} \exp (-\beta y - \alpha \cos y) \, dy
\]

where \( \phi \in I(n) \). Various methods for evaluating \( |I_{j\beta}(\alpha)|^2 \) numerically are given in Appendices II and IV, (also see Prob. 9-21).

Stratonovich (Ref. 3) originally obtained this result when studying the problem of oscillator instability in a tuned-circuit feedback oscillator. Tikhonov (Ref. 4) arrived at (9-38) for the first-order PLL. Here \( I_{j\beta}(x) \) is the Bessel function of imaginary order and imaginary index. In Appendix II we present a summary of mathematical formulas relative to Bessel functions of imaginary order and imaginary argument. The p.d.f. \( p(\phi) \) has the Fourier series representation

\[
p(\phi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} C_n \exp (jn\phi) \quad C_n = E[\exp (-jn\phi)]
\]

where the circular moments needed to define \( C_n \) are given by (see Appendix III)

\[
\cos n\phi = \text{Re} \left[ \frac{I_{n-j\beta}(\alpha)}{I_{-j\beta}(\alpha)} \right] \quad \sin n\phi = \text{Im} \left[ \frac{I_{n-j\beta}(\alpha)}{I_{-j\beta}(\alpha)} \right]
\]

with \( \text{Re} [\cdot] \) and \( \text{Im} [\cdot] \) denoting, respectively, the real and imaginary part of the bracketed quantities. Although we do not pursue the matter here, we point out that the circular moments are related to the wedge functions \( F_s(x) \) and \( G_s(x) \) discussed in Appendix II. They also serve to characterize carrier suppression in coherent transponder systems and the power and torque developed in a synchronous machine. The wedge functions are defined by

\[
F_s(x) \triangleq \frac{\pi}{2} \left[ \frac{I_{j\beta}(x) + I_{-j\beta}(x)}{\sinh \pi y} \right] = \frac{\pi}{2} \frac{\text{Re} [I_{j\beta}(x)]}{\sinh \pi y}
\]

\[
G_s(x) \triangleq \frac{j\pi}{2} \left[ \frac{I_{j\beta}(x) - I_{-j\beta}(x)}{\sinh \pi y} \right] = -\frac{\pi}{2} \frac{\text{Im} [I_{j\beta}(x)]}{\sinh \pi y}
\]

and their name is derived from the fact that in potential theory they show a certain analogy to the solutions of Legendre's equation called "cone functions." We also write
Characterization of the Statistical Behavior in the Presence of Noise

\[ I_f(x) = \frac{\sinh \pi y}{\pi} \left[ F_r(x) - jG_r(x) \right] \]  

(9-42)

which relates the Bessel function to the wedge functions.

![Graph showing the steady-state probability density function of the phase error process, \( \phi(t) \).](image)

**Fig. 9-6a.** Steady-State Probability Density Function of the Phase Error Process, \( \phi(t) \).

![Graph showing the steady-state probability density function of the phase error process, \( \phi(t) \).](image)

**Fig. 9-6b.** Steady-State Probability Density Function of the Phase Error Process, \( \phi(t) \).
The behavior of the p.d.f. \( p(\phi) \) as a function of the mismatch \( \gamma = \beta/\alpha = \Lambda_0/AK \) is illustrated in Figs. 9-6a and 9-6b for signal-to-noise ratios in the loop bandwidth of \( \alpha = \rho = 2 \) and 6 respectively. The cumulative distributions

\[
F(\phi) = \int_{-\pi}^{\phi} p(x) \, dx, \quad n = 0
\]  

(9-43)

are shown in Figs. 9-7a and 9-7b for various values of \( \beta/\alpha \) for \( \alpha = 2 \) and 6. It follows from an analysis of the graphs that for an increase in the normalized initial frequency detuning (i.e., the parameter \( \beta/\alpha \)), there is a resulting shift in the maximum of \( p(\phi) \) in the direction of the detuning and, as we shall later see, an increase in the variance of the phase error. In addition, the function \( p(\phi) \) becomes more asymmetric as the detuning approaches the synchronization band—that is, \( \Lambda_0 = B_r \).

Fig. 9-7a. Steady-State Probability Distribution Function of the Phase Error Process, \( \phi(t) \).
Let us investigate the behavior of the mean value $\bar{\phi} = m_\phi$ of the phase error and its variance $\sigma_\phi^2$ as a function of the parameter $\beta$. The mean value of the phase error may be found in terms of tabulated functions from (9-38) and the well-known Bessel function expansions for $\exp(\pm x \cos \phi)$. Without belaboring the details, we have (see Appendix I)

$$m_\phi = \bar{\phi} = \int_{-\pi}^{\pi} \phi p(\phi) \, d\phi$$

$$= \frac{2 \sinh \pi \beta}{\pi |I_{\beta}(\alpha)|^2} \sum_{m=1}^{\infty} \frac{m I_m(\alpha)}{m^2 + \beta^2}$$

$$\times \left[ \frac{I_0(\alpha)}{m} + \frac{I_m(\alpha)}{4m} + \sum_{k=1}^{\infty} \frac{2m(-1)^k I_k(\alpha)}{m^2 - k^2} \right]$$

(9-44)

where we have set $n = 0$ without loss in generality—that is, $E[(\phi - 2\pi n)^k] =$
$E(\phi^k)$. It is clear from (9-44) that with $\beta = 0$, $\bar{\phi} = 0$. Furthermore, $\bar{\phi^2}$ is given by (see Appendix I)

$$
\bar{\phi^2} = \int_{-\pi}^{\pi} \phi^2 p(\phi) d\phi \\
= \frac{\sinh \pi \beta}{\pi |I_{\beta}(\alpha)|^2} \left\{ \frac{I_0(\alpha)}{3} \left[ \frac{\pi^2 I_0(\alpha)}{k^2} + 4 \sum_{k=1}^{\infty} \frac{(-1)^k I_k(\alpha)}{k^2} \right] + 2\beta I_0(\alpha) \sum_{k=1}^{\infty} \frac{I_k(\alpha)}{k^2(\beta^2 + k^2)} + 2\beta \sum_{k=1}^{\infty} \frac{(-1)^k I_k(\alpha)}{\beta^2 + k^2} \times \left[ \left( \frac{\pi^2}{3} + \frac{1}{2k^2} \right) I_k(\alpha) + 4 \sum_{m \neq k} \frac{(-1)^m (k^2 + m^2) I_m(\alpha)}{(k^2 - m^2)^2} \right] \right\} 
$$

(9-45)

Fig. 9-8. Mean $m_{\phi}$ versus $|\gamma|$ for Various Values of $\alpha$.

Fig. 9-9. Standard Deviation $\sigma_{\phi}$ versus $|\gamma|$ for Various $\alpha$. 
The variance \( \sigma^2 = \bar{\delta}^2 - (\bar{\phi})^2 \) is minimized when \( \beta = 0 \) and \( \alpha \) is maximized. For \( \beta = 0 \) we have, from (9-45), that

\[
\sigma^2_{\min} = \frac{\pi^2}{3} + \frac{4}{I_0(\alpha)} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} I_k(\alpha)
\]  

(9-46)

Fig. 9-10. Mean \( \sin \bar{\phi} \) versus \( |\gamma| = |\beta|/\alpha \) for Various Values of \( \alpha \).
The relations $m_{\phi}$ and the standard deviation $\sigma_{\phi}$, obtained by means of numerical integration, are represented in Figs. 9-8 and 9-9 respectively. It is apparent from an analysis of the graphs that the mean value and variance of the phase error increase as the detuning decreases. As $\gamma$ approaches unity, the increase in the mean value slows down, whereas on the other hand, the increase

Fig. 9-11. Mean $\cos \phi$ versus $|\gamma|$ for Various Values of $\alpha$. 
in variance speeds up. The abscissa of the most probable value of $p(\phi)$ occurs at $\varphi_m = \sin^{-1} \gamma$. Finally, the moments of $\sin \phi$, $\cos \phi$ and their respective variances are plotted in Figs. 9-10 through 9-13 respectively.

It is convenient to give a physical interpretation of our solution in terms of the concept of “potential wells” containing a particle. As previously mentioned, the position of the trajectory $\varphi(t)$ at time $t$ can be thought of as representative of the position of a particle undergoing Brownian motion. In fact, the motion of the $\varphi$ trajectory along the $\varphi$ axis can be interpreted as the motion of a particle in an external force field $h_0(\varphi)$ whose dependence on the position is nonlinear. The function $U_0(\varphi) = -\int \varphi h_0(x)\,dx$ represents the potential at $\varphi$ relative to the noise, and $\{U_0(\varphi) - U_0(\varphi + 2\pi)\}$ represents the potential difference a distance of $2\pi$ radians apart.
Figure 9-13. Variance $\sigma_{\cos \phi}^2$ versus $|\gamma|$ for Various Values of $\alpha$.

Figure 9-14 illustrates the normalized function $\beta^{-1}U_0(\phi)$ vs. $\phi$ for various values of $\beta/\alpha$ and for the case of a PLL. In fact, all positions of possible phase lock are found when the restoring force $h_0(\phi)$ is zero. According to this theory, if $\alpha > \beta$, phase-lock is not possible, for no well bottoms $B_n$ exist. The particle
(phase error) will slip from well to well without coming to rest, and the probability flow is continuous to the right. If, however, $\beta = \alpha$, then the solutions to $\sin \varphi = 1$ specify inflection points of $U_\alpha(\varphi)$. If $\beta < \alpha$, well bottoms ($B_n$—stable points of potential minima) occur at $\varphi = 2n\pi + \sin^{-1}(\beta/\alpha)$ and well tops ($T_n$—unstable points of potential maxima) occur at $\varphi = (2n + 1)\pi - \sin^{-1}(\beta/\alpha)$. According to this theory, the slipping occurs more rapidly the greater the $\beta$ and the shallower the well depth $\alpha$. In other words, one wishes to design the loop such that, for any external conditions, $\beta$ is minimized and $\alpha$ is maximized. As we shall see, the graphical results presented thus far will serve as a set of universal curves for higher-order sinusoidal PLLs.

It is of interest to compare the variance of the phase error as determined by the linear PLL theory to that predicted by the nonlinear theory when $\beta = 0$. Figure 9-15 makes this comparison. It appears that for a loop signal-to-noise ratio $\alpha$ greater than 10 dB, the two theories agree when attempting to predict variance.
9-3.5 Statistical Dynamics of the Phase Error Rate, $\dot{\phi}$

In this section we study the first and second moments of the phase error rate. The phase error rate is important because its average value represents the residual frequency detuning of the loop which gives a measure of the frequency stability of the VCO. The fact that this is true is made obvious by noting from (3-20) with oscillator instabilities neglected, that the instantaneous VCO radian frequency is given by $\omega_V(t) \triangleq \hat{\theta}(t) + \omega_0 = \omega_0 + K_v(z + e)$. In the steady state its average value becomes $\bar{\omega}_V = \omega_0 + K_v(\bar{z} + e)$. Using (3-19) the instantaneous phase error rate is given by $\dot{\phi} = \dot{\Phi} - \dot{\hat{\theta}} = \omega - [\omega_0 + K_v(z + e)]$. Note now that $\dot{\phi}(t_0) = \omega - \Lambda_0$ and in the steady state $\bar{\dot{\phi}} = \omega - \bar{\omega}_V$. Consequently, the relative change in the VCO radian frequency from $t = t_0$ to the steady state is given by $\omega_V(t_0) \to \bar{\omega}_V = \bar{\phi} - \Lambda_0$. This says that $\bar{\dot{\phi}}/2\pi$ represents the average net number of cycles the VCO has changed from its initial value. Therefore, we can write $\bar{\dot{\phi}}/2\pi = N_u - N_L$ where $N_u$ ($N_L$) represents the average number of cycles per second the VCO has been pushed to the right (left) in the steady state. Since $\sin \varphi = \sin \phi$ and $\{N(t, \varphi)\}$ has zero mean, we have from (9-13) that

$$\bar{\dot{\phi}} = \bar{\dot{\Phi}} = \int_{-\pi}^{\pi} (\Lambda_0 - AK \sin \phi) p(\phi) d\phi = \Lambda_0 - AK \bar{\sin \phi}$$  \hspace{1cm} (9-47)
Using (9-40) to evaluate $\bar{\phi}$ we have

$$\bar{\phi} = \Lambda_0 - AK \text{Im} \left[ \frac{I_{1-j\beta}(\alpha)}{I_{-j\beta}(\alpha)} \right]$$

(9-48)

An alternate and more useful expression for $\bar{\phi}$ can be found from (9-26) by observing that in the steady state

$$(\Lambda_0 - AK \sin \phi) p(\phi) = \mathcal{J}(\phi) + \frac{N_0 K^2 d p(\phi)}{4 d\phi}$$

(9-49)

Substituting (9-49) in (9-47), keeping in mind that the probability current density $\mathcal{J}(\phi)$ is constant in the steady state, and using the fact that $p(\phi)$ is periodic, we get

$$\bar{\phi} = 2\pi \mathcal{J} = 2\pi [N_+ - N_-]$$

(9-50)

Using (9-26) and (9-38) to evaluate $\mathcal{J}$, we obtain

$$\bar{\phi} = \Lambda_0 \text{sinch} \frac{\pi \beta}{|I_{j\beta}(\alpha)|^2} = 2\pi [N_+ - N_-]$$

(9-51)

where $\text{sinch} x = \sinh x / x$ and $\beta = \alpha \gamma$. Equating (9-48) and (9-51), we have an identity relating the imaginary part of the ratio of two Bessel functions of imaginary order. Figure 9-16 illustrates a plot of (9-51) as obtained by means of a digital computer. We will now discuss three limiting cases of this result.

![Fig. 9-16. Ratio of the Residual Frequency Detuning to the Initial Frequency Detuning versus $|\gamma|$ for Various Values of $\alpha$.](image)
When the noise is large, $\beta$ and $\alpha$ both approach zero even though $\Lambda_0$ and $AK$ remain constant. Since $I_0(0) = 1$ we get from (9-51)

$$\bar{\phi} = \Lambda_0$$ (9-52)

Thus in the case where the noise is increased without limit, the frequency error equals $\Lambda_0$; consequently, the presence of the external signal produces no effect on VCO synchronization. At the other extreme—that is, very low noise—the relationship

$$\bar{\phi} = \begin{cases} 
0 & \text{if } \Lambda_0 \leq AK \\
\sqrt{\Lambda_0^2 - (AK)^2} & \text{if } \Lambda_0 > AK 
\end{cases}$$ (9-53)

holds good. Hence phase-lock is obtained for low noise conditions if $\Lambda_0 \leq AK$. For the case where $\alpha \gg 1$ and $\Lambda_0 < AK$, we may use the asymptotic expansion for $I_\nu(x)$ to obtain

$$\bar{\phi} = 2AK \sinh \pi \beta \exp(-2\alpha)$$ (9-54)

We now calculate the variance of the frequency error. From (9-13) we write

$$\bar{\phi}^2 \approx (\Lambda_0 - AK \sin \phi)^2 + K^2 \sigma_N^2$$ (9-55)

so that

$$\sigma_\phi^2 \approx (AK)^2 \sigma_{\sin \phi}^2 + K^2 \sigma_N^2$$ (9-56)

where we have made use of (9-51). In (9-56), $\sigma_{\sin \phi}^2 = \bar{\sin^2 \phi} - [(\sin \phi)]^2$ and the averages of sin $\phi$ can be obtained from (9-40). Note here that $\sigma_N^2$ represents the actual variance of $N(t, \phi)$ which is found from (9-22).

**9-3.6 Probability Current Density and the Average Rate of Cycle-Slapping**

Due to the presence of noise, discontinuities in synchronization occur. The VCO may slip a cycle—that is perform one more oscillation than the external signal. To calculate the average number of cycles-slimmed per unit of time, we make use of the concept of probability current. The average flow of probability through the section $\phi = \phi'$ in the positive direction per unit time is found from (9-50) and (9-51).

$$J = \frac{\Lambda_0}{2\pi} \frac{\sinh \pi \beta}{\pi \beta} \left[ \frac{I_{j\beta}(\alpha)}{I_{j\beta}(\alpha)} \right] = \frac{W_{k}}{\alpha} \left[ \frac{\sinh \pi \beta}{\pi^2 I_{j\beta}(\alpha)} \right]$$ (9-57)
If $\Lambda_0 = 0$, then $\mathcal{J} = 0$ and there is no net flow of probability through the section $\phi = \phi'$. From stochastic field theory we know that the curl of the stochastic magnetic field intensity $\mathcal{H}$ is equal to the probability current density $\mathcal{J}$ in the steady state. The curl is zero when the detuning is zero. Plotted in Fig. 9-17 is $\mathcal{J}/W_L$ vs $\alpha = \rho$ for various values of $|\gamma|$. Thus $\mathcal{J} = N_+ - N_-$ represents the average net number of cycles-slipped per unit time. The ratio $N_+/N_-$ can be obtained using results from the theory of statistical mechanics, which specifies the rate of escape of Brownian particles over a potential bar-

**Fig. 9-17.** Probability Current Density versus $\alpha$ for Various $|\gamma| = |\beta|/\alpha$. 
rier (see Chapter 7). For a cycle-slip to occur, it is necessary for the particle normally remaining in the plane of potential minimum at the position \( \phi(t) \) to overcome the potential barrier represented by \( U_0(\phi) \). As a result of \( \Lambda_0 \neq 0 \), the height of the barrier to the right is not equal to the height of the barrier to the left. Therefore from (7-149) or (7-232) the ratio of the current flow to the right to the current flow to the left for any \( \phi \) is given by

\[
\frac{N_+}{N_-} = \exp \left[ U_0(\phi) - U_0(\phi + 2\pi) \right] = \exp (2\pi\beta) \quad (9-58)
\]

For \( \beta = 0 \) we see from (9-58) that the to-the-left cycles slipped equal the to-the-right cycles slipped; that is, the flow of probability to the left equals the flow of probability to the right.

Knowing the difference \( \mathcal{J} = N_+ - N_- \) and the ratio in (9-58), it is possible to determine them independently, thus,

\[
N_\pm = \frac{\exp (\pm \pi\beta) \mathcal{J}}{2 \sinh \pi\beta} = \frac{W_L \exp (\pm \pi\beta)}{2\pi^2 \alpha |I_{1\beta}(\alpha)|^2} \quad (9-59)
\]

If \( \Lambda_0 < AK \) and \( \alpha \gg 1 \), then

\[
N_\pm = \frac{W_L}{\pi} \exp (\pm \pi\beta - 2\alpha) \quad (9-60)
\]

From this equation we see that when \( \Lambda_0 \neq 0 \), the number of phase-jumps is not the same in the opposite directions. Figure 9-18 shows the dependence of the relative number of cycles-slipped in opposite directions on the loop signal-to-noise ratio \( \alpha = \rho \). An analysis of the graph shows that the shaded area corresponds to the normalized difference \( (N_+ - N_-)/W_L \). It is interesting to note that in the presence of frequency detuning, the number of cycles-slipped in the direction corresponding to sign of \( \Lambda_0 \) is always larger than the number of slips in one direction when \( \Lambda_0 = 0 \), provided that \( \rho \) remains the same. Consequently, synchronization devices operate with highest frequency stability when \( \Lambda_0 \) is close to zero. Notice from this figure that once cycle-slips begin to appear, they increase rapidly for a small reduction in \( \alpha \). The threshold point in a loop could well be defined as that condition for which the number of slips per unit time remain below a certain level.

Knowing the difference \( N_+ - N_- \) and the ratio given in (9-58), we can evaluate the total number of cycles-slipped per unit of time independent of direction; that is,

\[
\tilde{S} \triangleq N_+ + N_- = (\coth \pi\beta) \mathcal{J}
\]

\[
= \frac{\Lambda_0 \left( \cosh \pi\beta \right)}{2\pi \left( \frac{\pi\beta}{|I_{1\beta}(\alpha)|^2} \right)} \frac{1}{\alpha \pi^2 |I_{1\beta}(\alpha)|^2} = \frac{W_L \cosh \pi\beta}{\alpha \pi^2 |I_{1\beta}(\alpha)|^2} \quad (9-61)
\]
Fig. 9-18. Average Number of Cycle-Slips in Opposite Directions, $\Lambda_0 \neq 0$.

which accounts for the diffusional spreading of the number of oscillations in the signal generated by the VCO. Moreover, the expected value of the time intervals between cycle-slipping events is given by

$$\Delta T = \frac{1}{\bar{S}}$$  \hspace{1cm} (9-62)
Numerical results are easily obtained from the theory and the use of a digital computer. In the next section we present results from which numerical values of (9-61) and (9-62) can be obtained.

9-3.7 Characterization of the Cycle-Slapping Probabilities

In this section we shall characterize the probability of slipping one or more cycles in \( t - t_0 \) seconds. We begin by first observing from Fig. 9-3 that the absolute phase is given by

\[
\phi(t) = \phi(t) + 2\pi J(t)
\]  

(9-63)

where at any instant \( t \), \( J(t) \) is a unique integer, \( j = 0, \pm 1, \pm 2, \ldots \), which accounts for the net number of phase-jumps which have occurred since \( t = t_0 \). Thus, \( J(t) \) is a discrete r.v. that takes on integer values at random points in time. The mean-squared value of \( \phi(t) \) is given by

\[
\sigma^2_\phi = \sigma^2_\phi(t) = (2\pi)^2 \sigma^2_\phi(t) + 4\pi [J(t)\phi(t) - J(t)\phi(t)]
\]  

(9-64)

and we see how the variance of \( \phi \) is related to the statistics of \( \phi \) and \( J \).

On the other hand, to characterize the cycle-slapping probabilities we shall assume that the cycle slips which tend to increase and decrease \( \phi \) by \( 2\pi \) form independent Poisson processes with rates of occurrence \( N_+ \) and \( N_- \), respectively, then the probability that a net number, say, \( N = n \), of cycle slips occur in \( t - t_0 \) seconds is given by

\[
P[N = n] = \left( \frac{N_+}{N_-} \right)^\frac{n}{2} \exp \left[ \frac{S(t - t_0)}{N_-/N_+} \right] I_n(2(t - t_0) \sqrt{N_+/N_-})
\]  

(9-65)

for \( n = 0, \pm 1, \pm 2, \ldots \). This is shown by forming the convolution of the difference of two independent Poisson processes. The fact that the p.d.f. for cycle-slapping can be approximated by a Poisson process has been observed experimentally.

9-3.8 The Diffusion Coefficient and Loop Threshold

The quantity \( \bar{S} \) is related to the phase error "diffusion coefficient" \( D_\phi \) which determines the rate at which the phase error undergoes diffusion and the variance \( \sigma^2_\phi(t) \) approaches infinity. To determine \( \sigma^2_\phi(t) \) we shall assume that the phase-jumps of \( \pm 2\pi \) from \( \phi = (2n \pm 1)\pi \) are Poisson distributed with a rate determined by the average number of crossings per second, \( \bar{f} \), of the axis \( \phi = (2n \pm 1)\pi \). If we say that the rate of phase-jumps tending to increase (decrease) \( \phi \) by \( 2\pi(-2\pi) \) is given by \( N_+(N_-) \) then from (9-50)
\[ E(J) = (N_+ - N_-)t = Jt \]  

and

\[ \sigma^2(t) = \tilde{S}t \]  

Using these facts in (9-64) we have

\[ \sigma^2(t) \approx \sigma^2 (t) + (2\pi)^2 \tilde{S}t + 4\pi [J(t)\phi(t) - \overline{J(t)}\phi(t)] \]  

\[ \frac{D_C}{W_L} = \frac{2\pi^2 S}{W_L} \]

**Fig. 9-19.** Normalized Diffusion Coefficient versus \( \alpha \) for Various |\( \gamma \)|. 
If we assume that the r.v.'s $J$ and $\phi$ are approximately independent, then in the steady state, we can write

$$\lim_{t \to \infty} \sigma^2_{\psi}(t) \approx \sigma^2_{\phi} + \lim_{t \to \infty} D_c t$$  \hspace{1cm} (9-69)

and this accounts for the fact that (9-16) is true. Moreover, we can write

$$\lim_{t \to \infty} \sigma^2_{\phi}(t + T) - \sigma^2_{\phi}(t) = D_c T$$  \hspace{1cm} (9-70)

where $D_c = (2\pi)^2 \tilde{S}$ and $T$ is the time required to make a phase measurement. Equation (9-70) accounts for the change in the variance of the phase error in $T$ seconds. If in $T$ seconds $D_c T$ is small in comparison with $\sigma^2_{\phi}$, then the phase measurement is a useful one. In fact, $D_c T$ is responsible for the threshold phenomenon familiar in PLLs. Obviously for a PLL to be useful in practice the rate $D_c$ at which the phase error process undergoes diffusion must be controlled. Figure 9-19 illustrates a plot of $D_c/W_L$ for various of $\alpha$ and $\gamma$.

### 9-3.9 Probability of Synchronization Failure (Loss of Lock)

It is also of interest to compute the probability of synchronization failure, i.e., loss of phase-lock. To do this we let $\mathcal{S} = n_+ + n_-$ be a r.v. whose value at time $t$ equals the total (independent of direction) number of cycles slipped per second. Assuming, as before that $n_+$ and $n_-$ are independent Poisson r.v.'s with rates $N_+$ and $N_-$, then from Section 1-13 of Chapter 1, we have that

$$P[n_+ = q] \approx \left[\frac{N_+(t - t_0)}{q!}\right]^q \exp\left[-N_+(t - t_0)\right]$$  \hspace{1cm} (9-71)

where $q = 0, 1, 2, \ldots$. Using this it is easy to show that

$$P[\mathcal{S} = m] \approx \left[\tilde{S}(t - t_0)\right]^m \exp\left[-\tilde{S}(t - t_0)\right]$$  \hspace{1cm} (9-72)

for $m = 0, 1, \ldots$. Thus the probability of losing phase lock or synchronization failure in $t - t_0$ seconds is equivalent to slipping one or more cycles in $t - t_0$ seconds, i.e.,

$$P(t) \triangleq P[\mathcal{S} = k \geq 1] \approx 1 - \exp\left[-\tilde{S}(t - t_0)\right]$$  \hspace{1cm} (9-73)

Figure 9-20 illustrates a plot of (9-73) for a PLL with $\gamma = 0, t_0 = 0$ and various values of $\alpha$. As we shall see, in the next section, $\tilde{S}$ is the reciprocal of the mean time to first slip of cycle.
Fig. 9-20. Probability of Losing Lock versus $W_L t$ for Various Values of $\alpha$.

9-4 Moments of the Mean Time to First Slip in a First-Order SCS and the Probability of Sync Failure

In this section we shall apply the theory developed in Chapter 7, Section 7-8, to the problem of finding the moments of the mean time to first slip or first loss of synchronization. The formula that we develop is recursive in nature, and the details of our development will be generalized in a later chapter when we study the nonlinear theory of higher-order SCSs. For this reason our development will be rather concise.

Before proceeding, it is interesting to note that the mean time to first slip is not as useful to the communications engineer as the average number of slips per unit time. The reason can be made clear by discussing the operation of synchronous motors. In electric power generation and regulation, the mean time to first skip a pole (analogous to mean first slip time) is very important in that it is indicative of the mean time before the motor will fall out of step and cease to operate. A SCS will frequently slip one or more cycles in succession, after which the oscillations of the VCO will become resynchronized with the oscillations of the incoming signal. Nevertheless, in the past, the communications engineer has used the first slip time frequently as a measure of loop performance.

Here we are interested in the $n$th moment of the time that it takes for the random function $\varphi(t)$ to first reach the boundaries $b_1 = \varphi_{II}$ and $b_2 = \varphi_{I2}$, given that at $t = t_0$, $\varphi(t_0) = \varphi_0$. We shall use $\tau^n(\varphi_0|\varphi_0)$ to denote the $n$th moment
of the first-passage time of \( \varphi \) to \( \varphi_{11} \) or \( \varphi_{12} \). As discussed in Section 7-8 of Chapter 7, the function \( Q = Q(\varphi; t) = Q(\varphi, t; \varphi_0, t_0) \) satisfies the differential equation

\[
\frac{\partial[(\beta - \alpha g(\varphi))Q]}{\partial \varphi} - \frac{\partial^2 Q}{\partial \varphi^2} + \frac{4}{N_0 K^2} \frac{\partial Q}{\partial t} = 0
\]

(9-74)

According to (7-111), (7-112), and (7-113), the boundary and initial conditions are given by

\[
Q(\varphi_{11}; t) = Q(\varphi_{12}; t) = 0 \\
\lim_{t \to 0} Q(\varphi; 0) = 1 \quad \lim_{t \to \infty} Q(\varphi; t) = 0
\]

(9-75)

so that the \( n \)th moment can be written from the rather general result given in (7-130). Identifying terms, we have

\[
W_L \tau^n(\varphi_1|\varphi_0) = \frac{\alpha}{2} \int_{\varphi_{11}}^{\varphi_{12}} \int_{\varphi_0}^{\varphi} n[C(n - 1) - \tau^{n-1}(x|\varphi_0)] \exp[U_0(x) - U_0(\varphi)] \, dx \, d\varphi
\]

(9-76)

where

\[
C(n - 1) = \frac{\int_{\varphi_{11}}^{\varphi_{12}} \tau^{n-1}(x|\varphi_0) \exp[U_0(x)] \, dx}{\int_{\varphi_{11}}^{\varphi_{12}} \exp[U_0(x)] \, dx}
\]

(9-77)

with \( \tau^0(x|\varphi_0) = u(x - \varphi_0) \).

**9-4.1 Moments of the First Slip Time of a Sinusoidal PLL**

Letting \( \varphi_{12} = \varphi_0 + 2\pi, \varphi_{11} = \varphi_0 - 2\pi, g(\varphi) = \sin \varphi \), and using (9-35) in (9-76), we have the \( n \)th moment of the mean time to first slip.

\[
W_L \tau^n(2\pi|\varphi_0) = \frac{\alpha}{2} \int_{\varphi_{11}}^{\varphi_{12}} \int_{\varphi_0}^{\varphi} n[C(n - 1) - \tau^{n-1}(x|\varphi_0)] \times \exp[\alpha[(\cos \varphi - \cos x) + \gamma(\varphi - x)]] \, dx \, d\varphi
\]

(9-78)

For \( n = 1 \), we have the mean time to first slip for a first-order PLL.

\[
W_L \tau(2\pi|\varphi_0) = \frac{\alpha}{2} \int_{\varphi_{11}}^{\varphi_{12}} \int_{\varphi_0}^{\varphi} [C(0) - u(x - \varphi_0)] \times \exp[\alpha[(\cos \varphi - \cos x) + \gamma(\varphi - x)]] \, dx \, d\varphi
\]

(9-79)
Carrying out the integration (Appendix IV) with \( \varphi_{t_2} = \varphi_0 + 2\pi \) and \( \varphi_{t_1} = \varphi_0 - 2\pi \) yields

\[
W_L(2\pi|\varphi_0) = \frac{W_L}{\bar{S}} = \frac{\pi \tanh (\pi \alpha \gamma)}{\gamma} \left[ I_0^2(\alpha) + 2 \sum_{n=1}^{\infty} (-1)^n \frac{I_n^2(\alpha)}{1 + (n/\alpha \gamma)^2} \right]
(9-80)
\]

Notice that the result is independent of \( \varphi_0 \) and that \( \bar{S} = 1/\tau(2\pi|\varphi_0)! \) In the limit as \( \Lambda_0 \) approaches zero, \( W_L(2\pi|\varphi_0) = \pi^2 \alpha \bar{I}_0^2(\alpha) \), which is originally due to Viterbi (Ref. 5). Plotted in Fig. 9-21 is (9-79) for various values of the loop signal-to-noise ratio \( \alpha \) and normalized detuning \( \gamma \). The numerical results are compared with those obtained by Holmes (Ref. 6), using the method of computer simulation.

![Fig. 9-21. Mean Time to First Slip versus \( \alpha \) for Various \( |\gamma| \).](image)

Table 9-1 compares the mean \( W_L(2\pi|\varphi_0) \) and the variance

\[
(W_L \sigma)^2 = \tau^2(2\pi|\varphi_0) - [\tau(2\pi|\varphi_0)]^2
(9-81)
\]

with the skewness
\[ s = \frac{E[T_{f,c}^{c}(\varphi_0)] - \tau(2\pi|\varphi_0)|^3}{\sigma^2} \]  \hspace{1cm} (9-82)

and excess

\[ e = \frac{E[T_{f,c}^{c}(\varphi_0)] - \tau(2\pi|\varphi_0)|^4}{\sigma^4} - 3 \]  \hspace{1cm} (9-83)

| \alpha | \tau(2\pi|\varphi_0) | \sigma | s   | e  |
|--------|----------------|------|-----|----|
| 1      | 15.82          | 15.15| 1.992 | 5.967|
| 2      | 102.57         | 100.11| 1.999 | 5.998|
| 3      | 704.34         | 702.38| 1.999 | 5.999|
| 4      | 5042.66        | 5039.40| 1.999 | 5.999|
| 5      | 36,616.40      | 36,612.88| 1.999 | 5.998|

for various values of \( \alpha \) with \( W_e = 1 \) and \( \Lambda_0 = 0 \). Notice that \( \tau(2\pi|\varphi_0)/\sigma \) is approximately one, whereas \( s \) and \( e \) are essentially independent of \( \alpha \). The numerical results in Table 9-1 are due to Schuhman (Ref. 7).

9.5 Signal Acquisition Probability and Moments of the Signal Acquisition Time

In this section we show how the general results pertaining to the first-passage time problem given in Section 7-8 of Chapter 7 can be applied to obtain moments of the time to first acquire the signal in phase and frequency. We then show how the signal acquisition probability is determined from the restricted p.d.f. \( Q(\varphi; t) = Q(\varphi, t|\varphi_0, t_0) \).

9.5.1 Acquisition Properties of the First-Order PLL in Noise

As noted earlier, a first-order loop is unstable if \( \gamma > 1 \). On the other hand, synchronization stability can be achieved if \( \gamma < 1 \) and that at \( t = t_0 \), \( \varphi_0 \in [2l\pi + \sin^{-1} \gamma, (2l + 2)\pi + \sin^{-1} \gamma] \), \( l \) any integer, then the \( n \)th moment of the acquisition time, \( T_{\text{acq}}^{n}(\varphi|\varphi_0) \), can be found by placing absorbing barriers at the stable lock points \( b_1 = \varphi_{11} = 2l\pi + \sin^{-1} \gamma \) and \( b_2 = \varphi_{12} = (2l + 2)\pi + \sin^{-1} \gamma \) illustrated in the phase-plane lot of Fig. 9-1 and defined mathematically in (9-3). Consequently, the general theory developed in Section 7-8 and applied in Section 9-4 to the first slip time problem, in particular, (9-76) and (9-77), can be used to find the moments of the first time to the barriers \( b_1 \) and \( b_2 \). In fact, from (9-76) we have that the \( n \)th moment of the time to first acquire
phase-lock is given (using the above definition of phase-lock) by

$$W_L T_{\text{acq}}(\varphi_0) = \frac{\alpha}{2} \int_{\varphi_1}^{\varphi_1} \int_{\varphi_1}^{\varphi_2} n[C(n - 1) - T_{\text{acq}}^{n-1}(x|\varphi_0)]$$

$$\times \exp [U_0(x) - U_0(\varphi)] \, dx \, d\varphi$$

(9-84)

$$\varphi_{i2} = (2l + 2)\pi + \sin^{-1} \gamma$$

$$\varphi_{i1} = 2l\pi + \sin^{-1} \gamma$$

(9-85)

and $C(n - 1)$ is defined in (9-77) by replacing $\tau^{n-1}(x|\varphi_0)$ by $T_{\text{acq}}^{n-1}(x|\varphi_0)$. The integration in the preceding equation can be carried out by using the methods given in Appendix IV of this chapter; however, the easiest way to obtain numerical results is by numerical integration on a digital computer; see Fig. 9-22. Under the above definition of lock, this figure clearly indicates why first-order loops are of no practical interest.

Based on the results given in Section 7-8.1, an alternate expression for the $n$th moment of the time to first acquire phase-lock is given by

![Diagram](image_url)

**Fig. 9-22.** Normalized Mean Acquisition Time versus $\alpha$ for Various Values of $\gamma$. 

where the probability current is defined in terms of the restricted p.d.f. $Q(\varphi; t) = Q(\varphi, t|\varphi_0, t_0)$. By evaluating $Q(\varphi; t)$ from (9-74), subject to the boundary and initial conditions defined in (9-75), we can find the probability currents $\mathcal{J}(\varphi_{12}; t)$ and $\mathcal{J}(\varphi_{11}; t)$ defined by (9-27) with $p(\hat{\varphi}; t)$ replaced by $Q(\varphi; t)$.

Having determined $Q(\varphi; t)$ from (9-75) subject to (9-76), we can evaluate the corresponding probability of signal acquisition, $P_{\text{acq}}(t|\varphi_0)$, in the time interval $[t_0, t]$ from (7-110). This turns out to be

$$P_{\text{acq}}(t|\varphi_0) = 1 - \int_{\varphi_{11}}^{\varphi_{12}} Q(\varphi, t|\varphi_0, t_0) \, d\varphi$$  \hspace{1cm} (9-87)

Thus we clearly see how the restricted p.d.f. $Q(\varphi; t)$ enters into the solution of the signal acquisition problem. If $\gamma > 1$, $T_{\text{acq}}^n(\varphi|\varphi_0)$ and $P_{\text{acq}}(t|\varphi_0)$ remain undefined, and synchronization stability cannot be achieved with a first-order loop. In a later chapter we shall present methods which can be used to determine $p(\hat{\varphi}; t)$ and $Q(\varphi, t|\varphi_0, t_0)$.

9-6 PLL Operation in the Presence of Impulsive and Gaussian Noise

Recently Ohlson (Ref. 15) has been concerned with the behavior of a PLL when the channel interference is non-Gaussian. A very important type of non-Gaussian noise is impulse noise, which was characterized in Section 1-13 of Chapter 1. In this section we shall be primarily concerned with characterizing the steady-state p.d.f. $p(\hat{\varphi})$ when the input noise is both impulsive and band-limited white Gaussian noise. It is important to note that when the average rate of the impulse noise is much greater than $W_L$ and the impulses cause phase error variations much less than $\pi/2$, the impulse noise can be approximated as Gaussian; however, when these assumptions cannot be invoked, then one must resort to a more general analysis.

Following the procedure given in Section 3-4, with $n_i(t)$ replaced by $n_i(t) + n_i(t)$, it is easy to show that the stochastic differential equation of loop operation is given by

$$\dot{\varphi} = \Lambda_0 - AK \sin \varphi - K[N(t, \varphi) + N_i(t, \varphi)]$$  \hspace{1cm} (9-88)

where

$$N_i(t, \varphi) = \sqrt{2} n_i(t) \cos \hat{\theta}$$  \hspace{1cm} (9-89)
with

\[ n_i(t) = \sum_j x_j \delta(t - t_j) \]  \hspace{1cm} (9-90)

The \( x_j \)'s in (9-90) are assumed to be independent, identically distributed r. v.'s with p.d.f. \( p(x_j) \) and the times, \( t_j \), are assumed to occur in the Poisson fashion described by (1-138). When an impulse occurs, its resultant amplitude in (9-89) will vary with time due to the cosine factor. Assuming that the rate of occurrence, \( f \), in (1-138) is much less than \( \omega_0 \) (i.e., \( f \ll f_0 \)), then \( \text{Var}(\omega_0 t_j) \gg 4\pi^2 \), so the modulo-2\( \pi \) argument \( \Phi \) of the cosine factor is approximately uniform on \([-\pi, \pi]\). Therefore, using (9-89) and (9-90), we can write

\[ N_i(t, \varphi) \approx \sqrt{2} \sum_j x_j \delta(t - t_j) \cos z_j \]  \hspace{1cm} (9-91)

where the \( z_j \)'s are uniformly distributed independent r.v.'s on \([-\pi, \pi]\).

### 9-6.1 The Stochastic Equation in the Conditional p.d.f. \( p(\phi; t) \)

Since \( \{\phi(t)\} \) is a first-order Markov process, the stochastic equation (7-40) applies. The intensity coefficients \( K_j(\phi) \) defined in (7-41) can be found from (9-88). Therefore \( K_j(\phi) = \Lambda_0 - AK \sin \phi \), and for \( q \geq 2 \) we use the binomial theorem and (7-41) to write

\[ K_j(\phi) = \lim_{\tau \to 0} \frac{-K}{\tau} \sum_{i=0}^{n} \binom{n}{i} E \left[ \left( \int_{t}^{t+\tau} N(t, \phi) \, dt \right)^i \right] \times E \left[ \left( \sqrt{2} \sum_j x_j \cos z_j \int_{t}^{t+\tau} \delta(t - t_j) \, dt \right)^{n-i} \right] \]  \hspace{1cm} (9-92)

Now in the limit as \( \tau \) approaches zero, the expectation, say \( I_{n-i} \), raised to the \( n - i > 0 \) power is, with high probability, either zero or one. Thus

\[ I_{n-i} = 2^{(n-i)/2} M_{n-i} Q_{m-i} \Pr [k = 1 \text{ in } \tau] \]  \hspace{1cm} (9-93)

where \( M_{n-i} = E(x^{n-i}) \) and

\[ Q_m = E[(\cos z)^m] = \begin{cases} \frac{\Gamma(m + 1)}{2^m [\Gamma(m/2 + 1)]^2} & \text{m even} \\ 0 & \text{m odd} \end{cases} \]  \hspace{1cm} (9-94)

Here \( \Gamma(x) \) is the well-known gamma function. Since \( \Pr [k = 1 \text{ in } \tau] = f\tau \) from (1-138) for small \( \tau \), we have
\[
K_q(\phi) = \begin{cases} 
\frac{K^2N_0}{2} + 2fK^2M_2Q_2 & q = 2 \\
2^{q/2}fK^qM_qQ_q & q \geq 4 \\
0 & \text{otherwise} 
\end{cases} \quad q \text{ odd}
\]

when we assume that \( N(t, \varphi) \) is white Gaussian noise and note that \( I_{n-i} = 1 \) when \( n - i = 0 \).

9-6.2 The Steady-State Probability Density Function, \( p(\phi) \)

Substitution of (9-95) and \( K_q(\phi) \) into (7-40) leads to

\[
\frac{d}{d\phi} [(\gamma - \sin \phi)p(\phi)] = \frac{1}{\alpha} \frac{d^2p(\phi)}{d\phi^2} + \Delta \sum_{q=1}^{\infty} G_q \frac{d^{2q}p(\phi)}{d\phi^{2q}}
\]

where \( G_q = K^{2q}M_{2q}/2^q(q!)^2 \) and \( \Delta = f/\alpha K \) is the normalized impulse rate of occurrence.

Now \( p(\phi) \) is representable by a Fourier series with period \( 2\pi \); that is,

\[
p(\phi) = \sum_{n=-\infty}^{\infty} c_n \exp(jn\phi)
\]

where

\[
c_n \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\phi) \exp(-jn\phi) \, d\phi
\]

are real since \( p(\phi) \) is real. Substitution of (9-97) into (9-96) and simplifying leads to

\[
0 = \sum_{n=-\infty}^{\infty} \exp(jn\phi) \left\{ \frac{n}{2}(c_{n+1} - c_{n-1}) + c_n \left[ \frac{n^2}{\alpha} + jn\beta + \Delta(1 - d_n) \right] \right\}
\]

where

\[
d_n \triangleq \sum_{q=0}^{\infty} (-1)^q(nK/\sqrt{2})^{2q}M_{2q} = E[J_0(\sqrt{2nK\alpha})]
\]

where \( J_0(y) \) is the Bessel function of order zero. Since \( \exp(jn\phi) \) are linearly independent, the coefficient of \( \exp(jn\phi) \) in (9-99) must vanish for every \( n \). Thus

\[
c_{n+1} = c_{n-1} - 2c_n \left[ \frac{n}{\alpha} + jr + \frac{\Delta(1 - d_n)}{n} \right]
\]
for all \( n \geq 0 \). The reduction of the infinite-order differential equation (9-96) to a set of second-order difference equations represents a major simplification to the problem.

Two boundary conditions are required for the solution of (9-101). From (9-89) with \( n = 0 \), it is clear that \( c_0 = 1/2\pi \), and by the Riemann Lebesgue lemma (Ref. 16) we have

\[
\lim_{n \to \infty} c_n = 0 \tag{9-102}
\]

Ohlson (Ref. 15) discusses numerical procedures for solving (9-101). Figure 9-23 illustrates a plot of the steady-state \( p(\phi) \) v. \( \phi \) for various values of \( \Delta \).

![Graph showing phase error probability density function for impulse noise](image)

**Fig. 9-23.** Phase Error p.d.f. When Impulse Noise Is Present (Courtesy of J. E. Ohlson).

### 9-6.3 An Integro-differential Equation for \( p(\phi) \)

Using (9-97) and (9-100), we note that the summation in (9-96) can be rewritten as

\[
-p(\phi) + \sum_{n=-\infty}^{\infty} c_n d_n \exp(jn\phi) \tag{9-103}
\]

where the sum on \( n \) represents a convolution in the \( \phi \) domain. Thus (9-96) and (9-103) can be written as
\[
\frac{d}{d\phi} [(\gamma - \sin \phi)p(\phi)] = \frac{1}{\alpha} \frac{d^2 p(\phi)}{d\phi^2} + \Delta [ - p(\phi) + \frac{1}{2\pi} D(\phi) \ast p(\phi) ] \tag{9-104}
\]

with boundary conditions defined in (9-30) and (9-32) with \( n = 0 \). Here \( D(\phi) \) is taken to be

\[
D(\phi) \triangleq \sum_{n=-\infty}^{\infty} d_n \exp (jn\phi) \tag{9-105}
\]

and the symbol \( \ast \) denotes convolution. Note also that \( D(\phi) \) is not unique.

Unfortunately, (9-104) does not permit a direct solution except for the case of large impulses. This arises when the Gaussian noise is absent and the impulse amplitudes are very large. The importance of this case is that the PLL is self-limiting; that is, if an impulse is large enough to cause the loop to slip a few cycles, being larger makes little difference. It must be noted that this is strictly a first-order PLL phenomenon. From (9-100), when \( K|x| \) is large with high probability, \( d_0 = 1 \) and \( d_k \) is approximately zero for \( k \neq 0 \). Thus \( D(\phi) = 1 \) from (9-105). The convolution in (9-104) is then unity and with \( \alpha = \infty \), (9-104) becomes

\[
\frac{d}{d\phi} [(\gamma - \sin \phi)p(\phi)] + \Delta p(\phi) = \frac{\Delta}{2\pi} \tag{9-106}
\]

By converting (9-106) to a first-order linear differential equation, the general solution may be found in integral form with complicated integration constants, but it is omitted here. For \( \gamma = 0 \) the solution to (9-106) becomes

\[
p(\phi) = \frac{\Delta |\tan (\phi/2)|^\Delta}{2\pi|\sin \phi|} \int_{|\phi|=1}^{\pi} [\cot (\phi/2)]^\Delta d\phi \tag{9-107}
\]

with \( |\phi| \leq \pi \). Analytic solutions can be found for integer \( \Delta \). For example, when \( \Delta = 1 \)

\[
p(\phi) = -\ln \left[ \frac{\sin (|\phi|/2)}{2\pi \cos^2 (\phi/2)} \right] \tag{9-108}
\]

and when \( \Delta = 2 \)

\[
p(\phi) = \frac{|\phi| - \pi |\sin \phi|}{4\pi \cos^2 (\phi/2)} + \frac{1}{\pi \cos^2 (\phi/2)} \tag{9-109}
\]

for \( |\phi| \leq \pi \).
9-7 Techniques for Approximating the Variance of the Phase Error in the Nonlinear Region of a Sinusoidal PLL

As discussed in Chapter 4, Section 4-13, various other approximating theories have been developed to explain the behavior of the variance of the steady-state phase error in the nonlinear region of loop operation. The emphasis on developing approximate theories was primarily due to the fact that the nonlinear theory, based on Fokker-Planck techniques, had not been satisfactorily advanced; and since loop design points are usually taken in the nonlinear region of operation, a satisfactory theory was required for purposes of accurate design, system evaluation, and testing. In this section we shall discuss these theories and their limitations.

9-7.1 Quasi-Linear Approximation

As the phase error grows too large, the linear approximation \( \sin \varphi \approx \varphi \) begins to fail. In order to extend the region of validity of this assumption, Develet (Ref. 17) defined an equivalent gain, \( K_{eq} \), as the average slope of the sinusoidal nonlinearity; that is, \( K_{eq} = E[\cos \varphi] \), where the average is taken with respect to a Gaussian r. v. with zero mean and variance \( \sigma^2_{\varphi} \). Then the approximation \( \sin \varphi \approx K_{eq} \varphi \) leads to the same equations as the linear PLL theory with \( A \) replaced by \( AK_{eq} \).

It is easy to show that \( K_{eq} = \exp(-\sigma^2_{\varphi}/2) \) and

\[
\sigma^2_{\varphi} \approx \frac{N_w W_L}{A^2 K_{eq}^2} \approx \frac{\exp(\sigma^2_{\varphi})}{\rho} \quad (9-110)
\]

Thus the response that allows us to compute \( \sigma^2_{\varphi} \), as opposed to \( \sigma^2_{\varphi} \), is itself a function of \( \sigma^2_{\varphi} \), but the transcendental equation for \( \sigma^2_{\varphi} \) is generally a tractable one, so the difficulty is only superficial. As a result of this quasi-linear method, the linear theoretic calculations can be extended to moderately large phase error excursions up to about \( \sigma^2_{\varphi} = 0.6 \).

A slightly different approach was followed by Margolis (Ref. 18), who linearized \( \sin \varphi \) about its steady-state phase error point \( \varphi_{ss} \),

\[
\sin \varphi \approx \sin \varphi_{ss} + (\varphi - \varphi_{ss}) \cos \varphi_{ss} \quad (9-111)
\]

The only significant difference is the change in loop gain by the factor \( \cos \varphi_{ss} \); that is, \( K \) in the linear PLL theory is replaced by \( K \cos \varphi_{ss} \). Notice that neither of the above approaches account for the many effects due to cycle-sliping.

9-7.2 Linear Spectral Approximation

A third approach, tailored for computing \( \sigma^2_{\varphi} \), as opposed to \( \sigma^2_{\varphi} \), and the phase error spectrum, is the so-called linear spectral method (Ref. 19). It is
formulated as follows: Suppose that (3-24), evaluated at time $t_1$, is multiplied by itself, evaluated at time $t_2$, and the product averaged. The result will be an equation relating various correlations between $\phi$ and $\sin \phi$, neither of which in general, is, a stationary process. However, Tausworthe (Ref. 19) assumed, in effect, that $\{\phi(t)\}$ was stationary and developed the approximate spectral equation

$$S_{\phi}(s) \approx \frac{-s^2 S_{\phi\phi}(s) + [K^2 N_0 F(s) F(-s)]/2}{-s^2 + AK\eta_0 [s F(-s) - s F(s)] + (AK\gamma_0)^2 F(s) F(-s)}$$  \hspace{1cm} (9-112)

where $\eta_0 = K_{eq} = \exp \left(-\sigma_\phi^2/2\right)$ and

$$\gamma_0 \approx \frac{K^2_{eq} \sinh \sigma_\phi^2}{\sigma_\phi^2}$$  \hspace{1cm} (9-113)

With $S_{\phi\phi}(s) = 0$ (i.e., exploring the term due to noise)

$$\sigma_\phi^2 \approx \frac{N_0 W_{L(eq)}}{A^2 \gamma_0^2}$$  \hspace{1cm} (9-114)

and

$$\sigma_\phi^2 = \frac{\pi^2}{3} \left[ 1 + \exp \left( -\frac{3\sigma_\phi^2}{\pi^2} \left( 1 + 0.13\sigma_\phi^2 \right) \right) \right]$$  \hspace{1cm} (9-115)

For a first-order loop $W_{L(eq)} = \gamma_0 AK/2$. Therefore $\sigma_\phi \exp \left(-\sigma_\phi^2/2\right) \sqrt{\sinh \sigma_\phi^2} = N_0 K/4A$ and this must be solved simultaneously with (9-115) to obtain $\sigma_\phi^2$.

**9-7.3 Volterra Functional Expansions**

Another approach has been explored in attempting to explain the nonlinear behavior of feedback systems, and, in particular, a phase-locked loop. It is the so-called *Volterra function* expansion technique, which was first applied to loop analysis by Van Trees (Ref. 20).

The concept of Volterra functionals in the analysis of nonlinear systems is a generalization of the convolution integral used in linear system analysis, the principle being that one desires to represent some arbitrary nonlinear system by a sequence of systems connected in parallel. In this parallel arrangement, the first system is a linear system. The output, say $y_1(t)$, is simply a convolution of the input, say $x(t)$, with an impulse response $h_1(\tau)$. The second system is of a quadratic nature whose output $y_2(t)$ is a two-dimensional convolution of the input $x(t)$ with an impulse response $h_2(\tau_1, \tau_2)$. The third system is of a cubic nature, characterized by a three-dimensional kernel $h_3(\tau_1, \tau_2, \tau_3)$, and so forth. In general, the total output, say $y(t)$, is the sum
Techniques for Approximating the Variance of the Phase Error

\[ y(t) = \sum_{k=1}^{\infty} y_k(t) \]  \hspace{1cm} (9-116)

Such a development is known as a Volterra expansion, and the various impulse responses \( h_{ik}(\tau_1, \tau_2, \ldots, \tau_k) \) are called Volterra kernels.

For the first-order PLL where no angle modulation exists (i.e., \( \theta(t) = 0 \)), Van Trees showed that

\[ \sigma^2 \approx \frac{1}{\rho} + \frac{1}{2\rho^2} + \frac{13}{24\rho^3} \]  \hspace{1cm} (9-117)

Notice in his result that the first term is directly traceable to the linear theory and that the other terms enter to compensate for the nonlinear behavior of the loop.

Finally, we note that Fig. 9-24 represents a comparison of the various nonlinear theories based on the evaluation of phase error variances. Unfortunately, none of these approximate theories accounts for cycle-slipping or problems associated with signal acquisition and synchronization stability.

![Graph](image)

**Fig. 9-24.** Comparison of the Variance of the Phase Error versus \( \alpha \) as Predicted by the Various Theories with \( \gamma = 0 \).
9-8 Related Studies

The transient solution of the Fokker-Planck equation (9-27) is covered in Chapter 16. For detailed references to the original work on the nonlinear theory of first-order PLLs, the reader is referred to Section 3-11 of Chapter 3. Stratonovich (Ref. 3) and Tikhonov (Ref. 4) apparently were the first to apply the Fokker-Planck equation to arrive at (9-38). The average number of slips per unit time was also investigated by Stratonovich (Ref. 3) and Nikitin (Ref. 10). The author (Ref. 11) unified and extended much of the earlier work, and a portion of this is given in this chapter. It is interesting to note that Uhran and Lindenlaub (Ref. 21) have experimentally verified formula (9-61) for the average number of slips per second when $\beta = 0$. Later Bozzi et al. (Ref. 22) justified (9-61) for various values of $\beta$ by means of computer simulation techniques.

Perhaps it is worth mentioning the interconnection between Rice’s (Ref. 23) “click” theory for FM and the results given here. In a SCS the threshold effect is due to the loop cycle-slipping phenomenon and can be accounted for by using Markov process theory. In FM receivers the threshold problem can be explained by the notion of “clicks”; however, Markov process theory cannot be used to account for the click phenomenon. The reader interested in further investigations should consult the papers by Rice (Ref. 23), Mazo and Salz (Ref. 24), and the book by Schilling and Taub (Ref. 25).

Based on (9-1), Lashinsky (Ref. 12) has studied the problem of periodic pulling and the transition to turbulence in a bounded plasma. It is interesting also to note that an equation like (9-1) is well known in the theory of elliptic orbits (Ref. 13). Be that as it may, Lashinsky shows that

$$\gamma \tan \frac{\varphi}{2} = \sqrt{\gamma^2 - 1} \tan \left( \sqrt{\frac{\gamma^2 - 1}{2}} \Lambda_{\varphi} \right) - \frac{1}{\gamma}$$

(9-118)

and that $\varphi(t)$ is a complex periodic function with the Fourier decomposition

$$\varphi = \xi + 2 \sum_{n=1}^{\infty} \frac{\kappa_n}{n} \sin n \xi$$

(9-119)

where $\xi = (\sqrt{\gamma^2 - 1})/\gamma^2 \Lambda_{\varphi}$ and $\kappa = (\sqrt{\gamma^2 - 1} - \gamma)/(\sqrt{\gamma^2 - 1} + \gamma)$. He also studies the spectrum of these oscillations just outside the synchronization band. It would be interesting to study similar results in the PLL context.

It may be interesting to make comparisons between the oscillating limiter and the first-order PLL discussed in Ref. 26, while an analysis of the performance of the oscillating limiter is given by Baghdady (Ref. 27) and Bozzi et al. (Ref. 28). The interested reader is referred to these studies for all details.
In this appendix we first establish the equivalence of (9-34) and (9-36) and then determine the normalization condition (9-37). Now (9-34) is given by

\[ p(\phi|\eta) = C_0 \exp \left[ \int^\phi h_0(x) \, dx \right] \left\{ 1 + D_0 \int_{(2n-1)\pi}^\phi \exp \left[ -\int^\phi h_0(x) \, dx \right] dy \right\} \] (I-1)

with \( C_0 = 2C/N_0K^2 \) and \( D_0 = -2\mathcal{F}/C \). Letting

\[ U_0(\phi) = \int^\phi h_0(x) \, dx \] (I-2)

we find from (9-30) and (I-1)

\[ D_0 = \frac{\exp \left\{ -U_0[(2n-1)\pi] + U_0[(2n + 1)\pi] \right\} - 1}{\int_{(2n-1)\pi}^{(2n+1)\pi} \exp \left[ U_0(y) \right] dy} \] (I-3)

Therefore

\[ p(\phi|\eta) = C_0 \left\{ \int_{(2n-1)\pi}^{(2n+1)\pi} \exp \left[ U_0(y) \right] dy 
+ \exp \left\{ -U_0[(2n-1)\pi] + U_0[(2n + 1)\pi] \right\} - 1 \right\} \int_{(2n-1)\pi}^\phi \exp \left[ U_0(y) \right] dy \] (I-4)
where

\[ C'_0 \triangleq \frac{C_0}{\int_{(2n-1)\pi}^{(2n+1)\pi} \exp[U_0(y)] \, dy} \quad (I-5) \]

Assume that the function \( h_0(\phi) \) admits of the Fourier series

\[ h_0(\phi) = \beta - \alpha g(\phi) \quad (I-7) \]

where \( g(\phi) \) is an odd function and \( \int_{\phi}^{\phi} g(x) \, dx \) is an even function of \( \phi \) so that

\[ -U_0[(2n - 1)\pi] + U_0[(2n + 1)\pi] = -2\beta \pi \quad (I-8) \]

Thus (I-1) becomes

\[
p(\phi|n) = C'_0 \exp[-U_0(\phi)] \left\{ \int_{\phi}^{(2n+1)\pi} \exp[U_0(y)] \, dy \\
+ \exp(-2\beta \pi) \int_{(2n-1)\pi}^{\phi} \exp[U_0(y)] \, dy \right\} \quad (I-9)
\]

Rewriting (I-9), using (I-2) and (I-6), gives

\[
p(\phi|n) = C'_0 \exp[-U_0(\phi)] \left\{ \int_{\phi}^{(2n+1)\pi} \exp[U_0(y)] \, dy \\
+ \int_{(2n-1)\pi}^{\phi} \exp[-\beta(y + 2\pi) + \alpha \int_{\phi}^{y} g(x) \, dx] \, dy \right\} \quad (I-10)
\]

Since \( g(\phi) \) is periodic in \( \phi \)

\[
\int_{\phi}^{\phi+2\pi} g(x) \, dx = \int_{\phi+2\pi}^{\phi+4\pi} g(x) \, dx 
\]

so that

\[
p(\phi|n) = C'_0 \exp[-U_0(\phi)] \left\{ \int_{\phi}^{(2n+1)\pi} \exp[U_0(y)] \, dy \\
+ \int_{(2n-1)\pi}^{\phi} \exp[-\beta(y + 2\pi) + \alpha \int_{\phi}^{y+2\pi} g(x) \, dx] \, dy \right\} \quad (I-12)
\]

Letting \( z = y + 2\pi \) in (I-12) and using (I-2) and (I-6) gives (9-36), that is,

\[
p(\phi|n) = C'_0 \exp[-U_0(\phi)] \int_{\phi}^{\phi+2\pi} \exp[U_0(y)] \, dy \quad (I-13)
\]
Appendix I / Properties of the Steady-State Probability Density

The normalization constant for a PLL is found from (I-13) with \( g(\phi) = \sin \phi \). For this case we write \((n = 0)\)

\[
\frac{1}{C_o} = \int_{-\pi}^{\pi} \int_{\phi}^{\phi + 2\pi} \exp \left[ \beta(\phi - x) + \alpha(\cos \phi - \cos x) \right] dx \, d\phi \quad (I-14)
\]

Noting that

\[
\cos A - \cos B = 2 \sin \left( \frac{A + B}{2} \right) \sin \left( \frac{A - B}{2} \right) \quad (I-15)
\]

and introducing the change of variables \( y = x - \phi \) gives

\[
\frac{1}{C_o} = \int_{-\pi}^{\pi} \int_{0}^{2\pi} \exp \left[ -\beta y + 2\alpha \sin \left( \frac{y}{2} \right) \sin \left( \frac{y + 2\phi}{2} \right) \right] dy \, d\phi \quad (I-16)
\]

Now

\[
\int_{-\pi}^{\pi} \exp \left[ 2\alpha \sin \left( \frac{y}{2} \right) \sin \left( \frac{y + 2\phi}{2} \right) \right] d\phi
\]

\[
= \int_{-\pi}^{\pi} \exp \left[ (2\alpha \sin \frac{y}{2}) \cos \left( \frac{y}{2} - \frac{\pi}{2} + \phi \right) \right] d\phi
\]

\[
= 2\pi I_0 \left( 2\alpha \sin \frac{y}{2} \right) \quad (I-17)
\]

so that

\[
\frac{1}{C_o} = \int_{0}^{2\pi} 2\pi \exp \left( -\beta y \right) I_0 \left( 2\alpha \sin \frac{y}{2} \right) dy \quad (I-18)
\]

Consider the integral

\[
2\pi \int_{0}^{2\pi} \exp \left[ -\beta y \right] I_0 \left( 2\alpha \sin \frac{y}{2} \right) dy = 4\pi \int_{0}^{\pi} \exp \left( -\beta y \right) I_0 \left( 2\alpha \sin \frac{y}{2} \right) dy \quad (I-19)
\]

and let \( z = y/2, \beta = -j\gamma', \) and \( \gamma' = j\beta. \) From Gradstheyn and Ryzhik (Ref. 8, p. 739) [combining 6.681(8) and 6.681(9)], we have

\[
\int_{-\pi}^{\pi} \exp \left( 2j \mu x \right) I_2 (2\alpha \sin x) dx = \pi \exp \left( j\mu \pi \right) I_{2-\mu}(\alpha) I_{2+\mu}(\alpha) \quad (I-20)
\]

so that
\[
\frac{1}{C_0'} = \int_0^\infty \exp(j2\gamma'z)I_0(2\alpha \sin z)\,dz = 4\pi^2 \exp(-\beta \pi)|I_{\beta}(\alpha)|^2 \tag{1-21}
\]

and

\[
C_0 = C_0' \int_{-\pi}^\pi \exp[-v(y)]\,dy
\]

Finally, we derive (9-44) and (9-45) using the expansions, familiar from Bessel function theory,

\[
\exp(\alpha \cos x) = \sum_{k=0}^\infty \epsilon_k I_k(\alpha) \cos kx
\]

\[
\exp(-\alpha \cos x) = \sum_{k=0}^\infty \epsilon_k(-1)^k I_k(\alpha) \cos kx
\]

with \( \epsilon_k = 1 \) for \( k = 0 \) and \( \epsilon_k = 2 \) for \( k \neq 0 \). We can write (9-38) as

\[
p(\phi) = \left[ 1 - \frac{\exp(-2\pi\beta)}{J} \right] \exp(\alpha \cos \phi) \left[ \frac{I_0(\alpha)}{\beta} \right]
\]

\[
+ 2 \sum_{k=1}^\infty \frac{(-1)^k I_k(\alpha)}{\beta^2 + k^2} (\beta \cos k\phi - k \sin k\phi)
\]

with \( J = 4\pi^2 \exp(-\pi \beta)|I_{\beta}(\alpha)|^2 \). The mean \( \bar{\phi} \) and mean square value \( \bar{\phi}^2 \) in (9-44) and (9-45) can be found by using (1-23).
APPENDIX II

BESSEL FUNCTIONS OF IMAGINARY ORDER AND IMAGINARY ARGUMENT

The following results pertaining to Bessel functions of imaginary order and imaginary argument are taken directly from Ref. 9.

Various problems from different branches of mathematical physics give rise to the differential equation

\[ \frac{v^2 d^2 w}{dv^2} + \frac{vdw}{dv} - (v^2 - v^2)w = 0 \]  \hspace{1cm} (II-1)

in which \( v \) and \( v \) are real quantities. Equation (II-1) is a special case of Bessel's equation

\[ \frac{z^2 d^2 w}{dz^2} + \frac{zdw}{dz} + (z^2 - \rho^2)w = 0 \]  \hspace{1cm} (II-2)

in which \( z = iv, \rho = iv \), and its solutions are therefore Bessel functions whose order and argument are both purely imaginary.

A fundamental real pair of solutions of (II-1) may be defined as follows:

\[ F_{\nu}(v) = \frac{\pi}{2} \frac{I_{\nu}(v) + I_{-\nu}(v)}{\sin \pi \nu} = \frac{\pi}{\sin \pi \nu} \text{Re} I_{\nu}(v) \]  \hspace{1cm} (II-3)
\[ G_s(v) = \frac{i\pi}{2} \frac{I_\nu(v) - I_{-\nu}(v)}{\sinh \pi v} = -\frac{\pi}{\sinh \pi v} \text{Im} \, I_\nu(v) \]
\[ = K_{1\nu}(iv) = \frac{1}{2} \pi e^{-\frac{1}{2} \nu^2} H_{1\nu}^{(1)}(iv) \]  

(II-4)

where \( v \) and \( \nu \) are real and positive and where \( \sinh \pi v = \sinh \nu v \). In these definitions \( I_\nu(v) \) is the modified Bessel function of the first kind of purely imaginary order, being related to the ordinary Bessel function \( J_\nu(iv) \) of imaginary order and imaginary argument by

\[ I_\nu(v) = e^{\nu iv} J_\nu(iv) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}v)^{\nu + 2m}}{m! \Gamma(iv + m + 1)} \]  

(II-5)

\( K_{1\nu}(v) \) is the modified Bessel function of the second kind of purely imaginary order, and \( H_{1\nu}^{(1)}(iv) \) is the Hankel function (Ref. 13) of the first kind with imaginary order and imaginary argument. For brevity, the functions \( F_\nu(v) \) and \( G_s(v) \) may be called “wedge functions” of the first and second kinds, respectively, since in potential theory they show a certain analogy to the solutions of Legendre’s equation called “cone functions.”

Representations of \( F_\nu(v) \) and \( G_s(v) \) in terms of series of modified Bessel functions of positive integral order are given by

\[ F_\nu(v) = \left( \frac{\nu \pi}{\sinh \nu \pi} \right)^{\frac{1}{2}} \left[ A(v, v) \cos \theta(v, v) + B(v, v) \sin \theta(v, v) \right] \]  

(II-6)

\[ G_s(v) = \left( \frac{\nu \pi}{\sinh \nu \pi} \right)^{\frac{1}{2}} \left[ B(v, v) \cos \theta(v, v) - A(v, v) \sin \theta(v, v) \right] \]  

(II-7)

where

\[ \theta(v, v) = v \log \frac{1}{2}v - \arg \Gamma(iv) \]  

(II-8)

\[ A(v, v) = \sum_{m=1}^{\infty} \frac{m(-1)^m(\frac{1}{2}v)^m}{m!(m^2 + v^2)} I_m(v) \]  

(II-9)

\[ B(v, v) = \sum_{m=0}^{\infty} \frac{v(-1)^m(\frac{1}{2}v)^m}{m!(m^2 + v^2)} I_m(v) \]  

(II-10)

\( A(v, v) \) and \( B(v, v) \) may also be expressed as power series in \( v \):

\[ A(v, v) = -\sum_{n=1}^{\infty} \sum_{k=0}^{[n/2]} \frac{(-1)^k(n)_{n-2k} v^{2k} n^{2n}}{4^n n!(1^2 + v^2) \cdots (n^2 + v^2)} \]  

(II-11)

\[ B(v, v) = \frac{1}{v} + \frac{1}{v} \sum_{n=1}^{[n/2]} \sum_{k=0}^{[n/2]} \frac{(-1)^k(n)_{n-2k} v^{2n}}{4^n n!(1^2 + v^3)(2^2 + v^3) \cdots (n^2 + v^2)} \]  

(II-12)
where \([s]\) represents the greatest integer contained in \(s\) and the symbol \((p)_q\), where \(p\) and \(q\) are any positive integers such that \(q \leq p\), denotes the sum of all the different products that can be formed by multiplying together \(q\) of the \(p\) factors \(1, 2, \ldots, p\), \((p)_0\) being equal to 1 by definition. A short table of values of \((p)_q\) has been given by Bocher (Ref. 14).

Definite integral representations of \(F_s(v)\) and \(G_s(v)\) are the following:

\[ F_s(v) = \frac{1}{s \nu \pi} \int_0^\pi e^{v \cos \theta} ch \theta \, d\theta - \int_0^\infty e^{-\nu \cosh} \sin vt \, dt \]  
\[ G_s(v) = \int_0^\infty e^{-\nu \cosh} \cos vt \, dt \]  
\[ \text{(II-13)} \]
\[ \text{(II-14)} \]

where \(ch t = \cosh (t)\).

When \(v\) is fixed and \(v\) is large and positive, we have the asymptotic series:

\[ F_s(v) \sim e^{\frac{v}{s \nu \pi}} \left( \frac{\pi}{2v} \right)^{1/2} \left[ 1 + \frac{(4v^2 + 1)^2}{1!(8v)^2} + \frac{(4v^2 + 1)^3(4v^2 + 3)^2}{2!(8v)^2} + \cdots \right] \]  
\[ G_s(v) \sim e^{-\frac{v}{s \nu \pi}} \left( \frac{\pi}{2v} \right)^{1/2} \left[ 1 - \frac{(4v^2 + 1)^2}{1!(8v)^2} + \frac{(4v^2 + 1)^3(4v^2 + 3)^2}{2!(8v)^2} - \cdots \right] \]  
\[ \text{(II-15)} \]
\[ \text{(II-16)} \]

while if \(v\) is fixed as \(v\) tends to zero through positive values.

\[ F_s(v) \to \left( \frac{\pi}{s \nu \nu \pi} \right)^{1/2} \sin \left[ v \log \frac{1}{2} v - \arg \Gamma(iv) \right] \]  
\[ G_s(v) \to \left( \frac{\pi}{s \nu \nu \pi} \right)^{1/2} \cos \left[ v \log \frac{1}{2} v - \arg \Gamma(iv) \right] \]  
\[ \text{(II-17)} \]
\[ \text{(II-18)} \]

When \(v\) is large and \(v\) is fixed,

\[ F_s(v) \sim e^{-1/2v \pi} \left( \frac{2\pi}{v} \right)^{1/2} \cos \left[ v \left( \log v - \log \frac{1}{2} v - 1 \right) + \frac{1}{4} \pi \right] \left( 1 + O \frac{1}{v} \right) \]  
\[ \text{(II-19)} \]
\[ G_s(v) \sim e^{-1/2v \pi} \left( \frac{2\pi}{v} \right)^{1/2} \sin \left[ v \left( \log v - \log \frac{1}{2} v - 1 \right) + \frac{1}{4} \pi \right] \left( 1 + O \frac{1}{v} \right) \]  
\[ \text{(II-20)} \]

while if \(v\) tends to zero, \(v\) being fixed,

\[ F_s(v) \to \frac{I_0(v)}{v} \to \infty \]  
\[ \text{(II-21)} \]
\[ G_{s}(v) \rightarrow K_{d}(v) \quad \text{(II-22)} \]

Finally, we note integral formulas that can be used to evaluate the modulus \(|I_{r}(v)|^2\) numerically. From Ref. 8, page 739, equation 11, we have

\[ I_{\mu}(v) I_{\nu}(v) = \frac{2}{\pi} \int_{0}^{\pi/2} \cos[(\mu - \nu)x] I_{\mu+\nu}(2v \cos x) \, dx \quad \text{(II-23)} \]

which can be used to derive the result

\[ |I_{n-j\beta}(\alpha)|^2 = I_{n-j\beta}(\alpha) I_{n+j\beta}(\alpha) \]

\[ = \frac{2}{\pi} \int_{0}^{\pi/2} \cosh(2\beta x) I_{2\alpha}(2\alpha \cos x) \, dx \quad \text{(II-24)} \]

which can be evaluated numerically. Using the results from Prob. 9-21 one can show that

\[ |I_{j\beta}(\alpha)|^2 = \frac{\sinh \pi \beta}{\pi \beta} \left[ I_{0}^{2}(\alpha) + 2 \sum_{m=1}^{\infty} (-1)^{m} \frac{I_{m}^{2}(\alpha)}{1 + \left( \frac{m}{\beta} \right)^{2}} \right] \quad \text{(II-25)} \]

which relates the modulus \(|I_{j\beta}(\alpha)|^2\) to the imaginary Bessel functions.
APPENDIX III

DERIVATION OF THE CIRCULAR MOMENTS OF $p(\phi)$

For completeness, we present in this appendix the derivation of the circular moments, $\sin (n\phi), \cos (n\phi)$. Consider the integral

$$ A(n) = \int_{-\pi}^{\pi} \cos n\phi p(\phi) \, d\phi $$

where $p(\phi)$ is defined in (9-38).

$$ A(n) = C' \int_{-\pi}^{\pi} \cos n\phi \, d\phi \int_{\phi}^{\phi+2\pi} \exp \left[ \beta (\phi - \Psi) + \alpha (\cos \phi - \cos \Psi) \right] \, d\Psi $$

where

$$ C' = \left[ 4\pi^2 \exp \left[ -\pi \beta \right] I_{j\rho}(\alpha) \right]^{-1} $$

Since

$$ \cos \phi - \cos \Psi = 2 \sin \left( \frac{\phi + \Psi}{2} \right) \sin \left( \frac{\Psi - \phi}{2} \right) $$

and letting $y = \Psi - \phi$,
\[ A(n) = C_0' \int_{-\pi}^{\pi} \cos n\phi \, d\phi \int_{0}^{2\pi} \exp \left\{ -\beta y + 2\alpha \sin \left( \frac{y}{2} \right) \sin \left( \frac{\phi}{2} + \phi \right) \right\} \, dy \]
\[ = C_0' \int_{0}^{2\pi} \exp \left\{ -\beta y \right\} \, dy \int_{-\pi}^{\pi} \cos n\phi \exp \left\{ 2\alpha \sin \frac{y}{2} \sin \left( \phi + \frac{y}{2} \right) \right\} \, d\phi \]  

(III-1)

Define

\[ G(y) = \int_{-\pi}^{\pi} \cos n\phi \exp \left[ A \sin \left( \phi + \frac{y}{2} \right) \right] \, d\phi \]  

(III-2)

where

\[ A = 2\alpha \sin \frac{y}{2} \]

Now

\[
\exp \left[ A \cos \left( \phi - \frac{\pi}{2} \right) \right] = \exp \left[ A \sin \phi \right] = \sum_{k=0}^{\infty} \epsilon_k I_k(A) \cos k\left( \phi - \frac{\pi}{2} \right)
\]

and

\[
\exp \left[ A \sin \left( \phi + \frac{y}{2} \right) \right] = \sum_{k=0}^{\infty} \epsilon_k I_k(A) \cos k\left( \phi + \frac{y}{2} - \frac{\pi}{2} \right)
= \sum_{k=0}^{\infty} \epsilon_k I_k(A) \left[ \cos \frac{k\pi}{2} \left( \cos k\phi \cos \frac{ky}{2} - \sin k\phi \sin \frac{ky}{2} \right) + \sin \frac{k\pi}{2} \left( \sin k\phi \cos \frac{ky}{2} + \cos k\phi \sin \frac{ky}{2} \right) \right]
\]  

(III-3)

Combining (III-2) and (III-3) and noting the orthogonal properties of \( \sin (ny) \) and \( \cos (ny) \) over the interval \([-\pi, \pi]\), we get

\[ G(y) = 2I_n(A) \sin \frac{n\pi}{2} \int_{-\pi}^{\pi} \sin \frac{ny}{2} \cos^2 n\phi \, d\phi = (-1)^{(n-1)/2} 2\pi I_n \left( 2\alpha \sin \frac{y}{2} \right) \sin \frac{ny}{2} \]

for \( n \) odd

\[ = 2I_n(A) \cos \frac{n\pi}{2} \int_{-\pi}^{\pi} \cos \frac{ny}{2} \cos^2 n\phi \, d\phi = (-1)^{n/2} 2\pi I_n \left( 2\alpha \sin \frac{y}{2} \right) \cos \frac{ny}{2} \]

for \( n \) even  

(III-4)

Returning to (III-1) and carrying both even and odd results as a two-element column vector,
\[
A(n) = 2\pi C_0 \int_0^{2\alpha} \left( (-1)^{(n-1)/2} \right) \exp \left\{ -\beta y \right\} I_n \left( 2\alpha \sin \frac{y}{2} \right) \left( \frac{\sin \frac{ny}{2}}{\cos \frac{ny}{2}} \right) dy
\]

\[
= 2\pi C_0 \int_0^{\alpha} \left( (-1)^{(n-1)/2} \right) \exp \left\{ -\beta y \right\} I_n \left( 2\alpha \sin \frac{y}{2} \right) \left( \frac{\sin \frac{ny}{2}}{\cos \frac{ny}{2}} \right) dy
\]

\[
+ 2\pi C_0 \int_{\alpha}^{2\alpha} \left( (-1)^{(n-1)/2} \right) \exp \left\{ -\beta y \right\} I_n \left( 2\alpha \sin \frac{y}{2} \right) \left( \frac{\sin \frac{ny}{2}}{\cos \frac{ny}{2}} \right) dy
\]

(III-5)

Letting \( y/2 = \pi/2 - x \) in the first integral and \( y/2 = \pi/2 + x \) in the second integral, we obtain

\[
A(n) = 4\pi C_0 e^{-\beta \pi} \int_0^{\pi/2} \left( (-1)^{(n-1)/2} \right) e^{2\beta x} I_n \left( 2\alpha \cos x \right) \left( \frac{\sin \left( \frac{\pi}{2} + x \right)}{\cos \left( \frac{\pi}{2} + x \right)} \right) dx
\]

\[
+ 4\pi C_0 e^{-2\beta \pi} \int_0^{\pi/2} \left( (-1)^{(n-1)/2} \right) e^{-2\beta x} I_n \left( 2\alpha \cos x \right) \left( \frac{\sin \left( \frac{\pi}{2} + x \right)}{\cos \left( \frac{\pi}{2} + x \right)} \right) dx
\]

But

\[
\sin \left( \frac{\pi}{2} \pm y \right) = (-1)^{(n-1)/2} \cos ny \quad n \text{ odd}
\]

\[
\cos \left( \frac{\pi}{2} \pm y \right) = (-1)^{n/2} \cos ny \quad n \text{ even}
\]

(III-6)

Thus

\[
A(n) = 4\pi C_0 e^{-\beta \pi} \int_0^{\pi/2} e^{2\beta x} I_n \left( 2\alpha \cos x \right) \cos nx \; dx \quad \text{for all } n
\]

\[
+ 4\pi C_0 e^{-2\beta \pi} \int_0^{\pi/2} I_n \left( 2\alpha \cos x \right) \cos nx \; dx
\]

(III-7)

Letting \( \beta = -jy \),

\[
A(n) = 4\pi C_0 e^{-\beta \pi} \int_0^{\pi/2} \left[ \cos (n + 2\gamma)x \right] I_n \left( 2\alpha \cos x \right) \; dx
\]

\[
+ 4\pi C_0 e^{-\beta \pi} \int_0^{\pi/2} \left[ \cos (n - 2\gamma)x \right] I_n \left( 2\alpha \cos x \right) \; dx
\]
Using the identity,
\[
\int_0^{\pi/2} \cos 2\mu x I_{2\mu}(2a \cos x) \, dx = \frac{\pi}{2} I_{\nu-\mu}(a)I_{\nu+\mu}(a) \quad \text{for } \Re \nu > \frac{1}{2} \quad (\text{III-8})
\]

\[
A(n) = 4\pi C_0 e^{-\beta n} \left[ I_{n/2-(\alpha+\gamma)}(\alpha) I_{n/2+\gamma}(\alpha) + I_{n/2-(\alpha-\gamma)}(\alpha) I_{n/2+\gamma}(\alpha) \right]
\]

substituting for \( C_0 \) and using the fact that \( \gamma = j\beta \),

\[
A(n) = \frac{1}{2} \left\{ \frac{I_{-j\beta}(\alpha)}{I_{j\beta}(\alpha)} \left[ I_{n+j\beta}(\alpha) + I_{j\beta}(\alpha) I_{n-j\beta}(\alpha) \right] \right\}
\]

\[
= \frac{1}{2} \left\{ \frac{I_{n+j\beta}(\alpha)}{I_{j\beta}(\alpha)} + \frac{I_{n-j\beta}(\alpha)}{I_{-j\beta}(\alpha)} \right\} = \text{Re} \left\{ \frac{I_{n-j\beta}(\alpha)}{I_{-j\beta}(\alpha)} \right\}
\]

Hence

\[
\cos n\phi = \text{Re} \left\{ \frac{I_{n-j\beta}(\alpha)}{I_{-j\beta}(\alpha)} \right\} \quad \text{for all } n \quad (\text{III-9})
\]

Following steps similar to the preceding ones, we can arrive at an equation for

\[
B(n) = \int_{-\pi}^{\pi} \sin np(\phi) \, d\phi \quad \text{which is analogous to (III-1)}.
\]

\[
B(n) = C_0 \int_0^{2\pi} G(y) \exp \left\{ -\beta y \right\} \, dy \quad (\text{III-10})
\]

where

\[
G(y) = \int_{-\pi}^{\pi} \sin n\phi \exp \left[ A \sin \left( \phi + \frac{y}{2} \right) \right] \, d\phi \quad (\text{III-11})
\]

Expanding as in (III-3), we get

\[
G(y) = 2I_n(A) \sin \frac{ny}{2} \int_{-\pi}^{\pi} \cos^2 \frac{ny}{2} \sin^2 n\phi \, d\phi = (-1)^{(n-1)/2} 2\pi I_n \left( 2a \sin \frac{y}{2} \right) \cos \frac{ny}{2}
\]

for \( n \) odd

\[
= -2I_n(A) \cos \frac{ny}{2} \int_{-\pi}^{\pi} \sin \frac{ny}{2} \cos^2 n\phi \, d\phi = (-1)^{n/2} 2\pi I_n \left( 2a \sin \frac{y}{2} \right) \sin \frac{ny}{2}
\]

for \( n \) even \quad (\text{III-12})

Breaking up the integral into two parts and making the same substitutions as following (III-5),
\[ B(n) = 4\pi C'_0 e^{-\beta n} \int_0^{\pi/2} \left( \frac{(-1)^{(n-1)/2}}{(-1)^{n/2}} \right) e^{2\beta x} I_n(2\alpha \cos x) \left( \frac{\cos n \left( \frac{\pi}{2} - x \right)}{\sin n \left( \frac{\pi}{2} - x \right)} \right) dx \]

\[ + 4\pi C'_0 e^{-\beta n} \int_0^{\pi/2} \left( \frac{(-1)^{(n-1)/2}}{(-1)^{n/2}} \right) e^{-2\beta x} I_n(2\alpha \cos x) \left( \frac{\cos n \left( \frac{\pi}{2} + x \right)}{\sin n \left( \frac{\pi}{2} + x \right)} \right) dx \]

(III-13)

But

\[ \cos n \left( \frac{\pi}{2} \pm y \right) = \mp (-1)^{(n-1)/2} \sin ny \quad n \text{ odd} \]

\[ \sin n \left( \frac{\pi}{2} \pm y \right) = \pm (-1)^{n/2} \sin ny \quad n \text{ even} \]

Thus

\[ (4\pi)^{-1} B(n) = C'_0 e^{-\beta n} \int_0^{\pi/2} e^{2\beta x} I_n(2\alpha \cos x) \sin nx \, dx \]

for all \( n \)

\[ - C'_0 e^{-\beta n} \int_0^{\pi/2} e^{-2\beta x} I_n(2\alpha \cos x) \sin nx \, dx \]

(III-14)

Letting \( \beta = -j\gamma \) as before

\[ B(n) = C'_0 e^{-\beta \pi j} \int_0^{\pi/2} \cos [(n + 2\gamma)x] I_n(2\alpha \cos x) \, dx \]

\[ - C'_0 e^{-\beta \pi j} \int_0^{\pi/2} \cos [(n - 2\gamma)x] I_n(2\alpha \cos x) \, dx \]

Following the same procedure as before,

\[ B(n) = \frac{j}{2} \left[ \frac{I_{n+\gamma\beta}(\alpha)}{I_{\beta}(\alpha)} - \frac{I_{n-\gamma\beta}(\alpha)}{I_{-\beta}(\alpha)} \right] = \text{Im} \left[ \frac{I_{n-\gamma\beta}(\alpha)}{I_{-\beta}(\alpha)} \right] \]

Thus

\[ \sin n\phi = \text{Im} \left[ \frac{I_{n-\gamma\beta}(\alpha)}{I_{-\beta}(\alpha)} \right] \quad \text{for all } n \]

(III-15)

where \( j = \sqrt{-1} \).
APPENDIX IV

DEVELOPMENT OF THE FIRST-SLIP TIME FORMULA

Using the Jacobi-Anger formula,

\[ \exp(\pm \alpha \cos \phi) = I_0(\alpha) + 2 \sum_{n=1}^{\infty} (\pm 1)^n I_n(\alpha) \cos n\phi \]  \hspace{1cm} (IV-1)

and the integral, with integrand \( f_{\pm}(\phi) = \exp[\pm \alpha(\cos \phi + \gamma \phi)] \),

\[ J_{\pm}(\phi) = \int f_{\pm}(\phi) \, d\phi = \frac{(\pm 1) \exp(\pm \alpha \gamma \phi)}{\alpha \gamma} [I_0(\alpha) + 2 \sum_{n=1}^{\infty} (\pm 1)^n I_n(\alpha) \psi_{\pm n}(\phi)] \]  \hspace{1cm} (IV-2)

where

\[ \psi_{\pm n}(\phi) \triangleq \frac{\cos n\phi + (n/\alpha \gamma) \sin n\phi}{1 + (n/\alpha \gamma)^2} \]  \hspace{1cm} (IV-3)

one can show that

\[ \int_{\phi_0}^{\phi_0 + 2\pi} \psi_{\pm n}(\phi) \exp(\alpha \cos \phi) \, d\phi = \frac{2\pi I_n(\alpha)}{1 + (n/\alpha \gamma)^2} \]  \hspace{1cm} (IV-4)

\[ \int_{\phi_0 - 2\pi}^{\phi_0 + 2\pi} \psi_{\pm n}(\phi) \exp(\alpha \cos \phi) \, d\phi = \frac{4\pi I_n(\alpha)}{1 + (n/\alpha \gamma)^2} \]
Appendix IV / Development of the First-Slip Time Formula

Now \( C(0) \), of (9-77), is given by

\[
C(0) = \frac{\int_{\phi_0}^{\phi_0 + 2\pi} \exp \left[ -\alpha (\cos \phi + \gamma \phi) \right] d\phi}{\int_{\phi_0 - 2\pi}^{\phi_0 + 2\pi} \exp \left[ -\alpha (\cos \phi + \gamma \phi) \right] d\phi} \tag{IV-5}
\]

when we set \( \gamma = \Lambda_0 / AK \) and use the definition (9-35) for \( U_0(\phi) \) with \( g(\phi) = \sin \phi \). This reduces to

\[
C(0) = \frac{1 - \exp (-2\pi \alpha \gamma)}{\exp (2\pi \alpha \gamma) - \exp (-2\pi \alpha \gamma)} = \frac{1}{1 + \exp (2\pi \alpha \gamma)}
\]

\[
= \frac{\exp (-\pi \alpha \gamma) \sinh (\pi \alpha \gamma)}{\sinh (2\pi \alpha \gamma)} = \frac{\exp (-\pi \alpha \gamma)}{2 \cosh (\pi \alpha \gamma)} \tag{IV-6}
\]

when we make use of (IV-2). Note, \( C(0) \) is in dependent of \( \phi_0 \). Now

\[
W_{L,\tau}(2\pi|\phi_0) = \int_{\phi_0 - 2\pi}^{\phi_0 + 2\pi} Q(\phi) \, d\phi \tag{IV-7}
\]

where

\[
Q(\phi) = \frac{\alpha f_+ (\phi)}{2} \left[ \int_{\phi_0 - 2\pi}^{\phi_0} C(0) f_-(\phi) \, d\phi - \int_{\phi_0}^{\phi_0 + 2\pi} f_+ (\phi) \, d\phi \right]
\]

\[
= \frac{\alpha}{2} \left[ (C(0) - 1)f_+(\phi)J_+(\phi) + [J_+ (\phi_0) - C(0)J_+(\phi_0 - 2\pi)]f_+ (\phi) \right]
\]

\[
(IV-8)
\]

Substitution of (IV-8) in (IV-7) and integrating leads to

\[
\int_{\phi_0 - 2\pi}^{\phi_0 + 2\pi} Q(\phi) \, d\phi = \frac{\alpha [C(0) - 1]}{2} \int_{\phi_0 - 2\pi}^{\phi_0 + 2\pi} f_+ (\phi)J_+(\phi) \, d\phi
\]

\[
+ \frac{\alpha}{2} [J_+ (\phi_0) - C(0)J_+(\phi_0 - 2\pi)][J_+ (\phi_0 + 2\pi) - J_+ (\phi_0 - 2\pi)]
\]

\[
(IV-9)
\]

Now

\[
\int_{\phi_0 - 2\pi}^{\phi_0 + 2\pi} \! f_+ (\phi)J_-(\phi) \, d\phi = -\frac{4\pi}{\alpha \gamma} \left[ \bar{I}_2 (\alpha) + \sum_{n=1}^{\infty} (-1)^n \frac{\bar{I}_2 (\alpha)}{1 + (n/\alpha \gamma)^2} \right] \tag{IV-10}
\]

so that by combining (IV-9) and (IV-10) with (IV-7) we have (9-80) when we use the facts
\[ J_\pm(\phi + 2\pi) = \exp(\pm 2\alpha \gamma \pi) J_\pm(\phi) \]
\[ J_\pm(\phi - 2\pi) = \exp(\mp 2\alpha \gamma \pi) J_\pm(\phi) \] (IV-11)

and \( C(0) \) of (IV-6). Moreover, using (II-25) of Appendix II and (9-61) it is easy to show that \( \tau(2\pi |\varphi_0|) = 1/S \).

Problems

9-1 A first-order PLL is to be mechanized with the phase detector (PD) characteristic illustrated in Fig. P9-1.

![Diagram of PD characteristic](image)

(a) In the absence of noise, determine the acquisition time for each PD characteristic.
(b) Sketch the phase-plane trajectories for each of the PD characteristics and discuss the synchronization stability of the loop.

9-2 Using the fact that

\[ \int \frac{dx}{a + b \sin x} = \frac{1}{\sqrt{b^2 - a^2}} \ln \left[ \frac{a \tan \frac{x}{2} + b - \sqrt{b^2 - a^2}}{a \tan \frac{x}{2} + b + \sqrt{b^2 - a^2}} \right] \]

(a) Develop an expression for the beat note \( \epsilon/K_1 K_m A = \sin \varphi \), which appears at the output of the phase detector in the absence of noise.
(b) Sketch this waveform.
(c) What is the significance of the positive and negative excursions of unequal area appearing in this waveform?

9-3 Using the relationship \(\tan x = -i \tanh ix\) and the notation \(\gamma \xi = \sqrt{1 - \gamma^2 \Lambda_0 t}\), show that (9-118) can be written in the form

\[
\exp(i\varphi) = \exp(i\xi) \left[ \frac{1 - K \exp(-i\xi)}{1 - K \exp(i\xi)} \right]
\]

Taking logarithms of both sides, show that the preceding relationship reduces to (9-119).

9-4 Show that the Fourier series expansion of \(\cos \varphi\) (where \(\varphi\) is given in Prob. 9-3) is given by

\[
\cos \varphi = \prod_{n=1}^{\infty} \left\{ \sum_{m=-\infty}^{\infty} J_m \left( \frac{Kn}{n} \right) \cos \left[ (1 + nm) \sqrt{\frac{\gamma^2 - 1}{\gamma^2} \Lambda_0 t} \right] \right\}
\]

9-5 A first-order PLL operates in the absence of noise with a VCO rest radian frequency of \(\omega_0\) rad/sec. The input phase is characterized by

\[
\theta(t) = \begin{cases} 
\Omega_0 t + \theta_0 & t < 0 \\
\Omega_0 t + \theta_0 + \pi & t \geq 0 
\end{cases}
\]

Assume that \(\sin^{-1}(\Omega_0/\Lambda K) = \pi/8\), \(e = 0\), and that the loop has been tracking for a long time so that it is operating in the steady state at \(t = 0^-\).

(a) Plot the trajectory for \(\varphi(t)\) on the phase plane (\(\dot{\varphi}\) vs. \(\varphi\)), carefully labeling the points for \(t = 0^-\) and \(t = 0^+\) and \(t\) approaching infinity.

(b) Find and sketch the function \(\varphi(t)\) vs. \(t\).

9-6 When \(\phi = -\Lambda K \sin \varphi + n(t)\), why cannot we simply substitute \(K_1(\varphi)\) and \(K_2(\varphi)\) into the stationary p.d.f. (7-83) to obtain \(p(\varphi)\)? Assume that \([n(t)]\) is a white Gaussian noise process.

9-7 Determine the effects on the acquisition range and synchronization stability of a first-order sinusoidal PLL with delay in the loop; that is, assume \(F(s) = \exp(-io\lambda)\), where \(\lambda\) is the delay in seconds. Compare your result with that obtained when \(\lambda = 0\).

9-8 Determine the intensity coefficients \(K_1(\varphi)\) and \(K_2(\varphi)\) which serve to characterize the FP equation (9-27). Hint: Note that

\[
\frac{\varphi(t + \tau) - \varphi(t)}{\tau} = \frac{1}{\tau} \int_t^{t+\tau} \left[ \Lambda_0 - AKg(\varphi) - KN(t, \varphi) \right] dt
\]

9-9 If \(h_0(\hat{\phi})\) is periodic, show that (9-34) reduces to (9-36). (Hint: See Appendix 1.)

9-10 For the rectangular PD characteristic illustrated in Fig. P9-1, determine the following:
(a) The steady-state p.d.f. \( p(\phi) \).
(b) Develop an expression for the variance of the phase error when \( \Lambda_0 = 0 \) and show that for large \( \alpha, \sigma_\phi^2 \) approaches \( 2/\alpha^2 \).
(c) Develop an expression for the mean time to first slip.
(d) What does (c) simplify to when \( \Lambda_0 = 0 \). [Hint: \( \dot{\phi} = \Lambda_0 - AKg(\phi) - Kn(t) \), where \( n(t) \) is a white Gaussian noise process \( (N_0/2) \).
(e) Find an expression for the probability current.
(f) Find expressions for \( N_+ \) and \( N_- \).
(g) Find an expression for the average number of slips \( \bar{S} \) per second.

9-11 Develop the steady-state p.d.f.'s for the phase error for all PLLs illustrated in Table 3-1.

9-12 Using the stochastic differential equation (9-13) and assuming that the covariance function of the input noise \( \{N(t, \varphi)\} \) is characterized by (9-22), develop an approximate expression for the variance \( \sigma_\phi^2 \) of the phase error rate. The circular moments can be used to simplify your answer when \( g(\phi) = \sin \phi \).

9-13 The signal-to-noise spectral density ratio at threshold in a sinusoidal PLL receiver is \( A^2/N_0 = 5.0 \). The loop bandwidth at threshold is 10 Hz.
(a) Using the linear PLL theory, determine the signal-to-noise ratio in the loop bandwidth if \( \gamma = 0 \).
(b) Repeat (a) using the nonlinear theory when \( \gamma = 0 \) and \( m_\phi = 0 \).
(c) Determine the increase in the threshold variance if the loop is stressed to the point where \( \gamma = 0.6 \) and \( \gamma = 1.0 \). Assume that \( m_\phi = 0 \).

9-14 Develop a formula for the frequency of skipping half cycles. For simplicity, assume that \( \varphi_0 = \Lambda_0 = 0 \).

9-15 A phase-coherent communication channel is designed such that, under worst conditions, \( A^2/N_0 = 20 \). A first-order sinusoidal PLL, with \( W_L = 20 \text{ Hz} \), is used to track the carrier. Find
(a) The loop signal-to-noise ratio.
(b) The variance of the phase error as specified by the linear and nonlinear PLL theory when \( \gamma = 0 \).
(c) The probability current if \( \gamma = 0.5 \).
(d) The diffusion coefficient if \( \gamma = 0 \) and \( 0.5 \).
(e) The probability of first loss of phase lock, \( \gamma = 0 \).
(f) The mean time to first slip when \( \gamma = 0 \) and \( 0.5 \).

9-16 Using (7-130), develop expressions (9-76) and (9-77) for moments of the first slip time. Letting \( n = 1 \), show that (9-76) reduces to (9-79).

9-17 Derive (9-80) using the results given in Appendix IV.

9-18 The process \( \{\varphi(t)\} \) generated by the differential equation \( d\varphi/dt = n(t) \) is Brownian motion when \( \{n(t)\} \) is white noise. Set absorbing barriers at \( b_2 = 2\pi \) and \( b_1 = -2\pi \) and find the mean of the first-passage time. (Note: This is the loop equation of a first-order loop when \( A = 0 \).)

9-19 Develop the expression (9-65) for the difference of two poisson r.v.'s when \( N_+ \neq N_- \). Repeat when \( N_+ = N_- \). Show that the first two moments of the r.v. \( N \) are given by \( E(N) = (N_+ - N_-)t \) and \( \sigma_N^2 = (N_+ + N_-)t = \bar{S}t \).
9-20 Show that all even semi-variants (cumulants) \( k_n \) of the r. v. \( N \) [with probabilities defined in (9-65)] are given by

\[
k_n = \begin{cases} 
(N_+ + N_-) & n \text{ even} \\
(N_+ - N_-) & n \text{ odd}
\end{cases}
\]

9-21 Using (IV-2) of Appendix IV, show that \( C_0' \) in (9-37) can be written as

\[
C_0' = \frac{\beta}{2\pi[1 - \exp(-2\pi\beta)]} \left\{ I_0(\alpha) + 2 \sum_{m=1}^{\infty} (-1)^m \frac{I_m(\alpha)}{1 + (m/\beta)^2} \right\}
\]

and that

\[
p(\phi) = \frac{\exp[i \cos \phi]}{2\pi I_0(\alpha)} (f(\phi, \alpha, \beta))
\]

where

\[
f(\phi, \alpha, \beta) = \frac{1 + 2 \sum_{m=1}^{\infty} (-1)^m \frac{I_m(\alpha)}{I_0(\alpha)} \Phi_m(\phi)}{1 + 2 \sum_{m=1}^{\infty} (-1)^m \frac{I_m(\alpha)}{I_0(\alpha)} \Phi_m(0)}
\]

and

\[
\Phi_m(\alpha) \triangleq \frac{\cos mx - \frac{m}{\beta} \sin mx}{1 + (m/\beta)^2}
\]

The constant \( C_0' \) is easy to compute as the series in the denominator converges quite rapidly. Numerical integration may also be used to evaluate \( C_0' \), see (II-24) of Appendix 9-II. Notice from this result one can develop an expression for the modulus \(|I_n(\alpha)|^2\) in terms of imaginary Bessel functions of real order.

References


9-20 Van Trees, H. L., Functional Techniques for the Analysis of the Nonlinear
References


10

NONLINEAR THEORY OF
SECOND-ORDER
SYNCHRONOUS
CONTROL SYSTEMS

10-1 Introduction

In this chapter we shall study the nonlinear behavior of second-order synchronous control systems (SCSs) with particular emphasis on those containing the perfect and imperfect integrating filters discussed in Chapter 4. The nonlinear theory of modulation tracking with second-order SCSs is reserved for a later chapter. We shall frequently allude to the use of various loop parameters specified by the linear PLL theory given in Chapter 4. Our interest is focused on second-order SCSs with an arbitrary periodic phase-detector characteristic because a variety of tracking receivers and synchronization systems have been mechanized this way.

This chapter covers a broad spectrum of topics. We begin by presenting discussions that pertain to the performance characterization of SCSs. Next we study, via the phase-plane, the signal acquisition properties of a sinusoidal PLL containing an imperfect integrating loop filter and follow by giving an analytical treatment that derives fundamental relations between signal acquisition
time, acquisition range, and various loop parameters. We then broaden our scope and develop formulas that specify the acquisition range for a SCS containing an arbitrary periodic, phase-detector (PD) characteristic. Comparison of the acquisition range is made for the sinusoidal, rectangular, and triangular PD characteristics. Next we determine the signal acquisition time in the absence of noise and develop an upper bound for the acquisition time for a sinusoidal PLL. The problem of false-lock in PLL receivers is studied next. Acquisition properties of second-order loops tracking accelerating targets is then discussed, followed by results pertaining to loops that are implemented with ideal integrating loop filters.

Finally, we present the theory of a second-order SCSs in the presence of noise; in particular, we give and discuss formulas for the p.d.f. of the phase error, moments of the mean time to first loss of phase synchronization, cycle-slip probabilities, frequency acquisition time, and average number of slips per unit time. The chief mathematical techniques needed in our study make use of the theory given in Chapter 5 (phase-plane methods) when noise is absent and that presented in Chapters 6, 7, and 8 when noise is present. Before we begin our analytical treatment, a few general comments pertaining to synchronization and tracking are in order.

### 10.1.1 Motivation

Synchronization and tracking are basic problems in modern digital data transmission and so is the problem of coherent demodulation of information. In radar and missile guidance systems, synchronization and tracking are connected with the ultimate aim of measuring range and vehicle velocity. In coherent communications, synchronization is required for high-quality reception of digital data. In the past, the problems of digital data transmission, synchronization and tracking have been studied separately. However, today the synchronization problem is recognized as basic to the study of statistical communication theory. In fact, most optimum detection methods developed to date require detecting the transmitted signal based on the availability of perfect synchronization at the point of reception. Consequently, it has been observed that sampling the output of the receiving device too early or too late produces a "mismatch" and introduces a timing noise component that causes degradation in the probability of correct reception (Ref. 1). Shannon's theorem is predicated on the fact that perfect timing information is available and may therefore be called the synchronized channel capacity theorem. If the timing noise were accounted for, we can envision a capacity theorem relative to non-ideal synchronization.

Basically, synchronization consists essentially of estimating two parameters in the received signal—frequency and time relative to a given origin. In digital data transmission we must distinguish between at least three types of
synchronization (Ref. 1). First, carrier and subcarrier reference synchronization is required for the operation of coherent reception. Second, symbol synchronization is required for purposes of estimating the instants when the modulation may change states. Third, word synchronization is required for determining the ending of one code word and the beginning of another. Other types, such as frame and group synchronization, can be classified with word synchronization.

As pointed out in Chapters 3 and 4, various types of phase-detector characteristics can be implemented by designing special synchronizing signals—for example, periodic oscillations, pseudo-noise waveforms, Barker sequences, etc. (Ref. 1). This method has the disadvantages of requiring the expenditure of extra power, time, and bandwidth to achieve synchronization. In addition, in certain applications, the channel may affect the information-bearing signal and the synchronizing signal differently. One can also use astute phase-detector mechanization methods to vary the phase-detector characteristic. This method has the advantage that the timing information can be obtained from the received signal itself. Although the method does not require extra power, it usually suffers from the disadvantage that the effective noise disturbance is no longer Gaussian, thereby creating analytical problems when analysis is required.

In what follows, the theory developed is sufficiently general to account for the design and analysis of second-order SCSs that employ the phase-lock or entrainment principle. This includes, among others, all the tracking and synchronizing systems discussed in Chapters 3 and 4, as well as those presented in Ref. 1.

10-1.2 Performance Characterization of SCSs

In this section we present a summary of the description of measures and parameters that characterize the performance of a SCS. Later we characterize these measures of performance for second-order SCSs in the absence and presence of noise.

It is necessary to point out that any SCS operates in two distinct modes, the signal acquisition mode and the synchronous or tracking mode.* The tracking mode refers to the phase stability attainable after the system has achieved a stable synchronized (phase-locked) condition. The signal acquisition mode refers to the performance associated with the system prior to phase-locking. Each of these modes has fundamental physical restrictions and characteristics. The full measure of performance permitted by the arriving signal component can be achieved by a SCS that makes these two modes of operation as independent as possible. Unfortunately, a sinusoidal PLL system uses the same mechanism for tracking and acquisition, and as we shall subsequently see, it is

*See the Prelude for a more qualitative description of the fundamental behavior of a SCS.
Introduction

fairly inefficient in its use of the frequency information contained in the arriving signal component. Other types of SCSs may use a multiplicity of mechanisms: one mechanism is designed for stable performance after synchronization; the other mechanism is designed to produce synchronization. The separation of the requirements of a sync system leads to the following principle: The limitation of a SCS, with respect to the signal acquisition mode, is the ability of the system, when out of sync, to recognize a frequency and phase difference and to distinguish this difference from noise.

In order to characterize and analyze the performance of a SCS, as well as carry out a design, one needs to develop a theory that specifies:

1. The noise bandwidth $W_L$, the integration time $1/W_L$, the system damping parameter, say $\zeta$, and the ratio $F_0$ of ac loop gain to dc loop gain. These are fundamental loop parameters and are nonstatistical in nature.
2. The maximum frequency acquisition (pull-in) range, $\gamma_m$. This represents the maximum normalized (to the loop gain) initial frequency detuning for which the system will automatically achieve the desired operating conditions.
3. The signal acquisition (phase sync stabilization) time, $T_{acq}$—that is, the time required for the operating characteristics of the system to reach stabilized conditions. One also needs to determine the probability $P_{acq}(t|\phi_0)$ of signal acquisition as a function of time.
4. The frequency acquisition time $T_f$—that is, the time necessary for the VCO to be changed from its rest frequency to a frequency from which the change of the phase between the arriving signal component and the VCO does not exceed one complete cycle.
5. The phase acquisition time $T_p$—that is, the transient time for the phase error to reach some definable term given frequency acquisition.
6. The steady-state (static) phase error. This depends on the effective initial frequency detuning $\Lambda_0$, the loop gain $AK$, and the loop signal-to-noise ratio.
7. The steady-state p.d.f. of the system phase error.
8. The mean $\bar{\phi}$ and the mean-squared value $\sigma^2_{\bar{\phi}}$ of the phase error due to noise and frequency detuning.
9. The $n$th moment, $\tau_n(2\pi|\phi_0)$, of the time to first loss of phase sync. This parameter depends on the initial phase error $\phi_0$, the effective detuning $\Lambda_0$ and the loop signal-to-noise ratio.
10. The average number of times the sync system fails per unit time, $S$. This parameter can be used to characterize system threshold.
11. The probability $P(t)$ of synchronization failure in $t - t_0$ sec.
12. The average number of cycle slips to the right and to the left, $N_+$ and $N_-$. 
13. The phase error diffusion constant $D_e$. 
14. Mean and variance of the frequency error. 
15. Signal hold-in range.

Among these fifteen measures of sync system performance, 2 through 5 have to do with characterizing the performance of the signal acquisition mode while measures 6 to 15 have to do with characterizing the performance of the synchronous mode. Some of the measures are deterministic when noise is absent and statistical in nature when noise is present; hence, an analysis based upon the theory given in Chapters 5, 6, 7, and 8 is required. We do not delineate further; the breakdown will be made clear in what follows. We note that these parameters were introduced in the Prelude and in Chapter 9 without elaborating upon their impact in the design of an efficient sync system. At that point in the development it was premature to discuss their significance; however, Table 10-1 presents a summary of the more significant measures of performance. It is important to note that the measures of performance of the acquisition mode can be accounted for only by taking into account the nonlinearity of the system. In the synchronous mode, performance measures one, two, 

<table>
<thead>
<tr>
<th>Signal Acquisition Mode</th>
<th>Synchronous or Tracking Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Signal acquisition range</td>
<td>1. Steady-state p.d.f. of the phase error</td>
</tr>
<tr>
<td>2. Signal acquisition time</td>
<td>2. Statistical moments of the phase error</td>
</tr>
<tr>
<td>(a) Phase acquisition time</td>
<td>3. Static and dynamic phase error</td>
</tr>
<tr>
<td>(b) Frequency acquisition time</td>
<td>4. Moments of time to first loss of synchronization (sync)</td>
</tr>
<tr>
<td>3. Probability of signal acquisition in the time interval $[t_0, t]$</td>
<td>5. Average time to first loss of phase sync</td>
</tr>
<tr>
<td>4. Signal acquisition techniques</td>
<td>6. Average number of cycles-slipped (sync failures) per unit time</td>
</tr>
<tr>
<td>5. Signal acquisition figure of merit</td>
<td>7. Probability of synchronization failure as a function of time</td>
</tr>
<tr>
<td>6. Signal acquisition behavior</td>
<td>8. Average number of cycles-slipped to the right and to the left per unit time</td>
</tr>
<tr>
<td></td>
<td>9. Probability of $k$ cycle slips in $T$ seconds</td>
</tr>
<tr>
<td></td>
<td>10. Average time between cycle-slipping events (system failures)</td>
</tr>
<tr>
<td></td>
<td>11. Average time duration of failure events</td>
</tr>
<tr>
<td></td>
<td>12. Mean and variance of the reference generator's frequency error</td>
</tr>
<tr>
<td></td>
<td>13. Phase (frequency) error diffusion constant</td>
</tr>
<tr>
<td></td>
<td>14. Signal hold-in range</td>
</tr>
<tr>
<td></td>
<td>15. Tracking mode figure of merit</td>
</tr>
</tbody>
</table>
and three can be approximated using the linear theory; however, an accurate account of measures four through fifteen require the use of the nonlinear theory.

The main purpose of this chapter is to present a theory which can be used to characterize and analyze the performance of a second-order SCS which has been designed on the basis of the linear theory given in Chapter 4; in particular, Sections 4-3 to 4-6. We begin by first considering the sinusoidal PLL and later turn our attention to systems containing an arbitrary periodic phase-detector characteristic.

10-1.3 System Representation and Loop Figures ofMerit

When (4-19) is substituted into (3-24) with \( \theta(t) = \Omega_0 t + \theta_0 \), the stochastic differential equation of system operation can be written as

\[
\dot{\phi} = \Lambda_0 - F_0 K [A g(\phi) + N(t, \phi)] + y_1
\]

(10-1)

where \( \Lambda_0 = \Omega_0 - K_v e \) and

\[
y_1 = \frac{-(1 - F_0)K}{1 + \tau_1 p} [A g(\phi) + N(t, \phi)]
\]

(10-2)

Here we have neglected any oscillator instabilities. It is important to note that the term \( \omega = \Lambda_0 + y_1 \) represents the effective instantaneous impressed radian frequency difference. Therefore the term \( y_1(t) \), due to the long-term time constant integration component in \( F(p) \), is responsible for the frequency pull-in mechanism of the system. This component is absent in a first-order system (\( F_0 = 1 \)) and makes it unattractive for use in practice. Later we shall be keenly concerned with the implications of this important observation. Figure 10-1 places into evidence a block diagram representation of (10-1) and (10-2) in the

![Fig. 10-1. Equivalent Model of a Second-Order Synchronous Control System.](image-url)
form of a closed loop. We shall therefore use the terms SCS and loop interchangeably.

From our discussions given in Chapter 4, Section 4-4, we know that the steady-state phase error, in the absence of noise, of a sinusoidal PLL is characterized by \( \sin \varphi_{ss} = \Lambda_0/\Delta K = \gamma = 2\pi \Delta f/\Delta K \), where \( \Lambda_0 \) is the effective loop detuning in radians (\( \Delta f \) in Hertz) and \( AK \) is the loop gain. Also, the loop bandwidth is given by \( W_L = 2B_L \approx (r + 1)/2\tau_2 \) when \( AK\tau_2 \gg 1 \). Loop damping \( \zeta \) is given by \( \zeta = \sqrt{r}/2 \), where \( r = \tau_2 AK\zeta_0 \). Since both small steady-state phase error and narrow bandwidth are desired in the tracking mode, it is possible to define a figure of merit for the tracking mode of the SCS of Fig. 10-1.

\[
(F \cdot M)_c \triangleq \begin{cases} 
\left| \frac{\sigma(\varphi_{ss})}{\Delta f} \right| B_L & \text{noise absent} \\
\left| \frac{\mathbb{E}[g(\varphi)]}{\Delta f} \right| B_L & \text{noise present}
\end{cases} 
\tag{10-3}
\]

The smaller this product, the better the tracking performance. It is easy to show that \( (F \cdot M)_c = \pi/2 \) for a sinusoidal PLL when \( \tau_2 = 0 \). This says that for the simplified loop filter (i.e., \( F_0 = 0 \)), the steady-state phase error and noise bandwidth product are independent; however, for the filter of Fig. 10-2, the loop can be designed for any figure of merit desired.

![Fig. 10-2. Frequency Response of the Imperfect Integrator Showing that for \( \omega \gg \frac{1}{\tau_2} \) the Gain of the Loop Filter is \( F_0 \).](image)

For the signal acquisition mode, one can define the figure of merit, for a given acquisition probability, as the product of the signal acquisition range in hertz and the signal acquisition time \( T_{acq} \).
Here $|\Lambda_0|_{\text{m}}/2\pi$ is signal acquisition range in hertz. Equations (10-3) and (10-4) can be used to access the effectiveness of a particular system design.

10-2 Signal Acquisition with Imperfect Second-Order Loops in the Absence of Noise

Illustrated in Fig. 10-2 is a sketch of the frequency response of the loop filter defined in (4-19). At high frequencies $\omega > 1/\tau_2$, the gain of the filter approaches $F_0 = \tau_2/\tau_1$; consequently, this loop is indistinguishable from a first-order PLL with gain $AKF_0$. As such, there also exists a synchronization band (the acquisition range) for which the loop will phase-lock to the externally applied signal. In fact, we shall later see that, for high-gain loops, $F_0$ determines the acquisition range. The loop acquisition range (sometimes called capture range, pull-in range) represents the largest frequency difference between the frequency of the applied oscillations and that of the VCO for which phase-lock is possible. We shall explore this region by analytical means at a later time.

The lock-up behavior of second-order loops with an imperfect integrator in the loop is best discussed by recognizing that the output of the phase-detector consists of a beat note at the frequency $1/T_{\text{bn}}$. As far as the beat note is concerned, the loop appears to be first order with gain $AKF_0$ if $\Lambda_0 > 1/\tau_2 = 2W_p/(r + 1)$ (see Chapter 4 for definitions of the parameters, and Fig. 10-1). That portion of the beat note which passes through the loop filter produces frequency modulation of the VCO. Hence the phase-detector output consists of the product of a sine wave and an FM wave. Since the spectral content of the FM wave frequently contains a myriad of components about the center frequency, it is possible for the loop to phase lock to one of the FM sidebands. When this takes place, the loop is said to be operating in a false-lock mode. As we shall see, this phenomenon shows up as limit cycles in the phase-plane. More frequently, the waveform appearing at the phase-detector output contains a dc component that is sufficiently strong to cause the loop to pull into phase lock. We presently discuss the phase-plane diagram and later explore the quantitative aspects of the phase-plane.

10-2.1 Phase-Plane Trajectories of a Sinusoidal PLL when Tracking a Frequency Offset

The equation governing loop behavior is found from (3-7) by inserting (4-19) and simplifying. Such a procedure yields

$$\Lambda_0 = \tau_1 \frac{d^2 \varphi}{dt^2} + (1 + AK\tau_2 \cos \varphi) \frac{d\varphi}{dt} + AK \sin \varphi$$  (10-5)
when \( \theta(t) = \theta_0 + \Omega_0 t \) and \( e \) in assumed constant. Normalizing time in (10-5) by setting \( \tau = AKt \), and setting \( d\varphi/d\tau = v \) yields

\[
\frac{dv}{d\varphi} + \left( \frac{1}{AK\tau_1} + F_0 \cos \varphi \right) = \frac{1}{AK\tau_1} v \left( \frac{A_0}{AK} - \sin \varphi \right)
\]

(10-6)

since

\[
\frac{d^2\varphi}{d\tau^2} = \frac{dv}{d\tau} = \frac{dv}{d\varphi} \frac{d\varphi}{d\tau} = v \frac{dv}{d\varphi}
\]

Fig. 10-3. Phase-Plane Trajectories for Second-Order Loop with Imperfect Integrator, \( \gamma = 0.4, \quad r = 2, \quad F_0 = 0.2 \). (Compliments of A. J. Viterbi.)
Setting $\gamma = \Lambda_0/AK$ and applying the methods discussed in Chapter 5 indicates that (10-6) has potential singularities at $\phi = 0$ for

$$\varphi_{ik} = \begin{cases} 
2n\pi + \sin^{-1}(\gamma) & \text{stable points} \\
(2n + 1)\pi - \sin^{-1}(\gamma) & \text{unstable points}
\end{cases} \quad (10-7)$$

which are identical with those of a first-order loop. Obviously, when $|\gamma| > 1$ there are no stable points and phase-lock is not possible.

Viterbi (Ref. 2) has produced the phase-plane plots of (10-6) reduced modulo-2$\pi$ for various initial conditions of the VCO (see Figs. 10-3 and 10-4). A major result from this phase-plane study is that even if $\Lambda_0 < AK$, phase-

Fig. 10-4. Phase-Plane Trajectories for Second-Order Loop with Imperfect Integrator, $\gamma = 0.9, r = 2, F_0 = 0.2$. (Compliments of A. J. Viterbi.)
lock may not occur even though stable points exist in the phase-plane. If \( \varphi(t) \) is taken to be the position of a particle along the \( \varphi \) axis at time \( t \), then for \( v > 0 \), motion of the particle is from left to right, while for \( v < 0 \), motion is from right to left in the phase-plane. If the initial conditions of the VCO are such that \( v < 0 \) for a given \( \varphi \), one follows the phase trajectory to the left until \( \varphi = -\pi \) is reached and then one skips back to \( \varphi = \pi \), starting at the next highest trajectory. This oscillatory motion—from \( \pi \) to \( -\pi \) to \( \pi \) to \( -\pi \)—is indicative of the fact the loop is slipping cycles; that is, periodic pulling or pushing is taking place between the frequency of the arriving oscillation and those produced by the VCO. When one no longer reaches the boundaries at \( \pi \) and \(-\pi \) by following the phase-plane trajectory from right to left, frequency lock is achieved and the motion continues by spiraling into a stable lock point where phase-lock is finally achieved.

Special attention should be given to Fig. 10-4, because, for values of \( \gamma > 0.4 \), it has been determined that lock-in may occur for some initial condition of the VCO but not for others. This figure shows that there is a limit cycle toward which all higher trajectories converge, as well as trajectories for which some of those from below converge. As discussed in Chapter 5, this limit cycle corresponds to time-periodic solutions in \( \varphi \) to the differential equation (10-5) of operation. We also observe the important fact that when attempting to acquire a positive Doppler-shifted frequency, it is best to make the VCO frequency lead the input frequency.

### 10-2.2 Signal Acquisition Time, Signal Acquisition Range, and Hold-In Range in Sinusoidal PLLs

In this section we are concerned with the development of a theory that can be used to characterize the performance of sinusoidal PLLs in the acquisition mode when noise is absent. In a later section we generalize our results. Our procedure here is to obtain an equation for the loop acquisition voltage that accounts for the loops acquisition mechanism. As we shall see, it is fairly inefficient. We then ask for the time it takes for the acquisition voltage waveform to reach its frequency lock value. A byproduct of our analysis is a bound on the signal acquisition range.

In the absence of noise, the dynamical equations (10-1) and (10-2) of loop operation become

\[
\dot{\varphi} = \omega - AKF_0 \sin \varphi
\]

\[
\omega \triangleq \Lambda_0 + y_1 = \Lambda_0 - \left[ \frac{AK(1 - F_0)}{1 + \tau_p} \right] \sin \varphi
\]

(10-8)

where \( y_1 \) is the Fourier transform of a time function that represents a slowly varying (\( \tau_i \gg 1 \)) acquisition voltage. When the loop filter time constants are
such that \( F_0 \ll 1 \) (i.e., \( \tau_1 \gg \tau_z \)), the average voltage \( y_1 \) (which may develop across the capacitor \( C \)) as a result of an unsymmetrical beat note appearing at the phase detector output) will not change appreciably over time intervals on the order of \( T_{bn} \), the period of the beat note. Thus \( y_1 \) may be assumed constant during this time period and the first equation in (10-8) looks like the equation of a first-order loop with detuning \( \Lambda_0 + y_1 \) and loop gain \( AKF_0 \). As shown in Chapter 9, Section 9-2.1, by averaging over a cycle of the beat note, \( \sin \varphi \), its average value can be found by integrating the first equation in (10-8),

\[
\langle \sin \varphi \rangle_a = \frac{1}{T_{bn}} \int_0^{T_{bn}} \sin \varphi \, dt \approx x + x_0 - \frac{2\pi}{AKF_0 T_{bn}}
\]

(10-9)

where \( x \triangleq \frac{\langle y_1 \rangle_a}{AKF_0}, x_0 \triangleq \Lambda_0/AKF_0 \), and

\[
T_{bn} \triangleq \frac{2\pi}{AKF_0 \sqrt{(x + x_0)^2 - 1}}
\]

(10-10)

Now

\[
\langle y_1 \rangle_a \triangleq -\left\langle \left\{ \frac{AK(1 - F_0)}{1 + \tau_1 p} \sin \varphi \right\} \right\rangle_a
\]

(10-11)

represents an average of the acquisition voltage over one cycle of the beat note. Contrast these results with those given for a first-order loop where the beat note period was determined by the loop gain and detuning.

At this point it is worth discussing the nature of the signal circulating in the loop. It is clear from Fig. 10-1 that there are two components. There is a cyclic component produced as a result of the direct transfer component \( F_0 \) in \( \bar{F}(p) \) that is analogous to the beat note appearing in a first-order loop. There is a slowly varying component \( y_1(t) \) that may cause the loop to pull into phase lock. Since this changes slowly when \( F_0 \ll 1 \), then

\[
\frac{1}{T_{bn}} \int_0^{T_{bn}} \left( \frac{1 - F_0}{1 + \tau_1 p} \right) \sin \varphi \, dt \approx \left( \frac{1 - F_0}{1 + \tau_1 p} \right) \langle \sin \varphi \rangle_a
\]

(10-12)

The average frequency shift \( \langle \sin \varphi \rangle_a \) can be eliminated from the foregoing equations to give a differential equation in the average acquisition voltage per cycle.

\[
\frac{dt}{\tau_i} = [(1 - F_0)/F_0] \frac{dx}{\sqrt{(x + x_0)^2 - 1} - (x + x_0)} - x
\]

(10-13)

Before the frequency acquisition time \( T_f \) can be specified, it remains to determine the limits of integration. Now the frequency of the beat note be-
comes infinite when \( x + x_0 = 1 \) so that \( \phi = 0 \) when \( x = 1 - x_0 \). Therefore the frequency acquisition time is given approximately by

\[
\frac{T_f}{\tau_1} = \int_0^{1-x_0} \frac{dx}{[(1 - F_0/F_0) [\sqrt{(x + x_0)^2 - 1} - (x + x_0)] - x} \quad (10-14)
\]

when the initial voltage across the loop filter capacitor \( C \) is zero. Letting \( z = \sqrt{(x + x_0)^2 - 1} - (x + x_0) \) and rearranging, we obtain

\[
\frac{T_f}{\tau_2} = \int_{-1}^{x_0} \frac{(z - 1/z) \, dz}{(2 - F_0) \, z^2 + 2yz + F_0} \quad (10-15)
\]

where \( z_0 = \sqrt{(\gamma/F_0)^2 - 1 - \gamma/F_0} \). The integration in (10-15) can be carried out; however, to obtain numerical results, it is simpler to perform numerical integration on a digital computer.

A lower bound for the frequency acquisition time is of interest for design work. When \( F_0 \ll 1 \), then letting \( z = x + x_0 \) in (10-14) gives

\[
\frac{T_f}{\tau_1} \gtrsim \int_{x_0}^{1} \frac{F_0 \, dz}{(1 - F_0) \, [\sqrt{z^2 - 1} - z]} \\
\gtrsim F_0 \left[ \frac{x_0^2 - 1}{2} + x_0 \sqrt{x_0^2 - 1} \cdot \frac{1}{2} - \frac{1}{2} \ln \left[ x_0 + \sqrt{x_0^2 - 1} \right] \right] \quad (10-16)
\]

and for large \( x_0 \) we have

\[
T_f \gtrsim \tau_2 \left[ \left( \frac{L_0}{AKF_0} \right)^2 - \frac{1}{2} \ln \left( \frac{2L_0}{AKF_0} \right) \right] \\
\gtrsim \pi^2(r + 1)^3(\Delta f)^2 \frac{1}{2r^2} - \left( \frac{r + 1}{4W_L} \right) \ln \left[ \left( \frac{2\pi(r + 1)}{r} \right) \frac{\Delta f}{W_L} \right] \quad (10-17)
\]

where \( \Delta f = L_0/2\pi \).

Neglecting the second term in this expression, we have the often-quoted formula for the time to acquire frequency lock developed first by Richman (Ref. 3) and later by Viterbi (Ref. 2). This lock up time is essentially minimized when \( r \approx 2 \).

The phase stabilization time \( T_p \) has only a small effect on the acquisition time \( T_{acq} \) when the initial frequency detuning is appreciable; however, as can be observed from Fig. 9-1c, normally \( AKF_0 T_p < 10 \) when the loop gain is large. Thus an approximation to the phase acquisition time is given by

\[
T_p < \frac{5}{W_L} \left( r + 1 \right) \quad (10-18)
\]
The acquisition range may be determined as a byproduct of our analysis by noting the condition that makes $T_f$ become infinite. This occurs when the denominator of the integrand in (10-15) has a real root; that is, the loop will lock if

$$|\gamma_m| \triangleq \left| \frac{\Lambda_0}{AK} \right|_m < \sqrt{2F_0 - F_0^2} \quad (10-19)$$

Thus by making $AK$ very large, the maximum acquisition range can be made to approach any desired value; however, this is not really satisfactory, for acquisition times are excessive in this extra range. It appears that Richman (Ref. 3) was the first American author to arrive at this result. Experimental evidence produced by Halliday (Ref. 38) indicates that (10-19) is not extremely accurate for $\omega_n/|AK| > 0.35$. As a consequence Halliday proposes the experimentally verified, empirically determined formulas

$$|\gamma_m| = \left| \frac{\Lambda_0}{AK} \right|_m < \frac{[rW_L/(r + 1)]}{(0.95rW_L)/(r + 1) + 0.04AK}$$

$$|\gamma_{mh}| = \left| \frac{\Lambda_0}{AK} \right|_m < \frac{[rW_L/(r + 1)]}{[(0.85rW_L)/(r + 1)] + 0.16AK}$$

for the acquisition range $\gamma_m$ and the single hold-in range, $\gamma_{mh}$. Those formulas appear to be valid for $0 < \omega_n/|AK| < 2$ where $\omega_n$ is defined in (4-24).

When the pole at $x_0^2 = 1 - 2/F_0$ is included in (10-17), one can write the simplified equation for frequency acquisition time as

$$B_L T_f \approx \frac{r + 1}{4} \left[ \frac{(\frac{\Lambda_0}{AKF_0})^2 - \frac{1}{2} \ln \left( \frac{2\Lambda_0}{AKF_0} \right)}{1 - \frac{F_0}{2F_0} (\frac{\Lambda_0}{AKF_0})^2} \right] \quad 1 < \left( \frac{\Lambda_0}{AKF_0} \right)^2 < \frac{2}{F_0} - 1$$

(10-20)

when $\Lambda_0/|AKF_0| \gg 1$ and $F_0 \ll 1$. Figures 10-5 and 10-6 represent plots of (10-20) for various loop parameters and signal conditions. The dimensionless parameter $T_f/B_L$ is used as the ordinate and $\Delta f/B_L = \Lambda_0/2\pi B_L$ is the abscissa in the two sets of performance curves illustrated in Figs. 10-5 and 10-6 where $\gamma/F_0 = \pi(r + 1/2r)\Delta f/B_L$. From these figures we note that a sinusoidal PLL exchanges frequency acquisition time for sync stability at an expensive rate, i.e., $T_f$ varies as the square of the initial frequency detuning interval. Since $AKF_0 = 4rB_L/(r + 1)$, $T_f$ varies inversely as the cube of $B_L$. For example, the time required to pull the VCO 1.0 kHz will be greater than 100 seconds if $B_L = 100 \text{ Hz}$, $F_0 = 0.001$ and $r = 1$. We reiterate that (10-20) gives best results for extremely narrowband loops; the reason, of course, is due to the
assumption that the voltage across $C$ does not change appreciably over time intervals on the order of $T_{br}$. We also note that the voltage across $C$ is zero at $t = t_0 = 0$. Frequently, the loop is allowed to run freely when no input signal is available and a large voltage will develop across $C$ as a result of integrating
Fig. 10-6. Acquisition Time versus $|\Delta f|/B_L$ for Various Values of $F_0$ with $r = 2$.

The noise. This alters the acquisition time and (10-20) does not account for this.

We now turn to the problem of studying the acquisition range and time for an arbitrary periodic nonlinearity. In fact, a special case of our result will
be an upper bound for the acquisition time of a sinusoidal PLL. The method we shall use is to construct the periodic motion that is present in a loop that is false-locked and then determine loop parameters that break up this periodic motion. Notice that we are using the term false-lock to imply any type of oscillations that yield unstable conditions in the loop.

10-2.3 Signal Acquisition Range for an Arbitrary Periodic Phase-Detector Characteristic

In this section we broaden the scope of our study to include SCSs with various phase-detector (S curve) characteristics (see Chapter 3, Table 3-1, for example). We do require, however, that \( g(\phi) \) be periodic in \( \phi \) and have a maximum value of unity.

If we replace \( \sin \phi \) by \( g(\phi) \) in (3-7), insert (4-19) and normalize time by setting \( \tau = AKt \), then (10-5) generalizes to

\[
\frac{dv}{d\phi} = \frac{1}{AK\tau} \left[ \frac{\gamma - g(\phi)}{v(\phi)} - 1 \right] - F_0 \frac{dg(\phi)}{d\phi} \tag{10-21}
\]

where \( v(\phi) = d\phi/d\tau \). This expression represents the tangent line to the phase-plane trajectories on the surface of a phase cylinder with coordinates \( (v, \phi) \). The acquisition range represents the largest normalized detuning \( \gamma \) at whose boundary synchronous operation becomes impossible for any given initial conditions. In order to determine the acquisition range, our procedure will be first to find the condition that is satisfied by all solutions that are periodic on the cylinder; that is, \( v(\phi) = v(\phi + 2\pi) \), which says the SCS is in a false-lock condition. This establishes the condition of limit cycles encompassing the phase cylinder; that is, \( \phi \) is periodic with the passage of time. Once we find such, we then determine the relationship between loop parameters that will cause these periodic solutions \( v(\phi) \) to fail. Doing so also allows us to study how \( g(\phi) \) should be chosen in order to obtain the largest signal acquisition range for a given loop design.

If we multiply both sides of (10-21) by \( v(\phi) d\phi/2\pi \), take into account the periodicity of \( g(\phi) \) and the limit cycle \( v(\phi) \), we obtain

\[
\left\langle v(\phi) \right\rangle_a - AK\tau_a \left\langle \frac{dv(\phi)}{d\phi} \right\rangle_a = \gamma \tag{10-22}
\]

Here the symbol \( \left\langle \cdot \right\rangle_a \) denotes the average of the enclosed quantity over one period \( 0 \leq \phi \leq 2\pi \). Substitution for \( dv/d\phi \) from (10-21) into (10-22) gives the condition that all periodic solutions \( v(\phi) \) must satisfy—namely,

\[
\left\langle v(\phi) \right\rangle_a - F_0 \left\langle \left( \frac{\gamma - g(\phi)}{v(\phi)} \right) g(\phi) \right\rangle_a = \gamma \tag{10-23}
\]
For an arbitrary $F_0, \gamma$, and $g(\varphi)$ it does not seem possible to find exact analytical solutions $v(\varphi)$ of (10-21). This is due to the nonlinearity of $g(\varphi)$. In general, solutions must therefore be obtained via the digital computer. However, for high-gain loops, we can find accurate asymptotic solutions $v_{as}(\varphi)$ for the case where $AK_1$ approaches infinity. From the phase-plane plot of Fig. 10-4 we note that the limit cycle, say $v(\varphi)$, is positive for all $\varphi$; that is, $v(\varphi) > 0$. Thus we seek asymptotic solutions for the case where $F_0 \neq 0, \gamma > 0$, and $AK_1 \gg 1$ with $v_{as}(\varphi) > 0$. This is equivalent to assuming that $\Lambda_0 > 0$; however, we note that a completely symmetrical argument will hold for $\Lambda_0 < 0$. Under these assumptions, the first term in the right-hand side of (10-21) approaches zero, and the resulting equation may be solved to give

$$ v_{as}(\varphi) = F_0 [C - g(\varphi)] \quad (10-24) $$

where $C$ is the constant of integration and $CF_0$ is the average of the limit cycle on the phase cylinder $v(\varphi); 0 \leq \varphi \leq 2\pi$. If $g(\varphi) = \sin \varphi$, our result is intuitively pleasing with the limit cycle illustrated in Fig. 10-4. Under the assumption that $v_{as}(\varphi)$ is positive on the cylinder, then, if $|g(\varphi)| < 1$ for $0 \leq \varphi \leq 2\pi$, we have that $C > 1$. Substitution of (10-24) into (10-23) gives

$$ \gamma(C) = \frac{C \left( F_0 - \langle x \rangle_a + \langle \frac{x}{1-x} \rangle_a \right)}{1 + \langle \frac{x}{1-x} \rangle_a} \quad (10-25) $$

where $x \triangleq g(\varphi)/C$ and $|x| < 1$. This determines the condition for an asymptotic limit cycle, and hence $\gamma$ as a function of $C$ for a given $F_0$. Now the locus, $\gamma(C)$ vs. $C$ for a given $F_0$, determines the conditions for limit cycles to occur in a SCS. Using the expansion

$$ \frac{1}{1-x} = 1 + x + x^2 + \cdots \quad (10-26) $$

which is valid for $|x| < 1$ in (10-25) and noting that $g_a \triangleq \langle g^n(\varphi) \rangle_a = 0$ for $n$ odd gives

$$ \gamma(C) = \frac{C \left( F_0 + \sum_{n=1}^{\infty} \langle x^{2n} \rangle_a \right)}{1 + \sum_{n=1}^{\infty} \langle x^{2n} \rangle_a} \quad (10-27) $$

Notice that for any periodic $g(\varphi)$ we have

$$ g_{2(n+1)} \leq g_{2n} \leq 1 \quad (10-28) $$
so that

\[ \gamma(1) \leq 1 \quad \lim_{C \to \infty} \gamma(C) \to F_0 C \quad (10-29) \]

represent useful boundary conditions. It follows from this that for any \( g(\phi) \) the curves \( \gamma(C) \) vs. \( C \) has a minimum at \( C = C_m > 1 \) when \( F_0 \) is sufficiently small. In fact, if \( F_0 = 1 \) we see that \( \gamma(C) = C \) and no minimum occurs. This corresponds to the case of a first-order loop. As noted in Chapter 9, \( \gamma \) must be less than one for phase-lock to occur. Obviously, the value

\[ \gamma(C_m) \triangleq \left( \frac{A_0}{AK} \right)_m \triangleq \gamma_m \quad (10-30) \]

corresponds to the boundary of the signal acquisition range because \( \gamma(C_m) \) corresponds to the smallest normalized detuning that satisfies conditions for the periodic solutions \( v(\phi) \). If \( \gamma > \gamma_m \), phase-lock will not occur; whereas when \( \gamma < \gamma_m \), phase-lock will occur. In fact, if \( \gamma > \gamma_m \), a phenomenon frequently referred to as frequency pushing (VCO frequency is forced away from the signal frequency) will be observed; while for \( \gamma < \gamma_m \), frequency pulling occurs. Therefore, in order to determine the acquisition range for an arbitrary \( g(\phi) \), it is necessary to compute the derivative \( d\gamma/dC \) from (10-25) and then equate this derivative to zero in order to evaluate \( C_m \). This value of \( C_m \) is inserted into (10-25) to determine \( \gamma_m \).

Unfortunately, this method does not yield closed-form expressions for the acquisition range for all cases of practical interest. Nevertheless, one example that does admit a closed-form solution is the case where \( g(\phi) = -g(-\phi) \) and

\[ g(\phi) = \begin{cases} 1 & 0 < \phi < \pi d \\ 0 & \pi d \leq \phi \leq \pi \end{cases} \quad (10-31) \]

where \( 0 < d \leq 1 \) and \( g(0) = 0 \). The average values needed in (10-27) become

\[ \left\langle x^2 \right\rangle_a = \left\langle \frac{x}{1-x} \right\rangle_a = \frac{d}{C^2 - 1} \quad (10-32) \]

so that

\[ \gamma(C) = \frac{C[F_0(C^2 - 1) + d]}{C^2 - 1 + d} \quad (10-33) \]

Graphs of \( \gamma(C) \) vs. \( C \) are plotted in Fig. 10-7 for \( d = \frac{1}{2} \) and \( d = 1 \). Note the effects as \( d \) is varied. It can be seen from Fig. 10-7 that the value
Fig. 10-7. Normalized Detuning, $\gamma(C)$, versus $C$ of the Limit Cycle for Two Values of $d$.

$\gamma_m = \gamma(C_m)$ represents the smallest normalized detuning that satisfies (10-33) for periodic solutions. The value $\gamma_m$ is found when $d\gamma/dC = 0$, so that

$$F_0C_m^4 - [2F_0 + d(1 - 3F_0)]C_m^2 + (d - 1)(d - F_0) = 0 \quad (10-34)$$

For $d = 1$ this reduces to

$$C_m = \sqrt{(1 - F_0)/F_0} \quad (10-35)$$

and the asymptotic synchronization boundary is defined by

$$|\gamma_m| = 2\sqrt{F_0(1 - F_0)} \quad (10-36)$$

Thus, according to this theory, frequency pulling will occur if

$$\frac{|\Lambda_0|}{AK} < |\gamma_m| = 2\sqrt{F_0(1 - F_0)} \quad (10-37)$$

and frequency pushing will occur if
\[
\frac{\left| \Lambda_0 \right|}{AK} > \left| \gamma_m \right| = 2\sqrt{F_0(1 - F_0)}
\] (10-38)

That is, phase synchronization will not be possible. Accurate plots of (10-37) are illustrated in Fig. 10-8 for various values of \( F_0 \). Figure 10-8 also illustrates how \( \gamma_m \) vs. \( F_0 \) varies for the rectangular characteristic in (10-31) for various values of \( d \). Using (10-33) and (10-34), it is easy to show that

\[
\lim_{d \to 0} \gamma_m = F_0
\] (10-39)

**Fig. 10-8.** Asymptotic Normalized Acquisition Range \( \gamma_m \) versus \( F_0 \) for Various Values of \( d \).

Since the mean-squared value of any \( g(\phi) \), in the class considered, is bounded by \( 0 < g_2 \leq 1 \), the curves of Fig. 10-8 for \( d = 0 \) and \( d = 1 \) are the upper and lower boundaries of the asymptotic acquisition range achievable with an arbitrary phase-detector characteristic.

For the simple “triangular” characteristic
\[ g(\varphi) = \begin{cases} \frac{\varphi}{\pi(1-d)} & 0 < |\varphi| < \pi(1-d) \\ \frac{(\pi - \varphi)}{\pi d} & \pi(1 - d) \leq |\varphi| \leq \pi \end{cases} \]  

(10-40)

an expression for the capture range can be obtained only from numerical solution of two transcendental equations derived in the same way as (10-33) and (10-34). When \( d = 0 \), (10-40) represents the sawtooth phase-detector characteristic.

### 10-2.4 Design of the Phase-Detector Characteristic to Maximize Acquisition Range

In order to obtain tractable formulas for an arbitrary \( g(\varphi) \), we consider those cases in (10-27) for which \( F_0 \ll 1 \) and \( C_m^2 \gg 1 \). If \( C_m^2 \gg 1 \), we have \( |x| \ll 1 \) and only the first term becomes significant in the sums of (10-27). Thus \( \gamma(C) \) reduces to

\[ \gamma(C) \approx C \left( \frac{C^2 F_0 + g_2}{C^2 + g_2} \right) \]  

(10-41)

Applying our procedure produces the values

\[ C_m \approx \sqrt{\frac{g_2}{F_0}} \quad \gamma_m \approx 2 \sqrt{F_0 g_2} \]  

(10-42)

where \( g_2 = \langle g^2(\varphi) \rangle_n \) is the mean-squared value of \( g(\varphi) \). From these results we note that, for \( F_0 \ll 1 \) and \( g_2 \gg F_0 \), the asymptotic acquisition range depends only on the mean-squared value of the phase-detector characteristic \( g(\varphi) \). Consequently, we conclude that the same acquisition range can be obtained with any \( g(\varphi) \) having the same mean-squared value. It is therefore possible, for a given acquisition range, to choose the form of \( g(\varphi) \) that minimizes the mean-squared value of the phase error in noise or the smallest number of cycle-slips per second due to the noise. Notice that the range of applicability of (10-42) should shrink as \( g_2 \) becomes small and when the coefficients \( g_n \) decay slowly as \( n \) is increased in (10-27).

For a sinusoidal PLL, \( g(\varphi) = \sin \varphi \) and \( g_2 = \frac{1}{2} \) so that (10-42) reduces to

\[ \gamma_m \approx \sqrt{2 F_0} = \sqrt{\frac{2 \tau_2}{\tau_1}} \quad C_m \approx \sqrt{\frac{\tau_1}{2 \tau_2}} \]  

(10-43)

This says that \( F_0 \) limits the acquisition range when the loop gain is large; Fig.
10-2 shows that $F_0$ represents the gain of $F(i\omega)$ when $\omega \gg 1$. According to this theory, the loop will lock if

$$\frac{|\Lambda_0|}{AK} < \sqrt{\frac{2\tau_2}{\tau_1}} = |\gamma_m| \tag{10-44}$$

and in terms of loop parameters for a high-gain, second-order loop (see Chapter, 4), we find that false-lock will not occur—that is, the loop will not go into a stable limit cycle—if

$$\frac{|\Lambda_0|}{W_L} < \frac{4}{1 + r} \cdot \sqrt{\frac{1}{2F_0}} = |\gamma_m| \tag{10-45}$$

Equations (10-44) and (10-45) are interesting to compare with (10-19) for the sinusoidal PLL. In fact, using (10-19), it is easy to show that

$$\frac{\Lambda_0}{W_L} < \left( \frac{4}{r + 1} \right) \sqrt{r \left( 1 + \frac{r}{2F_0} \right)} \tag{10-46}$$

for all initial conditions of the VCO. If $r/2F_0 \gg 1$, this upper bound agrees with (10-45).

Figure 10-9 illustrates plots of $\gamma_m$ vs. $F_0$ for the rectangular characteristic given in (10-31) with $d = 1$, the triangular characteristic in (10-40) with $d = 0$, and the sinusoidal characteristic when numerical methods are applied to solving (10-27). The dashed lines show the results of computations in accordance with (10-42). Comparison of the results indicates that for these three

![Fig. 10-9. Comparison of the Asymptotic Acquisition Range $\gamma_m$ versus $F_0$ for Various Types of Phase-Detector Characteristics.](image-url)
characteristics the approximations in (10-42) give a satisfactory agreement with those obtained via (10-27) over the range $0 < \gamma_m < 0.8$. On the basis of these results, we see that the rectangular phase-detector characteristic produces the largest acquisition range, whereas the sinusoidal phase-detector characteristic is better than a triangular phase-detector characteristic.

It is interesting to compare our results with the phase-plane analysis given in Figs. 10-3 and 10-4. For $F_0 = 0.2$ we find from (10-42) that $\gamma_m = 0.63$. According to this theory, the loop will lock if $\Lambda_0/AK < \gamma_m = 0.63$. Figure 9-3 illustrates the acquisition (phase-plane) characteristics when $\Lambda_0/AK = 0.4$. As predicted, no limit cycle shows up in the phase plane. Contrast this case with that shown in Fig. 10-4, where $\Lambda_0/AK = 0.9 > \gamma_m = 0.63$ and, as predicted by the theory, a limit cycle shows up in the phase-plane.

### 10-2.5 Acquisition Time and the Time to Slip a Cycle for an Arbitrary, Periodic Phase-Detector Characteristic

The time required to slip a cycle when the loop is false-locked (i.e., the phase error is periodic in time; see Fig. 10-4 for a sinusoidal PLL), can be evaluated in a manner similar to that used for the first-order PLL. Now (10-24) represents an approximation to the limit cycle behavior when the loop is false-locked. We also note that for the sinusoidal PLL (10-24) is in close agreement with the form of the limit cycle shown in Fig. 10-4. From (10-24) we can write

$$\frac{d\phi}{dt} = AK \frac{d\phi}{d\tau} = AKF_0 [C_m - g(\phi)]$$  \hspace{1cm} (10-47)

so that the time $T_{2\pi}$ to slip one cycle is given by

$$\int_{0}^{T_{2\pi}} dt \approx \int_{-\pi}^{\pi} \frac{d\phi}{AKF_0 [C_m - g(\phi)]}$$  \hspace{1cm} (10-48)

The problem of evaluating the time to reach frequency lock—that is, the time required until no more cycle slipping occurs—appears to be a formidable task due to the nonlinearity $g(\phi)$. However, if $\Lambda_0 < AK\gamma_m$, then the time to acquire frequency lock can be bounded from above by multiplying the time required to slip one cycle by the square of the maximum detuning that can be tolerated for phase-lock to occur. Thus for *conditional-lock* (i.e., $\Lambda_0 < AK\gamma_m$)

$$T_f < \left(\frac{\Lambda_0}{2\pi}\right)^2 T_{2\pi}$$  \hspace{1cm} (10-49)
represents an upper bound on the acquisition time for a second-order SCS with an arbitrary phase-detector characteristic.* In general, numerical integration is required to evaluate (10-48); however, for a sinusoidal PLL (10-48) is easily shown to reduce to

\[ T_{2n} \approx \frac{1}{AKF_0} \left[ \frac{2\pi}{\sqrt{C_m^2 - 1}} \right] \]  

(10-50)

so that the time required to slip a cycle during pull-in is bounded from above by (10-50). This fact is obvious from Fig. 10-4, for we see that the amount of reduction in \( \nu(\nu) \) per cycle becomes larger as the loop VCO pulls closer to a phase-lock position. Using (10-43) in (10-50), we obtain

\[ T_{2n} \approx \frac{2\pi}{AK} \sqrt{\frac{2}{F_0(1 - 2F_0)}} \approx \frac{2\pi}{AK} \sqrt{\frac{2}{F_0}} \]  

(10-51)

for \( F_0 \ll 1 \). On the other hand, when \( F_0 = 1 \) in (10-50), we have the time to slip a cycle in a first-order sinusoidal PLL, \( C_m = \Lambda_0 / AK > 1 \).

From (10-49) and (10-51) we note that the time to frequency lock is upper bounded by

\[ T_f \leq \left( \frac{\Lambda_0^2 / 2\pi}{AK} \right) \sqrt{\frac{2}{F_0(1 - 2F_0)}} \]  

(10-52)

when \( \Lambda_0 < AK\gamma_m \). When the pole in (10-15), which makes the acquisition time infinity, is inserted in (10-52), and the basic loop parameters introduced we have

\[ T_f \approx \left[ \frac{\pi}{2} \frac{(\Delta f)^2}{B_L} \left( \frac{r + 1}{r} \right) \sqrt{\frac{2F_0}{1 - 2F_0}} \right] \left[ 1 - \left( \frac{\pi^2 F_0}{2 - F_0} \right) \left( \frac{r + 1}{2r} \right) \left( \frac{\Delta f}{B_L} \right) \right]^{-1} \]  

(10-53)

It is conjectured that the time required to achieve phase lock, given frequency lock, is of the order \( T_p \approx 5(r + 1) / r W_L \) seconds. Thus the acquisition time, \( T_{acq}(2) \), of a second-order sinusoidal PLL is upper bounded by

\[ T_{acq}(2) \leq \left[ \frac{\pi}{2} \frac{(\Delta f)^2}{B_L} \left( \frac{r + 1}{r} \right) \sqrt{\frac{2F_0}{1 - 2F_0} + \frac{5(r + 1)}{2rB_L}} \right] \left[ 1 - \left( \frac{\pi^2 F_0}{2 - F_0} \right) \left( \frac{r + 1}{2r} \right) \left( \frac{\Delta f}{B_L} \right) \right]^{-1} \]  

(10-54)

*The fact that \( T_f \) is proportional to \( (\Lambda_0/2\pi)^2 \) has been verified experimentally; in fact, for large loop signal-to-noise ratios, i.e., \( \rho \geq 10 \), (10-49) and (10-53) agree with experimental measurements obtained for a sinusoidal PLL.
According to those who handle the experimental side of this problem, (10-54) agrees with laboratory measurements when the noise level is low. We note that (10-20) and (10-53) represent approximate upper and lower bounds on the frequency acquisition time.

Figure 10-10 illustrates plots of (10-53) for various values of $F_0$ with $r = 2$. Based on experimental evidence, it appears that (10-54) is very accurate for $9 \leq B_L < 20$ Hz; however, (10-20) gives more accurate results for $B_L < 1$ Hz. These facts should be kept in mind when attempting to apply (10-20), (10-53) and (10-54).

It is interesting to compare the time required to slip a cycle for various

\[ T_V / B_L \]

\[ g(\varphi) = \sin \varphi \]

\[ C = 0 \text{ at } t = 0 \]

**Fig. 10-10.** Acquisition Time versus $|\Delta f|/B_L$ for Various Values of $F_0$. 

\[ F_0 = 0.01 \]

\[ F_0 = 0.001 \]
PD characteristics. In fact, for the rectangular, sinusoidal, and sawtooth characteristics we have, using (10-42), (10-49) and (10-50) that

\[
\gamma_m \approx \begin{cases} 
2\sqrt{F_0} & \text{if } T_{2\pi} \approx \frac{2\pi}{AK} \sqrt{\frac{1}{F_0}}, \\
2\sqrt{\frac{F_0}{2}} & \text{if } g(\phi) = \sin \phi, \\
2\sqrt{\frac{F_0}{3}} & \text{if } g(\phi) = \frac{\phi}{\pi} \text{ for } |\phi| \leq \pi,
\end{cases}
\]

when \( F_0 \ll 1 \). From this approximation and (10-49), we conclude that the acquisition time \( T_f \) for the rectangular characteristic is smaller than the sinusoidal PLL by \( \sqrt{2} \) and smaller than the sawtooth PLL by \( \sqrt{3} \). It should be noted that the slopes of the PD characteristics at the origin are different. It would be interesting to justify these approximate results; particularly, from the viewpoint of symbol synchronizer design.

**10.2.6 False-Lock in Sinusoidal PLL Receivers**

In practice, the acquisition range can be further degraded from (10-45) by narrowing the IF filter bandwidth beyond a certain limit. Narrowing the IF filter bandwidth introduces additional phase shift in the loop and, together with other system delays, may cause the loop filter response to assume the form

\[
F(s) = \left( \frac{1}{1 + \frac{\tau_2 s}{\tau_1 s}} \right) T(s)
\]

(10-55)

Here the “additional transfer function,” \( T(s) \), is somewhat arbitrary. By investigating only time-periodic solutions in \( \phi \) to the loop equation, the normalized acquisition range can be determined from (see Appendix I)

\[
\gamma(\omega_f) \approx \frac{\omega_f}{AK} + \frac{a_1 AK}{2\tau_1 \omega_f} (\tau_2 \omega_f \cos \epsilon_1 - \sin \epsilon_1)
\]

(10-56)

where \( a_1 \) and \( \epsilon_1 \) represent the amplitude \( a(\omega) \) and phase shift \( \epsilon(\omega) \) of the additional filter

\[
T(i \omega) = a(\omega) \exp [i \epsilon(\omega)]
\]

(10-57)

evaluated at the fundamental beat frequency \( \omega_f \) of a Fourier series expansion of \( \phi(t) \). In arriving at (10-56), all terms except the first two in this expansion have been neglected. Here the normalized acquisition range is obtained by
finding that value of $\gamma(\omega_f)$ for which the loop equation has no solutions of fundamental frequency $\omega_f$. Stated another way, the acquisition range is obtained by finding that value of $\omega_f$ which minimizes (10-56) and, on substituting this value of $\omega_f$ into (10-56), produces the minimum value $\gamma_m(\omega_f)$. If $\Lambda_0/AK < \gamma_m(\omega_f)$, the loop will lock, whereas if $\Lambda_0/AK > \gamma_m(\omega_f)$, frequency pushing will occur and the loop will false-lock to an internally generated sideband. Due to the oscillatory nature of $\cos x$ and $\sin x$, there are both stable and unstable false-lock frequencies much like the stable and unstable phase-lock points defined in (10-7). Frequently solutions to this transcendental equation must be determined graphically or via a digital computer.

To illustrate our result (10-56) further, let us consider the following examples.

**Example 1.** Let $T(s) = 1$ in (10-55). Thus (10-56) reduces to

$$\gamma(\omega_f) \approx \frac{\omega_f}{AK} + \frac{\tau_2 AK}{2\tau_1 \omega_f} \tag{10-58}$$

The general form of (10-58) is shown in Fig. 10-11. For sufficiently large values of $\gamma(\omega_f)$, two equilibrium states exist with corresponding beat frequencies $\omega_1$ and $\omega_2$. The equilibrium state $\omega_1$, corresponding to a positive slope, is stable. The state $\omega_2$, with negative slope, is unstable. From this it can be seen that the beat frequency $\omega_1$ is always smaller than the normalized detuning $\Lambda_0/AK$. Taking the derivative with respect to $\omega_f$ and equating it to zero produces $\omega_f/AK = \sqrt{F_0/2}$. Substitution of this into (10-58) yields the acquisition range $\gamma_m(\omega_f)$ in agreement with (10-43). It is interesting to note that the method of solution in this section yields results that agree with those developed in Section 10-2.4.

![Graph of $\gamma(\omega_f)$](image)

**Fig. 10-11.** Principal Form of the Synchronization Boundary $\gamma(\omega_f)$ for the Imperfect Integrator.
Example 2. Let \( T(s) = 1/(1 + \tau_3 s) \) in (10-55). Then
\[
a(\omega) = \frac{1}{\sqrt{1 + (\omega \tau_3)^2}} = \cos [\epsilon(\omega)]
\]
\[
\sin [\epsilon(\omega)] = \frac{\omega \tau_3}{\sqrt{1 + \omega^2 \tau_3^2}}
\]
Thus (10-56) becomes
\[
\gamma(\omega_f) \approx \frac{\omega_f}{AK} - \frac{AK}{\omega_f} \left[ \frac{1}{1 + (\omega_f \tau_3)^2} \right] \left( \frac{\tau_2 - \tau_3}{2\tau_1} \right) \tag{10-60}
\]
This result can be studied in general, using a digital computer, however, if \( \tau_3 \) is small, the factor \( 1 + (\omega_f \tau_3)^2 \) may be treated as constant in the vicinity of the minimum. To a good approximation, one can easily show that
\[
\gamma_m(\omega_f) \approx \gamma_m \sqrt{\frac{1 - \tau_3/\tau_2}{1 + (AK\gamma_m \tau_3)^2}} \tag{10-61}
\]
where \( \gamma_m \) is the normalized acquisition range when \( \tau_3 = 0 \). From this result we see that for \( \tau_3 > \tau_2 \) and \( \tau_3 \) small, the second term in (10-61) is negative and the synchronized state of the loop is unstable. For \( \tau_2 = \tau_3 \), (10-61) fails to determine the acquisition range. For \( \tau_3 < \tau_2 \) a degradation in the acquisition range is observed.

Example 3. Let us examine the behavior of (10-56) when \( T(i\omega) = \exp(-i\omega \lambda) \). Here \( a(\omega) = 1 \) and \( \epsilon(\omega) = -\omega \lambda \), where \( \lambda \) designates the corresponding delay time. Thus the synchronization boundary defined in (10-56) reduces to
\[
\gamma(\omega_f) \approx \frac{\omega_f}{AK} + \frac{r\lambda W_f}{r + 1} \left[ \cos \frac{\omega_f \lambda}{\omega_f \lambda} + \sin \frac{\omega_f \lambda}{\tau_2 \omega_f^2 \lambda} \right] \tag{10-62}
\]
Notice here the effect of changing the sign of \( \lambda \) in (10-62). In order to determine \( \gamma_m(\omega_f) \) accurately, this transcendental equation must be studied via a digital computer. It is of interest to write (10-62) as a function of the acquisition range \( \gamma_m \) when \( \lambda = 0 \). Using (10-43) in (10-62), we can write
\[
\gamma(\omega_f) \approx \frac{\omega_f}{AK} + \frac{AK}{\omega_f} \left[ \frac{\gamma_m^2}{2} \cos \omega_f \lambda + \frac{\lambda}{2\tau_1} \sin \omega_f \lambda \right] \tag{10-63}
\]
If we assume that the delay \( \lambda \ll \tau_1 \), then the second term in the bracket can be neglected. The general form of (10-63) is illustrated in Fig. 10-12 for the case where \( \lambda \ll \tau_1 \). Curve I corresponds to the case \( \lambda = 0 \) of Fig. 10-11. Curve
II corresponds to the case \( AK\lambda < 1 \), and we see that the delay introduces changes in the asymptotic behavior for \( \omega_f > AK\gamma_m \). For \( AK\lambda \ll 1 \), we see that the acquisition range is only slightly decreased by the delay.

As the value of \( AK\lambda \) is increased (curves III, IV, and V), we see that oscillations with decreasing amplitude are found around the dotted line \( \gamma(\omega_f) = \omega_f/AK \). Physically this implies that as the signal frequency approaches the VCO frequency, we note a periodic change of attraction (frequency pulling) and repulsion (frequency pushing) of the VCO frequency. Thus, for larger values of \( AK\lambda \), the decrease in the acquisition range becomes more pronounced.

Setting \( \frac{d\gamma(\omega_f)}{d\omega_f} = 0 \) gives, in general, several values of \( \omega_f \), say \( \omega_{f_k}, k = 0, 1, \ldots \), which satisfy (10-62) (see Fig. 10-12). Calling the smallest of these values \( \omega_{f_0} \) and denoting the effective frequency offset by \( \Lambda_0 \), then the following conclusion may be drawn: If \( \Lambda_0/AK < \gamma(\omega_{f_0}) \), the loop will lock. If \( \Lambda_0/AK > \gamma(\omega_{f_0}) \), then frequency pushing occurs and the loop goes into a stable limit cycle.
The most stringent requirement placed on \( \lambda \) is that value at which the loop will not lock even if the VCO is at its quiescent frequency and the offset in frequency is zero. This corresponds to curve IV in Fig. 10-12. In this situation, the loop will always become unstable. In order to arrive at this value of delay, we must solve the equation \( \gamma(\omega_{\tau_0}) = 0 \) for \( \lambda \). Substituting the normalized parameters \( b = \omega_f \tau_2 \) and \( \eta = \lambda/\tau_2 \) into (10-63), differentiating with respect to \( b \), and equating the result to zero, gives

\[
\frac{2\tau_2}{\tau_1 r^2} = \frac{(2b_k + \eta b_k^2)}{b_k^2} \sin b_k \eta + \frac{(b_k^2 - \eta b_k^2)}{b_k^2} \cos b_k \eta
\]  

(10-64)

where \( b_k \) corresponds to the normalized value of \( \omega_{\tau_k} \); that is, \( b_k = \tau_2 \omega_{\tau_k} \). Setting \( k = 0 \) in (10-64), we see from (10-62) that

\[
\gamma(b_0) = 0 = \frac{\tau_2 b_0}{r \tau_1} + \frac{r}{2} \left[ \frac{\cos b_0 \eta + \sin b_0 \eta}{b_0} \right]
\]  

(10-65)

Solving (10-64) and (10-65) simultaneously, with \( k = 0 \) in (10-64), gives an equation for the critical value (see curve IV, Fig. 10-12) of normalized delay, \( \eta_c \). Using the fact that the slopes of \( 1/b_0 \) and \( 1/b_0^2 \) are approximately equal in the range of interest then

\[
\tan \eta_c b_0 \approx -\frac{2 - b_0 \eta_c}{2 + b_0 \eta_c}
\]  

(10-66)

whose principal value is \( b_0 \eta_c = 3.394 \). Substituting (10-66) into (10-64) gives

\[
\eta_c \approx \sqrt{\frac{19.34 \tau_2}{\tau_1 r^2}}
\]  

(10-67)

and the absolute instability region is then given by \( \eta > \eta_c \). Consequently, for values of delay such that \( \lambda < \tau_2 \eta_c \), the loop may or may not be stable, depending, of course, on the initial frequency offset, as already discussed. For \( r = 2 \), \( \tau_1/\tau_2 = 200 \), we find that \( \eta_c \approx 0.156 \). We note here that in the data-aided loop discussed in Chapter 3, Section 3-8.1, the problem of loop stability becomes more important due to the presence of a delay in the open-loop transfer function.

As already mentioned, if \( \lambda \ll \tau \), then the second term in the bracket of (10-63) can be neglected. Furthermore, when \( \lambda \ll 1 \), then \( \cos \omega_f \lambda \approx 1 - (\omega_f \lambda)^2/2 \), and the values \( \omega_f \) and \( \gamma_m(\omega_f) \) become
\[ \omega_f = \sqrt{\frac{rW_L}{r + 1} \left( \frac{2AK}{2 - AK\lambda} \right)} \]

\[ \gamma_m(\omega_f) = \sqrt{\frac{2F_0}{2} - \frac{2rW_L}{r + 1}} \] (10-68)

so that the loop will lock if

\[ \frac{|\Lambda_0|}{W_L} < \frac{4r}{r + 1} \sqrt{\frac{1}{2F_0} - \frac{r\lambda W_L}{2F_0(r + 1)}} = \gamma_m(\omega_f) \] (10-69)

and \( \lambda \ll 1 \). We note that the smaller \( \lambda \), the larger is the acquisition range. As \( \lambda \) approaches zero, (10-69) reduces to (10-45) as it should.

False lock or the stable limit cycle can be broken by either tuning the VCO to reduce \( \Lambda_0 \) or by resorting to opening the loop, tuning the VCO to reduce \( \Lambda_0 \) to the point where the signal lies within the synchronization band of the loop, and then closing the loop.

### 10-2.7 Tracking Accelerating Targets with Imperfect Second-Order Loops

Frequently second-order SCVs are required to track accelerating targets, whereby

\[ \theta(t) = d(t) = \theta_0 + \Omega_0 t + \frac{1}{2} \Omega_1 t^2 \] (10-70)

Substitution of (4-19) and (10-70) into (3-7) produces

\[ \frac{dv}{d\phi} = \frac{1}{AK\tau_1} \left[ \frac{[\Omega_0 + \Omega_1(\tau_1 + t) - K_v(e + \tau_1 \dot{e})]}{v} / 2K - g - 1 \right] - \frac{F_0 dg}{d\phi} \] (10-71)

when we normalize time and replace \( \sin \phi \) by \( g(\phi) = g \). From this equation we see the importance of the presence of a manual operator who produces tuning of the VCO such that the differential equation

\[ \tau_1 \dot{\dot{e}} + e = \frac{\Omega_1}{K_v} (\tau_1 + t) \] (10-72)

is satisfied. If \( e \) does not satisfy this equation, not only will the loop not achieve lock, but even if it is initially in lock it will ultimately fall out of lock with the passage of time. When (10-72) is satisfied for all \( t \), the tracking properties are identical with those of a loop tracking a frequency offset of \( \Omega_0 \) rad/ sec [see (10-21)].
10-3 Signal Acquisition Properties of Perfect Second-Order Loops in the Absence of Noise

As we have seen, first- and second-order SCs with an imperfect integrating filter are stable and phase-lock to the incoming signal provided the frequency of the external signal is not too far from the VCO rest frequency. In this section we discuss a loop filter mechanism that ultimately locks to any frequency offset. Substitution of (4-25) into (3-7) with \( \theta(t) = \Omega_0 t + \theta_0 \), and normalizing time as before, produces the loop equation

\[
\tau_1 (AK)^2 \frac{d^2 \phi}{dT^2} = -\tau_2 (AK)^2 v \frac{dg}{d\phi} = AK g(\phi) \tag{10-73}
\]

where \( v(\phi) = d\phi/dT \) and \( \dot{e} = 0 \). Dividing both sides of (10-73) by \( v(\phi) \) and re-arranging gives

\[
\frac{dv}{d\phi} = -\frac{g(\phi)}{AK\tau_1 v} - F_0 \frac{dg}{d\phi} \tag{10-74}
\]

For large \( AK\tau_1 \) or \( v(\phi) \) we see that the phase trajectories are nearly equal to \( g(\phi) \). The singular points occur when \( g(\phi) = 0 \) and \( v(\phi) = 0 \). When \( v(\phi) = 0 \) and \( g(\phi) = \sin \phi \), we note the singularities occur at

\[
\phi_{ik} = \begin{cases} 
2n\pi & \text{stable points} \\
(2n + 1)\pi & \text{unstable points}
\end{cases} \tag{10-75}
\]

where \( n \) is any integer. It can be shown that the acquisition range is infinite (see Prob. 10-6). Viterbi (Ref. 2) illustrates various phase-plane plots for the case \( g(\phi) = \sin \phi \). Since they behave, for all \( \Lambda_0 \), much the same way as those illustrated in Fig. 10-3, we do not reproduce the results here.

10-3.1 Tracking Accelerating Targets with Perfect Second-Order Loops

If the loop is required to track an accelerating target that produces phase modulation of the transmitter given by (10-70), the slope of the phase trajectories becomes

\[
\frac{dv}{d\phi} = \frac{1}{AK\tau_1} \left[ \frac{\tau_1 \Omega_1 - K_v \tau_i \dot{e}}{v} \right] / AK - g - F_0 \frac{dg}{d\phi} \tag{10-76}
\]

When \( \dot{e} = 0 \) and \( AK\tau_1 \gg 1 \), we note that this equation is of the same form as that given in (10-21) for the case of tracking a frequency offset \( \Omega_0 \) rad/sec.
Thus the tracking properties are identical with those of a second-order loop with an imperfect integrator provided that we replace $\Omega_0$ by $\tau_i \Omega_i$ in (10-21) and assume that $AK\tau_i \gg 1$. Phase-plane plots of third-order loops in the absence of noise is given in Viterbi (Ref. 10-2, pp. 64-72.). Viterbi also shows that if $F(s) = 1 + a/s$, $g(\phi) = \sin \phi$, then the maximum rate at which the VCO can be swept to acquire with certainty is $\Omega_i = a(2W_L - a)/2$. The approximate acquisition range, in terms of the loop natural frequency $\omega_n$ and loop damping $\zeta_i$, is $\Omega_0 < \zeta \omega_n^2/\Omega_i$.

10-4 Signal Acquisition Aids and Techniques

As discussed earlier, one of the major problems associated with the operation of a SCS is that of effecting rapid signal acquisition. We have also observed that the first- and second-order loops are stable and lock onto the incoming carrier provided that loop detuning is not too large; however, the acquisition time can be prohibitive, particularly if the loop signal-to-noise ratio is near threshold or if $\Omega_0$ is too large.

On the other hand, if the frequency difference $\Omega_0$ were known a priori, the loop could be pretuned so as to minimize the signal acquisition time. Since this is not usually the case in practice, some form of signal acquisition device is normally required to rapidly aid a SCS to the tracking mode. When the loop signal-to-noise ratio is not too small, the following methods are currently used to acquire a frequency offset:

1. VCO tuning, in the form of a sweep voltage, can be supplied by an operator.
2. VCO tuning, based on an estimate of $\Omega_0$, can be derived from the signal itself.
3. VCO tuning can be programmed according to a known Doppler ephemeris.
4. Sweep the transmitted center frequency through the frequency uncertainty interval in the receiver.
5. Offsetting the VCO frequency in the direction of the Doppler shift and allowing the signal to pass through the loop bandwidth.

A SCS that uses an auxiliary frequency difference detector (FDD) to improve signal acquisition performance is illustrated in Fig. 10-13. The acquisition aid output $e(t)$ provides a suitable control voltage that affects acquisition performance. When $\Lambda_0 = \Omega_0 - K_v e(t)$ is within the flat portion of the FDD characteristic, it turns itself off automatically. The width of this flat portion is determined by the frequency difference within which the SCS can produce
effectively instantaneous pull-in. The composite system functions as a two-mode system; that is, when phase locked, the system "operates" as a PLL; and when out of lock, it operates as an automatic frequency control system. Richman (Ref. 4) has suggested a quadricorrelator as one possible implementation of a device that operates as a frequency difference detector. The sketches in Fig. 10-14 present signal acquisition characteristics that indicate the improvement obtained with one form of acquisition aid. The sketches show the acquisition time to frequency lock as a function of the interval over which the oscillator is pulled. The narrow curve in the center represents a PLL system, while the wide curve illustrates loop performance when the loop is aided to an earlier synchronized condition.
10-4.1 Signal Acquisition by the VCO Sweep Method, Maximum Sweep Rate

Another method for improving the performance of the signal acquisition mode (particularly for tracking accelerating targets) is to tune the VCO by applying, either manually or via a computer, a sweep voltage \( e(t) \) and effecting a search for the input frequency. If done properly, the loop will lock up as the VCO sweeps into the input frequency. Frequency sweep can be obtained by applying a ramp of voltage to the VCO input from an independent sweep generator. On the other hand, if the loop is second-order, a voltage ramp can be generated by applying a step function at the loop filter input. The slope (sweep rate) of the ramp is controlled by adjusting the magnitude of the step of voltage that is applied to the loop filter input.

When a step function is applied to the input of an imperfect integrating loop filter, approximately \( F_r \) times the amplitude of the step appears directly at the output of \( f(s) \). The result is a corresponding jump in the VCO frequency just as the sweep begins. The particular application must be able to tolerate the jump. If it cannot, an auxiliary sweep generator without such a jump must be used. Ordinarily the direction of the sweep is periodically reversed at some predetermined limit. In many applications it is necessary for the sweep voltage to be removed after phase-lock is achieved in order that the VCO or loop filter amplifier not be saturated. This step can be avoided by not sweeping outside the linear range of the loop components.

Frazier and Page (Ref. 5) have obtained an empirical equation that predicts the maximum sweep rate, say \((\Delta \omega)_m\), that will provide an “acquisition probability” of 0.9 in the presence of noise. This equation is given by

\[
(\Delta \omega)_m = \begin{cases} \frac{\omega_n^2}{\mu} & \text{no limiting} \\ \frac{\omega_e}{\mu} \left[ 1 - \frac{1}{\sqrt{\alpha_i}} \right] & \text{IF limiting} \end{cases}
\]  

(10-77)

where \( \rho \) and \( \omega_n \) are defined in (4-15) and (4-24), respectively, and \( \mu \) and \( \alpha_i \) are defined in (4-80) and (4-83). The parameter \( d \) is approximated by

\[
d = \begin{cases} \exp \left[ -\frac{\pi}{2} \sqrt{\frac{r_0}{\mu} \left( 4 - \frac{r_0}{2\mu} \right)} \right] & \text{no limiting} \\ \exp \left[ -\frac{\pi}{2} \sqrt{\frac{r_0}{\mu} \left( 4 - \frac{r_0}{2\mu} \right)} \right] & \text{IF limiting} \end{cases}
\]  

(10-78)

if \( r < 4 \); if \( r > 4 \), then \( d \approx 0 \). For large \( \rho \) or \( \alpha_i \) we note that \((\Delta \omega)_m \approx \omega_n^2 \) or \( \omega_e^2/\mu \). In order to provide a safety margin, it is advisable to sweep at a lower rate, for example, \((\Delta \omega)_m \leq 0.386 \omega_n^2 \) insures 100% lock in no noise.
10-4.2 Signal Acquisition Aids that Use Frequency Difference Measurements

A particular frequency difference measurement technique that supplements the performance of a PLL system during signal acquisition is suggested by the instrumentation illustrated in Fig. 10-15. Its elements consist of a pair of phase detectors (one that belongs to the PLL) that are fed with reference signals derived from the VCO output.

![Diagram](image)

**Fig. 10-15. PLL-Signal Acquisition Aid Mechanization.**

The PD (multiplier) outputs are bandpass filtered to remove any extraneous signals near dc. The output of the lower phase detector is passed through a differentiating circuit. The two signals are then heterodyned in another phase detector and the output is used as the input to a low-pass filter. This filter exchanges brevity of integration time for reliability of frequency measurement. The resulting output \( e(t) \) serves as an estimate of the frequency difference and can be used to rapidly move the VCO rest frequency to that of the incoming signal. The loop gain is made high enough to hold static phase errors to a preselected small limit.

Operation of the acquisition aid can be explained as follows: Assume that a frequency difference \( \Delta f \) Hz exists between the signal component and the VCO frequency and that the input noise consists of the sum of two uncorrelated noise components. The output of the lower phase detector consists of a cosine beat note and the noise along one reference axis. The output of the upper phase detector consists of a sine beat note and the noise along an axis in
quadrature with the reference axis. The noise voltages appearing at the phase-detecter output are independent of each other.

The cosine beat note is converted by differentiation to a sine beat note having an amplitude proportional to its frequency difference $\Delta f'$; the differentiated noise voltages are still independent of each other. The output of the cross-multiplying phase detector will contain a dc term proportional to and polarized according to the frequency difference $\Delta f'$ plus random noise. Thus the voltage $\epsilon(t)$ serves to provide an early estimate of $\Delta f'$, both in sign and magnitude. This two-mode action permits the loop bandwidth to be made narrow enough to give the desired tracking performance without affecting acquisition performance. It would undoubtedly have poor performance at low signal-to-noise ratios.

### 10-4.3 Computer-Aided Acquisition for Low Signal-to-Noise Ratios

In order to acquire a signal near threshold, it may be necessary to use a digital computer to determine the frequency difference $\Omega_0$. Once the computer has determined $\Omega_0$, it can be programmed into the VCO and acquisition can be accomplished. There are two approaches, coherent and noncoherent.

In the coherent case, the FDD in Fig. 10-13 operates on the phasedetector output by sampling it at the Nyquist rate. This rate is determined by the predetection bandwidth of the receiver. The sample points are fed into an analog-to-digital converter that feeds a digital computer. When all the data, of duration $T$, have been stored in the computer, a fast Fourier transform (FFT) is performed and the most likely value for $\Omega_0$ is outputed. Since the FFT preserves phase, a phase estimate of the signal is also provided as a by-product of the procedure and it can be used to feed the VCO along with the estimate of $\Omega_0$. The computer program mode is then switched from the acquisition mode to the tracking mode and the loop acquires the signal.

Another approach is to use the FFT to estimate $\Omega_0$ by noncoherent methods. Assuming that the frequency of the signal appearing at the output of the loop's phase detector belongs to an uncertainty band of $W$ Hz and that one observes the phase detector output for an acquisition time of $T$ seconds, one can employ maximum likelihood frequency estimate techniques to obtain an estimate of $\Omega_0$. This estimator consists of a continuum of devices (see Fig. 10-16), each of which consists of a “correlation detector” of the signal envelope. The output functions $M^2(\omega)$ is related to the correlation function of $\epsilon(t)$ through

$$M^2(\omega) = 4 \int_0^T R_\epsilon(\tau) \cos \omega \tau \, d\tau$$  \hspace{1cm} (10-79)

where
so that \( M^2(\omega) \) is proportional to the transform of the finite-time autocorrelation function, \( R_\varepsilon(\tau) \), of \( \{\varepsilon(t)\} \). Although neither \( R_\varepsilon(\tau) \) nor its correlation function can be generated in real time, the fast Fourier transform can again be used to process the signal in the digital domain.

Since the optimum estimator requires an uncountable number of the elements illustrated in Fig. 10-16, it is clearly physically unrealizable; however, it can be shown that a reasonable approximation to this estimator can be achieved by using no more than \( 2WT + 1 \) elements, each tuned to frequencies spaced \( \pi/T \) rad/sec. For \( T \) seconds of acquisition time the acquisition probability is bounded by \( P_{\text{acq}} < WT \exp(-4A^2T/\pi^2N_0) \). The element that gives rise to the large output is then chosen as the one most closely matched to the frequency difference. The VCO is then shifted in frequency by this amount.

### 10-4.4 Signal Acquisition Aid by Means of Nonlinear Loop Filter Design

Another approach to improving the performance of a PLL during the signal acquisition mode is to employ a loop filter consisting of a nonlinear resistance \( R \) and a nonlinear capacitance \( C \). This fact is suggested by the contradiction between the requirements of a wide acquisition range and narrow loop bandwidth. Since the normalized acquisition range, \( \gamma_m < \sqrt{2\tau_2/\tau_1} \), for a SCS is dependent on \( \tau_1 \) and \( \tau_2 \), it appears that it may be possible to incorporate nonlinear elements into the loop filter such that, during acquisition, \( \tau_1 \) is effectively smaller, while during the tracking mode, the value of \( \tau_1 \) automatically increases itself through the incorporation of the nonlinear loop filter design. The \( RC \) long-term integration component of the loop filter illustrated in Fig. 10-2 is used to effect the nonlinear design. This filter is designed to operate
directly on the voltage, }y(t)\text{, representing the frequency acquisition mechanism. The details of this approach are beyond the scope of the present discussion; however, it would be interesting to carry out further analysis. Figure 10-17 illustrates the voltage-current } (v-i) \text{ relationship of the nonlinear resistance and the dependence of the capacitance on the voltage } (C-v) .

![Figure 10-17. Voltage-Current and Capacitance-Voltage Characteristics.](image)

10-5 Nonlinear Theory of Second-Order SCSs in Noise

In this section we begin our study of the nonlinear behavior of the second-order SCS in noise. In particular, we first present the p.d.f. of the phase error process and discuss some of its properties. Although the main results hold for an arbitrary periodic nonlinearity } g(\psi) \text{, we shall study in greater detail the case of a sinusoidal PLL that is required to track a frequency offset of } \Omega_0 \text{ rad/sec. Since our results are special cases of the more general results to be derived in the next chapter, we shall not present the detailed developments here. We begin by substituting (4-19) into (3-24) with } \theta(t) = \Omega_0 t + \theta_0 \text{ and } \psi_2 = 0 \text{, which produces the equivalent system of two first-order, stochastic differential equations.}

\[ \dot{\psi} = \Lambda_0 - F_o K [A g(\psi) + N(t, \psi)] + y_1 \]
\[ \dot{y}_1 = \frac{\psi_1}{\tau_1} - \left( 1 - F_o \right) K [A g(\psi) + N(t, \psi)] \tau_1 \]

(10-81)

This essentially represents a rearrangement of (10-1) and (10-2) with } \varphi \text{ replaced by the modulo-2\pi version, } \psi, \text{ discussed in Chapter 9, Section 9-3.}^* \text{ If we make}

*The modulo-2\pi process } [\phi(t)] \text{ is explicitly defined by } [\phi(t)] = [\phi(t) + \pi] \text{ mod } 2\pi - (2n - 1)\pi \text{ where } n \text{ is any fixed integer. Notice that if } |\phi(t)| \leq \infty \text{ then } \phi(t) \text{ belongs to the interval } I(n) = [(2n - 1)\pi, (2n + 1)\pi].
the assumption that the correlation time of the phase error process \(\tau_\phi\) is much larger than the correlation time of the noise process \(\tau_N\), then the diffusion approximation applies and we can make use of the Fokker-Planck equation given in Chapters 7 and 8 to obtain the p.d.f. of the phase error. Using (8-40) and (10-81), it is easy to show that the intensity coefficients due to the signal are given by

\[
K_0(\phi, y_1, t) = \Lambda_0 - AKF_0 g(\phi) + y_1,
K_1(\phi, y_1, t) = -y_1 - \frac{(1 - F_0)AKg(\phi)}{\tau_1} \tag{10-82}
\]

while those producing diffusion are given by, using (8-41) and (10-81),

\[
K_{10}(\phi, y_1, t) = K_{01}(\phi, y_1, t) = \frac{F_0(1 - F_0)N_0K^2}{2\tau_1},
K_{00}(\phi, y_1, t) = \frac{F_0^2K^2N_0}{2},
K_{11}(\phi, y_1, t) = \frac{(1 - F_0)^2K^2N_0}{2\tau_1} \tag{10-83}
\]

Using (10-82), we can produce the probability current densities, (7-184),

\[
\mathcal{J}_0(\phi, y_1; t) = \left[\Lambda_0 - AKF_0 g(\phi) + y_1 - \frac{1}{2} \left( K_{00} \frac{\partial}{\partial \phi} + K_{01} \frac{\partial}{\partial y_1} \right) \right] p \\
\mathcal{J}_1(\phi, y_1; t) = \left( \left( -y_1 - \frac{(1 - F_0)AKg(\phi)}{\tau_1} \right) - \frac{1}{2} \left( K_{10} \frac{\partial}{\partial \phi} + K_{11} \frac{\partial}{\partial y_1} \right) \right) p \tag{10-84}
\]

where \( p(\phi, y_1, t|\phi_0, y_{10}, t_0, n) = p(\phi, y_1; t) \) is the conditional transition p.d.f. discussed in Chapter 9. Thus the Fokker-Planck equation (7-185) reduces to

\[
\frac{\partial \mathcal{J}_0(\phi, y_1; t)}{\partial \phi} + \frac{\partial \mathcal{J}_1(\phi, y_1; t)}{\partial y_1} + \frac{\partial p}{\partial t} = 0 \tag{10-85}
\]

For \( F_0 = 0 \) we have the probability current densities for the case of simple RC filter. If we integrate (10-85) on \( y_1 \) from minus infinity to infinity, we will show in Chapter 11 that (10-85) can be written as

\[
\frac{\partial \mathcal{J}_0(\phi; t)}{\partial \phi} + \frac{\partial p(\phi; t)}{\partial t} = 0 \tag{10-86}
\]

with

\[
\mathcal{J}_0(\phi; t) = -\frac{K_{00}}{2} \exp \left[ -U_0(\phi; t) \right] \frac{\partial}{\partial \phi} \left[ p(\phi; t) \exp \left[ U_0(\phi; t) \right] \right] \tag{10-87}
\]
In (10-87) we have introduced the nonlinear restoring force

\[ h_0(\phi; t) = \frac{2K_0(\phi, t)}{K_{00}} = \frac{4[\Lambda_0 - AKF_0 g(\phi) + E(y_1, l(\phi, \phi_0, y_{10})]}{N_0 K^2 F_0^2} \]  

(10-88)

and the potential function

\[ U_0(\phi; t) \ dx = -\int^\phi h_0(x; t) \ dx \]  

(10-89)

where \( K_{00} = K_{00}(\phi, t) \). Assuming in the limit as \( t \) approaches infinity that \( p(\phi; t) \) approaches the steady-state conditional p.d.f. \( p(\phi|n) \), the stationary diffusion current is constant and obeys the law

\[ \mathcal{J}_0(\phi) = -\frac{K_{00}}{2} \exp \left[ -U_0(\phi) \right] \frac{\partial}{\partial \phi} \left[ p(\phi|n) \exp \left[ U_0(\phi) \right] \right] \]  

(10-90)

since \( U_0(\phi; t) \) is time dependent. The result of Chapter 7 [i.e., (7-83)], can be used to solve for \( p(\phi|n) \). Thus, from (7-83) we can write

\[ p(\phi|n) = C_0 \exp \left[ -U_0(\phi) \right] \left\{ 1 + D_0 \int_{(2n-1)\pi}^{\phi} \exp \left[ U_0(x) \right] \ dx \right\} \]  

(10-91)

where \( \phi \in I(n) \triangleq [(2n - 1)\pi, (2n + 1)\pi] \) and \( D_0 = -2 \mathcal{J}_0/C_0 K_{00} \). To evaluate the constant \( D_0 \), we can set \( n = 0 \), without loss in generality, and use the fact that \( p(\pi) = p(-\pi) \). Thus

\[ D_0 = \frac{\exp \left[ -U_0(-\pi) \right] - \exp \left[ -U_0(\pi) \right]}{\exp \left[ -U_0(\pi) \right] \int_{-\pi}^{\pi} \exp \left[ U_0(x) \right] \ dx} \]  

(10-92)

Since \( E(y_1|\phi) \) is periodic in \( \phi \) and since \( g(\phi) \) is continuous, the results in Appendix I of Chapter 9 apply. Thus

\[ p(\phi) = p(\phi|n) = C'_0 \exp \left[ -U_0(\phi) \right] \int_{\phi}^{\phi + 2\pi} \exp \left[ U_0(x) \right] \ dx \]  

(10-93)

for \( \phi \in I(n) \), \( n \) any fixed integer. This result represents a canonical form for the class of second-order loops for which \( p(\phi) \) is periodic. In (10-93) \( C'_0 \) is the normalization constant. This result is exact, and we see that a complete knowledge of \( p(\phi) \) is embedded in a knowledge of the conditional expectation \( E(y_1|\phi) \) which cannot be obtained analytically.

In order to proceed, it appears that the conditional expectation \( E(y_1|\phi) \) must be approximated. In Appendix II we show, to a good approximation, that
\[
\hat{E}(y_1|\phi) = \frac{\tau_0 AK (1 - F_0) G(\phi)}{2\tau_1} - AK (1 - F_0) \hat{g}(1 + \tau_0/2\tau_1) \quad (10-94)
\]

where \( \tau_0 = S_0(0)/\sigma_0^2 \) is the correlation time (see Chapter 1) of the process \( G = g - \hat{g} \). At this point in the development of a working theory, it appears that the goodness of the assumptions that lead one to (10-94) must be justified by direct measurement of \( E(y_1|\phi) \). The measurement of \( E(y_1|\phi) \) can readily be adapted for simulation on a digital computer. The fact that \( E(y_1|\phi) \) is approximately sinusoidal in the steady state for a sinusoidal PLL with \( \Lambda_0 = 0 \) has been verified by computer-simulation techniques. Typical results from the simulation for a low signal-to-noise ratio case are shown in Fig. 10-18, along with a plot of \( \hat{E}(y_1|\phi) \) to accentuate the agreement.

![Fig. 10-18. Comparison of Approximate Conditional Expectation with Computer Simulation Results.](image)

Using (10-88), (10-89), and (10-94), the steady-state restoring force can be written as

\[
\hat{h}(\phi) = \beta(1) - \alpha(1)g(\phi)
\]

\[
\hat{U}_o(\phi) = -\int^\phi \hat{h}_o(x) \, dx
\]

\[(10-95)\]
where \( \sim \) signifies the use of the estimate in (10-94) and

\[
\beta(1) \triangleq \frac{2}{K_{00}} \left[ \Lambda_0 - AKg(1 - F_0)(1 + \tau_g/2\tau_i) \right] = \frac{2\tilde{\phi}}{K_{00}} + \alpha(1)g
\]

\[
\alpha(1) \triangleq \frac{2}{K_{00}} \left[ AKF_0 - \frac{AK\tau_g}{2\tau_i}(1 - F_0) \right]
\]

(10-96)

The steady-state p.d.f. given in (10-93) for a second-order loop is therefore approximated by

\[
p(\phi) \approx C'_0 \exp \left[ \beta(1)\phi - \alpha(1) \int_{\phi}^{\phi+2\pi} g(x) \, dx \right] \int_{\phi}^{\phi+2\pi} \exp \left[ -\beta(1)x + \int_{x}^{\pi} g(y) \, dy \right] \, dx
\]

(10-97)

for \( \phi \in I(n) \). Here \( C'_0 \) is the normalization constant. If the loop is operating such that \( \beta(1) \approx 0 \), then

\[
p(\phi) \approx C'_0 \exp \left[ -\alpha(1) \int_{\phi}^{\phi+2\pi} g(x) \, dx \right]
\]

(10-98)

These rather general results can be used to obtain the phase error p.d.f. for various second-order SCSs—for example, the delay-locked loop, hybrid loops, data-aided loops, various symbol synchronizer mechanizations, and the double-loop tracking device discussed in Chapter 3, Table 3-1. Next we consider the p.d.f. \( p(\phi) \) for the sinusoidal PLL.

### 10-5.1 Steady-State Probability Density of the Phase Error for the Sinusoidal PLL

For the second-order loop, \( h_0(\phi) \) in (10-88) can be written as

\[
h_0(\phi) = -\left(\frac{r + 1}{r}\right) \rho \sin \phi + \left(\frac{r + 1}{r}\right)^2 \left[ \frac{\rho \Lambda_0}{2W_L} + \frac{\rho E(y_1|\phi)}{2W_L} \right]
\]

(10-99)

where we have used the definitions for \( r, \rho, \) and \( W_L \) given in Chapter 4, Section 4. Using the estimate \( \tilde{E}(y_1|\phi) \) in (10-94) to approximate \( E(y_1|\phi) \), the potential function can be found from (10-95). Thus

\[
\hat{U}_0(\phi) = -\beta \phi - \alpha \cos \phi
\]

(10-100)

where, for convenience, we have set
\[ \beta = \beta(1) = \left( \frac{r + 1}{r} \right) \frac{\rho}{F_0} \left[ \frac{\Lambda_0}{AK} - (1 - F_0) \sin \bar{\phi} \right] \left[ 1 + \frac{F_0}{\rho(r + 1)\sigma_0^2} \right] \]

\[ \alpha = \alpha(1) = \left( \frac{r + 1}{r} \right) \rho - \frac{1}{r\sigma_0^2} \]  

(10-101)

and we have made use of the linear PLL theory set \( S_0(0) = N_0/2\bar{A}^2 \). Substitution of (10-100) into (10-93) leads to

\[ p(\phi) \approx \frac{\exp[\beta \phi + \alpha \cos \phi]}{4\pi^2 \exp(-\pi\beta)|I_{\beta}(\alpha)|^2} \int_{\phi}^{\phi + 2\pi} \exp[-\beta x - \alpha \cos x] \, dx \]  

(10-102)

since \( C_0 \) is equivalent to that of a first-order PLL with parameters \( \beta \) and \( \alpha \). In fact, as \( r \) approaches infinity and \( F_0 \) approaches one in (10-102), we obtain the p.d.f. of the phase error in a first-order loop. Consequently, the mean \( \bar{\phi} \), the variance \( \sigma_0^2 \), and the circular moments \( \sin n\bar{\phi} \) and \( \cos n\bar{\phi} \), when considered as functions of \( \alpha \) and \( \beta \), may be obtained using the formulas and graphs given in Chapter 9. When these moments are considered as functions of \( \rho \) and \( \Lambda_0/\bar{A}K \), one must resort to a particular loop design. Figure 10-19 represents a plot of \( \rho \) vs \( |\gamma| \) with \( \beta/\alpha \) and \( \alpha \) as parameters. Through the use of such a grid and the graphical results given in Chapter 9 one can determine the performance characteristics for a sinusoidal PLL. Thus for a given \( \rho \) and \( \gamma \) one finds that point in the \( (\rho, \gamma) \) plane corresponding to \( (\beta/\alpha, \alpha) \) in the loop. If a limiter precedes the loop, one can model the resulting phase error p.d.f. by replacing \( \rho \) by \( \alpha \), in (4-83) and \( r \) by \( r_0/\mu \) of (4-78) and (4-80).

It is of interest to produce a canonical expansion for \( p(\phi) \) other than that given in Chapter 9. Now (10-102) can be written as

\[ p(\phi) \approx \frac{K(\alpha, \beta)}{2 \exp(-\pi\beta) \sinh \pi\beta} \left[ F_-(\phi) \exp (\beta \phi + \alpha \cos \phi) \right] \]  

(10-103)

where

\[ K(\alpha, \beta) \triangleq \frac{\sinh \pi\beta}{2\pi^2 |I_{\beta}(\alpha)|^2} \quad F_-(\phi) \triangleq \int_{\phi}^{\phi + 2\pi} \exp(-\beta x - \alpha \cos x) \, dx \]  

(10-104)

Making use of the Fourier series expansion for \( \exp(-\alpha \cos \phi) \) and doing a term-by-term integration gives (see Appendix IV, Chapter 9)

\[ F_-(\phi) = 2 \sinh(\pi\beta) \exp(-\beta\phi - \pi\beta) \left[ \frac{I_0(\alpha)}{\beta} + 2\beta P_1(\phi) - 2P_2(\phi) \right] \]  

(10-105)
Fig. 10-19. Relating the Parameters $\alpha$ and $\beta/\alpha$ to $\rho$ and $\gamma$ in the Loop.
where

\[ P_1(\phi) \triangleq \sum_{n=1}^{\infty} (-1)^n \frac{I_n(\alpha) \cos n\phi}{n^2 + \beta^2} \]

\[ P_2(\phi) \triangleq \sum_{n=1}^{\infty} (-1)^n \frac{I_n(\alpha) n \sin n\phi}{n^2 + \beta^2} \]  \hspace{1cm} (10-106)

Using (10-104) and (10-105) in (10-103), we have

\[ p(\phi) \approx K(\alpha, \beta) \exp(\alpha \cos \phi) \left[ \frac{I_0(\alpha)}{\beta} + 2\beta P_1(\phi) - 2P_2(\phi) \right] \]  \hspace{1cm} (10-107)

for \( \phi \in I(n) \).

Now the absolute value of \( I_{j\beta}(\alpha) \) can be expressed in terms of the wedge functions defined in Chapter 9; that is,

\[ |I_{j\beta}(\alpha)|^2 = \frac{\sinh^2 \pi \beta}{\pi^2} \left[ |F_{\beta}(\alpha)|^2 + |G_{\beta}(\alpha)|^2 \right] = \int_0^{\frac{\pi}{\alpha}} \cosh (2\beta x) I_0(2\alpha \cos x) \, dx \]  \hspace{1cm} (10-108)

and the wedge functions have the asymptotic expansions (see Appendix II of Chapter 9)

\[ \lim_{\alpha \to \infty} F_{\beta}(\alpha) = \frac{\exp(\alpha)}{\sinh \frac{\pi \beta}{2\alpha}} \sqrt{\frac{\pi}{2\alpha}} \quad \lim_{\alpha \to \beta} G_{\beta}(\alpha) = \exp(-\alpha) \sqrt{\frac{\pi}{2\alpha}} \]  \hspace{1cm} (10-109)

so that

\[ \lim_{\alpha \to \infty} |I_{j\beta}(\alpha)|^2 = \frac{\pi}{2\alpha} \left[ \frac{\exp(2\alpha)}{\sinh^2 \frac{\pi \beta}{2\alpha}} + \exp(-2\alpha) \right] \frac{\sinh^2 \pi \beta}{\pi^2} \]  \hspace{1cm} (10-110)

If we neglect the second term in brackets, then

\[ \lim_{\alpha \to \infty} |I_{j\beta}(\alpha)|^2 = \frac{\exp(2\alpha)}{2\pi \alpha} \]  \hspace{1cm} (10-111)

and we notice that this is the asymptotic expansion for \( I_0(\alpha) \). Thus

\[ \lim_{\alpha \to \infty} |I_{j\beta}(\alpha)|^2 \approx I_0(\alpha) \quad \lim_{\alpha \to \infty} K(\alpha, \beta) \approx \frac{\sinh \pi \beta}{2\pi^2 I_0^2(\alpha)} \]  \hspace{1cm} (10-112)
and since \( I_n(x) \approx I_0(x) \) for large \( x \), the series expressions for \( P_1(\phi) \) and \( P_2(\phi) \) reduce to

\[
\lim_{\alpha \to \infty} P_1(\phi) = I_0(\alpha) \left( \frac{\pi \cosh \beta \phi}{2\beta \sinh \beta \pi} - \frac{1}{2\beta^2} \right)
\]
\[
\lim_{\alpha \to \infty} P_2(\phi) = -I_0(\alpha) \left( \frac{\pi \sinh \beta \phi}{2 \sinh \pi \beta} \right)
\] (10-113)

Substitution of (10-112) and (10-113) into (10-107) gives

\[
p(\phi) \approx \exp \left[ \frac{\beta \phi + \alpha \cos \phi}{2\pi I_0(\alpha)} \right] \quad \alpha \gg 1
\] (10-114)

with \(|\phi| \leq \pi\) when we assume that \( n = 0 \).

For small \( \phi \), we can replace \( \phi \) by \( \sin \phi \) and write

\[
p(\phi) \approx \exp \left[ \frac{\alpha \cos \left( \phi - \tan^{-1}(\beta/\alpha) \right)}{2\pi I_0(\alpha)} \right] \approx \exp \left[ \frac{\alpha \cos (\phi - \beta/\alpha)}{2\pi I_0(\alpha)} \right]
\] (10-115)

for large \( \alpha \). Expanding the cosine in a Taylor series shows that \( p(\phi) \) is Gaussian with mean \( \beta/\alpha \) and variance \( 1/\alpha \) when \( \alpha \gg 1 \). At the other extreme, \( \alpha \ll 1 \), \( p(\phi) \) becomes uniformly distributed.

If the loop is operating such that \( \beta \approx 0 \) then

\[
p(\phi) \approx \frac{\exp \left[ \frac{\alpha \cos \phi}{2\pi I_0[\alpha]} \right]}{\alpha}, \quad |\phi| \leq \pi
\] (10-116)

when \( F_0 \ll 1 \). The variance of \( \phi \) can be obtained by first solving for \( \sin^2 \phi \) from

\[
\overline{\sin^2 \phi} = \int_{-\pi}^{\pi} \sin^2 \phi p(\phi) \, d\phi = \frac{1}{\alpha} \left[ \frac{I_1(\alpha)}{I_0(\alpha)} \right]
\] (10-117)

and the corresponding variance can be obtained from

\[
\sigma^2_\phi = \int_{-\pi}^{\pi} \phi^2 p(\phi) \, d\phi
\] (10-118)

by numerical integration on a digital computer. Figure 10-20 illustrates the results for various values of \( r \). The circles in Fig. 10-20 correspond to points obtained in the laboratory (Ref. 6) by means of a hardware mechanization with \( r = 2 \).
It is also instructive to approximate the conditional expectation by using linear PLL tracking theory. If we replace \( g(\phi) \) by \( \phi \) in (10-81), then \( \phi \) and \( y_1 \) are jointly Gaussian r.v.'s and the conditional expectation can be approximated by using elementary results that pertain to such r.v.'s. Thus

\[
\hat{E}(y_1|\phi) = (1 - F_0) \left[ \frac{2rW_k}{(1 + r)^2} \right] \phi - \Lambda_0 \left( 1 + \frac{F_0}{1 + r} \right)
\]  

(10-119)

If \( F_0 \ll 1 \), we can write

\[
\hat{h}_0(\phi) = -\left( \frac{r + 1}{r} \right) \rho g(\phi) + \frac{\rho}{r} \phi + \frac{\Lambda_0 \rho}{AK}
\]  

(10-120)

so that

\[
p(\phi) \approx C_0 \exp \left[ -\left( \frac{r + 1}{r} \right) \rho \int_0^\phi g(x) \, dx + \frac{\rho}{2r} \phi^2 + \frac{\Lambda_0 \rho}{AK} \phi \right]
\]  

(10-121)

which is independent of \( \Lambda_0 \) for large \( AK \). If we replace \( g(x) \) by \( x \), then (10-121) is Gaussian and it is seen that the loop will always pull in lock so
that $p(\phi)$ is symmetric about $\Lambda_0/AK$. For reasonable damping factors and $\Lambda_0$, a small $F_0$ produces very little mean-phase error. This, of course, is why second-order loops are used in practice. Contrast this with the solution for a first-order loop with $\Lambda_0 \neq 0$ where it is noted that the ratio of (linear) mean-phase error for a second-order loop to a first order is $\ddot{\phi}_2/\ddot{\phi}_1 = F_0(r + 1)/r \ll 1$ for equal loop bandwidths.

Finally, we note from (10-81) that in the steady state

$$\ddot{y}_1 = -(1 - F_0)AK\ddot{g}(\phi)$$

(10-122)

By integrating (10-90) over the interval $[(2n - 1)\pi, (2n + 1)\pi]$ using (10-81), (10-88), (10-89), and (10-122) the residual frequency detuning becomes

$$\ddot{\phi} = \int_{(2n-1)\pi}^{(2n+1)\pi} K_0(\phi)p(\phi) \, d\phi = 2\pi J_0$$

$$= 2\pi [N_+ - N_-] = AK(\gamma - \ddot{g})$$

(10-123)

where $N_+(N_-)$ represents the average number of cycles per second slipped to the right (left). Note $\ddot{\phi} = 0$ when $\ddot{g} = \gamma$. As was done in the case of a first-order loop (Section 9-3.5), (10-123) will be used later to determine the average cycle slipping rate in the presence of noise. Evaluating the variance of the phase error rate is left as an exercise for the reader.

### 10-5.2 Moments of the Mean Time to First Loss of Phase Synchronization

Since SCS systems find wide application in many different branches of engineering, it is difficult to point to one performance criterion of such a system that is satisfactory for all possible applications. Theoretically speaking, cycle slipping in second-order SCSs is not completely understood. Although the average number of slips per unit time is a more meaningful parameter in practice, communication engineers for some time have been interested in the first-slip time. No doubt the reason is that the development of first-slip theory preceded the development of the theory that accounts for the average number of slips per unit time. Since the problem of evaluating the moments of the first-slip time will be derived in Chapter 11 for a more general problem, we only quote the main results of interest here.

In terms of the potential function, which is found from (10-88) and (10-89),

$$U_0(\varphi; \tilde{\psi}) = -\frac{2}{K_{00}} \left[ \Lambda_0 \varphi - AKF_0 \int g(x) \, dx + \int E(y_1, \tilde{\psi}, \varphi_0, y_{10}) \, d\psi \right]$$

(10-124)
the \( n \)th moment of the first time to pass barriers located at \( b_1 = \varphi_{i1} \) and \( b_2 = \varphi_{i2} \) is given by the recursive formula:

\[
\tau^n(\varphi|\varphi_0) = \frac{2n}{K_{00}} \int_{\varphi_{i1}}^{\varphi_{i2}} \int_{\varphi_{i1}}^{\varphi} [C_0(n - 1) - \tau^{n-1}(x|\varphi_0)] \exp \left[ U_0(x; \bar{t}) - U_0(\varphi; \bar{t}) \right] dx \, d\varphi
\]

\[
\tau^0(x|\varphi_0) = u(x - \varphi_0)
\] (10-125)

where \( \bar{t} \) is some point belonging to the interval \((0, \infty)\) and \( u(x) \) is the unit step function. In (10-125)

\[
C_0(n) = \frac{\int_{\varphi_{i1}}^{\varphi_{i2}} \tau^n(x|\varphi_0) \exp \left[ U_0(x; \bar{t}) \right] dx}{\int_{\varphi_{i1}}^{\varphi_{i2}} \exp \left[ U_0(x; \bar{t}) \right] dx}
\] (10-126)

For the sinusoidal PLL, the potential function (10-89) becomes

\[
U_0(\varphi; \bar{t}) = -\left\{ \left( \frac{r + 1}{r} \right) \rho \cos \varphi + \left( \frac{r + 1}{r} \right)^2 \frac{\rho}{2 W_L} \right. \times \left[ \int_{\varphi_{i1}}^{\varphi_{i2}} E(y_1, \bar{\tau}|\psi, \varphi_0, y_{10}) \, d\psi + \Lambda_0 \varphi \right] \right\}
\] (10-127)

when we use the definitions for \( r, \rho, \) and \( W_L \) defined in Chapter 4. For large \( \rho, \) the conditional expectation can be approximated using linear PLL theory. Thus, for \( F_0 \ll 1, \)

\[
U_0(\varphi; \bar{t}) \approx -\left( \frac{r + 1}{r} \right) \rho \cos \varphi - \frac{\rho \varphi^2}{2r} - \rho \varphi
\] (10-128)

since the initial conditions do not enter into the calculation of \( E(y_1, \bar{\tau}|\varphi, \varphi_0, y_{10}) \) if \( \bar{t} \) is large. Consequently, the \( n \)th moment of the mean time to first slip is approximated by

\[
W_L \tau^n(\varphi|\varphi_0) \approx \left( \frac{r + 1}{r} \right)^2 \frac{\rho}{2} \int_{\varphi_{i1}}^{\varphi_{i2}} \int_{\varphi_{i1}}^{\varphi} n [C_0(n - 1) - \tau^{n-1}(x|\varphi_0)]
\]

\[
\times \exp \left[ \rho \left( \frac{r + 1}{r} \right) (\cos \varphi - \cos x) \right.
\]

\[
+ \left. \frac{\rho}{2r} (\varphi^2 - x^2) + \rho \varphi (\varphi - x) \right] dx \, d\varphi
\] (10-129)

*Actually the \( n \)th moment of the first-slip time also depends on \( y_{10} \); however, for simplicity of notation, we suppress this dependence or assume that \( y_{10} = 0 \). This implies zero charge on the capacitor in the loop filter. Also rather than showing the dependence on \( \varphi_{i1} \) and \( \varphi_{i2} \) we write \( \tau^n(\varphi_{i2}, \varphi_{i1}|\varphi_0) = \tau^n(\varphi_{i1}|\varphi_0). \)
where $\varphi_{12} = \varphi_0 + 2\pi$ and $\varphi_{11} = \varphi_0 - 2\pi$. This generalizes Tausworth's result (Ref. 7) via another method. In fact, for $n = 1$, $\Lambda_0 = 0$, and $C'(0) = 1/2$, (10-129) reduces to his result. When $r$ approaches infinity, (10-129) reduces to (9-78) as it should. Figure 10-21 represents a plot of (10-129) for various values of $\rho$ and $\gamma$ with $n = 1$ and $\varphi_0 = 0$. Superposed are computer-simulation results obtained by Holmes (Ref. 8).

In evaluating (10-129), with $n = 1$, on a digital computer, one is faced with the problem of taking the difference between exponentially large quan-

---

Fig. 10-21. Mean Time to First Slip versus $\rho$ for Various Values of $\gamma$. 
Tausworthe (Ref. 9) has used the linear PLL to approximate the conditional expectation and includes the initial condition by setting \( \varphi_0 = \varphi_{ss} = \sin^{-1}(\Lambda_0/AK) \). He further manipulates the result into the form

\[
W_L \tau (2\pi |\varphi_0) = \left( \frac{r + 1}{r} \right)^2 \frac{\rho}{2} \left\{ D \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e(\varphi, y) \, d\varphi \, dy \right. \\
+ \left[ 1 - D \right] \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e(\varphi, y) \, d\varphi \, dy \right\}  
\]

(10-130)

which alleviates the computation problem mentioned above. Here

\[
e(\varphi, y) \triangleq \exp \left\{ \frac{4A}{N_6F_6K} [q(\varphi) - q(y)] \right\}  
\]

(10-131)

where \( q(z) \approx ax + \cos x + bx^2/2 \) and

\[
\begin{align*}
a &= \Lambda_0 - b\varphi_0, \\
b &= \frac{(1 - F_0) \cos \varphi_0}{1 + r \cos \varphi_0}, \\
D &= \frac{\int_{-\pi}^{\pi} e(0, y) \, dy}{\int_{-\pi}^{\pi} e(0, y) \, dy} 
\end{align*}  
\]

(10-132)

**10-5.3 Cycle Slipping Rate, Cycle Slipping Probabilities, and Probability of Sync Failure**

In this section we shall present general results for the cycle slipping rate as well as the cycle slipping probabilities and the probability of sync failure. From (7-139) we have the ratio \( N_+/N_- \) given and the difference \( N_+ - N_- \) is given in (10-123). Using these results it is easy to show that

\[
N_+ = \left[ \frac{1 - C_0(0)}{1 - 2C_0(0)} \right] J_0  
\]

\[
N_- = \left[ \frac{C_0(0)}{1 - 2C_0(0)} \right] J_0  
\]

(10-133)

so that the cycle slipping rate, i.e., diffusional spreading of the number of oscillations in the local reference, is given exactly by

\[
\tilde{S} \triangleq N_+ + N_- = \left[ \frac{1}{1 - 2C_0(0)} \right] J_0  
\]

(10-134)

where \( C_0(0) \) is defined in (10-126) with \( \varphi_{11} = \varphi_0 - 2\pi \) and \( \varphi_{12} = \varphi_0 + 2\pi \). Now the probability current density can be found from (10-90) using (10-93). Thus

\[
J_0 = C_0' K_0 \exp (-\Delta U/2) \sinh (\Delta U/2)  
\]

(10-135)
where $\Delta U \triangleq U_0(\phi) - U_0(\phi + 2\pi)$. Using (10-100) to approximate $U_0(\phi)$ then, for a PLL,

$$J_0 \approx \frac{K_0 \sinh \frac{\pi \beta}{\xi^2} |J_{\beta}(\alpha)|^2}{4\pi^2 |J_{\beta}(\alpha)|^2}$$  \hspace{1cm} (10-136)$$

and

$$\tilde{S} \approx \left[ \frac{K_0 \sinh \frac{\pi \beta}{\xi^2} |J_{\beta}(\alpha)|^2}{4\pi^2 |J_{\beta}(\alpha)|^2} \right] \left[ \frac{1}{1 - 2C_0(0)} \right]$$ \hspace{1cm} (10-137)$$

where $C_0(0)$ can be approximated using (10-128) in (10-126). The expected value of the time intervals between cycle slipping event is approximated by

$$\Delta T \approx \frac{1}{\tilde{S}}$$ \hspace{1cm} (10-138)$$

A special case of interest occurs when $\gamma = 0$ and $g(\phi) = \sin \phi$. In this case the evaluation of (10-137) with $\gamma = 0$ leads to the indeterminate form $0/0$. To evaluate this indeterminate form seems to be a formidable problem; however, one can write

$$\tilde{S} \approx \left( \frac{r}{r + 1} \right)^2 \frac{d\beta}{d\gamma} \bigg|_{\gamma = 0} \int_0^{2\pi} \exp \left[ U_0(x, \bar{\beta}) \right] dx$$

$$\frac{S}{W_L} \approx \left( \frac{r}{r + 1} \right)^2 \frac{d\beta}{d\gamma} \bigg|_{\gamma = 0} \int_0^{2\pi} \exp \left[ U_0(x, \bar{\beta}) \right] dx$$ \hspace{1cm} (10-139)$$

as an approximate expression where $U_0(x, \bar{\beta})$ is defined in (10-128) with $\gamma = 0$ and $F_0 \ll 1$. Figure 10-22 illustrates results obtained by computer simulation. The results given in this figure are compatible with experimental work due to Lindenlaub and Uhlan, Ref. 10.

Furthermore, the cycle slipping probabilities are of interest. Let $N(t)$ be a discrete r.v., taking on values $n = 0, \pm 1, \pm 2, \ldots$, which accounts for the net number of cycles slipped during $[t_0, t]$. If we decompose $N(t)$ into the difference of the two Poisson r.v.'s $n_+(t)$ and $n_-(t)$ with respective rates of occurrence of $N_+$ and $N_-$, then

$$P[N = n] = \frac{(N_+)^n}{N_-} \exp \left[ -\tilde{S}(t - t_0) \right] U_n[2(t - t_0) \sqrt{N_+ N_-}]$$ \hspace{1cm} (10-140)$$

Moreover, the probability that a total of $K = n_+ + n_-$ cycles will be slipped independent of direction at time $t$ is given by
Fig. 10-22. Normalized Average Number of Slips Per Unit Time versus $\rho$ for Various Values of $r$. (Courtesy of M. K. Simon)

\[ P[\mathcal{S} = k] \approx \frac{(\bar{S}t)^k \exp[-\bar{S}t]}{k!} \]  

(10-141)

while the probability of sync failure (loss of phase lock) is given by

\[ P(t) \approx P[\mathcal{S} = k \geq 1] = 1 - \exp[-\bar{S}(t - t_0)] \]  

(10-142)
An exact expression for \( P(t) \) during \([t_0, t]\) is given by

\[
P(t) = 1 - \int_{-2\pi}^{2\pi} Q(\varphi; t) \, d\varphi
\]

and \( Q(\varphi; t) \) is the solution to the two point boundary-valued problem defined by (11-87) and (11-97) with \( y_k = \varphi \).

10-6 Frequency Acquisition Time and Signal Acquisition Probability in the Presence of Noise

In the previous sections we have been concerned with the performance of SCSs when operating in the tracking mode. We now turn our attention to the acquisition mode by presenting a theory that specifies the moments of the frequency acquisition time when noise is present. As we shall see, this problem is related to the first-passage time problem.

As already noted, the voltage \( y_i(t) \) (see Fig. 10-1) accounts for the loop’s acquisition mechanism. In fact, we have agreed to say that the loop is locked when the average value of the phase error rate is zero—that is, \( \bar{\varphi} = 0 \). Noting from (10-81) that in the steady state, \( \bar{\varphi} = 0 \) implies \( \bar{y}_1 = -(\Lambda_0 - AKF_0g) \); furthermore, from the dynamical equation involving \( y_1(t) \), we note that \( \bar{y}_1 = 0 \) implies \( \bar{y}_1 = -(1 - F_0)AKg \). Eliminating \( g \) from these expressions for \( \bar{y}_1 \), we have that the loop will be locked (on the average) when the average value of the pull-in voltage becomes \( \bar{y}_1 \triangleq y_{\text{ul}} = -\Lambda_0(1 - F_0) \). This represents the dc voltage appearing on the capacitor of the RC filter in Fig. 10-1. In fact, if this voltage is zero at \( t = t_0 = 0 \), then \( y_{\text{ul}} = 0 \) and the capacitor charges to within \( \Lambda_0F_0 \) rad of the frequency of the incoming signal. Thus when operation is well within the loop’s pull-in range, the average frequency acquisition time \( T_f(y_{\text{ul}}) \) is the time for the VCO to be pulled from \( |\Delta f = \Lambda_0/2\pi| \) Hz to within \( \Lambda_0F_0/2\pi \) cycles of the carrier.

Therefore the \( m \)th moment of the time \( T_f(y_{\text{ul}}) \) to acquire frequency lock can be obtained by asking for the \( m \)th moment of the time required for \( y_i(t) \) to pass to \( y_{\text{ul}} = -\Lambda_0(1 - F_0) \) for the first time, given that at \( t = 0, \varphi = \varphi_0, y_{\text{ul}} = 0 \). Chapter 11, Section 11-11, treats this problem more generally; however, for the case of interest here it is shown there that

\[
T_f(y_{\text{ul}}) = \frac{2n}{K_{\text{11}}} \int \int \left[ C_i(n - 1) - T_f^{n-1}(x) \right] \exp \left[ U_i(x; \tilde{\varphi}) - U_i(y_{\text{ul}}; \tilde{\varphi}) \right] \, dx \, dy_i
\]

(10-143)

when \( \Lambda_0 > 0 \) and \( T_f^0(x) = u(x) \). Here
\[ U_i(y, \tilde{t}) = -\frac{2}{K_{11}} \left[ \frac{y_1 + AK(1 - F_0)}{\tau} E(g(\phi, \tilde{t}, y_1, \varphi_0) \right] \] (10-144)

represents the potential function generated by the loop's pull-in mechanism. If \( \lambda_0 < 0 \), the limits of integration change in an obvious way. In order to obtain numerical results, one could approximate the conditional expectation in much the same manner as was done for the first-slip time and evaluate the results on a digital computer. Needless to say, further work is required before this problem can be considered solved. It represents a large part of the signal acquisition problem.

The probability of signal acquisition, \( P_{\text{acq}}(t|\varphi_0) \), during the time interval \([t_0, t]\), is given by

\[ P_{\text{acq}}(t|\varphi_0) = 1 - \int_{-\infty}^{y_1} Q(y, t|\varphi_0, t_0) dy \] (10-145)

if we assume \( y_{10} = 0 \). The integrand is the solution to the boundary-valued problem defined in Chapter 11, Section 11-10 for which no solution has presently been found.

### 10-6.1 Optimum Signal Acquisition and Search Procedures

Search optimality criteria basically envisions maximizing the probability of signal acquisition for a specified acquisition time or minimizing the average time required to achieve a specified probability of acquisition. In most cases, the signal search can be carried out discretely by subdividing the frequency-time uncertainty plane into elementary cells (Fig. 10-23a), and the signal parameters can be estimated in each cell by binary detection using samples from each \((\delta f, \delta t)\) element of a two-dimensional space. The search is carried out in frequency in the interval \( \Delta F \) of possible frequency values and in the time interval \( \Delta T \) equal to the largest signal repetition period.

One optimal signal-search procedure (Ref. 39) consists of the following: First, one examines samples from the cell having maximum value a priori probability of containing the signal. Samples are taken until the a posteriori probability that the signal is contained in this cell becomes smaller than the a priori probability that the signal is contained in the next most probable cell. Then the sampling process is continued in this cell, the a posteriori probability being assumed as the new a priori probability. Thus, at each sample, the results of all previous samples are taken into account and the next cell investigated is always the most probable from the viewpoint of containing the signal.

If the signal probability distribution in the frequency-time plane is Gaussian, the search can be carried out using the receiver described in Ref. 40. The search starts from the intervals with largest a priori probability (Fig. 10-23b) (from \(-n_i\) to \(n_i\)) and was developed in Refs. 41, 42, 43, 44, 45. If no
Fig. 10-23. Signal Acquisition Search Strategy.

signal is found in this interval, the search is continued in the interval from $-n_2$ to $n_2$ then, from $-n_3$ to $n_3$ etc. For a known a priori distribution of signal parameters the acquisition time is, of course, smaller than the time required for sequentially examining the space of possible values of the signal frequency and time of arrival. In the case of a uniform a priori distribution of the signal in the frequency-time plane, one can adopt a two-stage search consisting first of a “coarse” inspection, and, then, of the first search procedure suggested above.

10-6.2 Signal Acquisition Receivers

Frequently, the acquisition problem has spatial as well as frequency uncertainty. The spatial uncertainty arises due to limited orbital data or due to the uncertainties during powered flight. Since this spatial uncertainty usually extends over several antenna beamwidths, a spatial search is required. The frequency uncertainty is due to the oscillator frequency drift and possibly Doppler shift. Such difficulties have accelerated the development of receiving systems whose sole function is acquisition. A model for one such receiver is given in Fig. 10-24. Various frequency searching strategies can be incorporated
into the receiver. Aside from the search techniques mentioned above it is possible to accomplish frequency search by the swept IF, wide-open, or multichannel configuration.

The swept IF receiver mixes the incoming signal with a local oscillator signal whose frequency varies as a periodic sawtooth. A relatively narrowband IF filter follows and “looks” at successively different portions of the frequency uncertainty. The wide-open receiver employs an IF filter which is sufficiently wide to accommodate the entire frequency uncertainty and thus no actual sweep is performed. The multichannel receiver employs parallel sweep in which a bank of IF filters simultaneously “looks” at different portions of the frequency uncertainty. Performance analysis for the three receivers can be found in Ref. 46.

10-7 Approximate Theories for Evaluating the Steady-State Variance of the Phase Error

For second-order loops the variance of the phase error, as specified by the quasi-linear theory given in Section 9-7.1, remains unchanged in form. However, \( \rho \) is now interpreted as the loop SNR defined in (4-15) and (4-22) for the second-order PLL.

Based on the linear spectral theory discussed in Section 9-7.2, the variance of the phase error is given by (9-114) with

\[
W_{L(\text{eq})} \approx \frac{1 + r \gamma_0}{2 \tau_2 \sqrt{1 + 2(\gamma_0 - \eta_0)/r \gamma_0}}
\]

(10-146)

where \( \eta_0 = \exp(-\sigma_v^2/2) \) and \( \gamma_0 = [1 - \exp(-\sigma_v^2)]/\sigma_v^2 \) with \( \sigma_v^2 = N_0 W_{L(\text{eq})}/2(A \gamma_0)^2 \).

Using the Volterra expansion method discussed in 9-7.3 with \( F(s) = \)
(1 + τ₂s)/τ₁s and damping factor equal to 0.707, Van Trees obtained the approximate formula

\[ \sigma_0^2 \approx \frac{1}{\rho} + \frac{2}{3\rho^2} \]  

(10-147)

As before, the first term is due to the linear theory, whereas the second term enters to compensate for the nonlinear behavior of the loop. Conceptually the Volterra expansion technique solves the problem of finding the variance; however, in cases of practical interest the evaluation of the Volterra kernels is extremely difficult. Figure 10-25 represents a comparison of the various theories for \( r = 2 \).

![Graph showing comparison of various theories](image)

**Fig. 10-25.** Comparison of the Variance of the Phase-Error versus \( \rho \) as Predicted by the Various Theories and Experimentally Measured Points.

10-8 Conclusions Regarding the Application of First- and Second-Order SCSs

The use of an RC filter \( F(p) = 1/(1 + \tau p) \) increases the order of the equation of operation, and we can conclude that a synchronization system with an RC filter is more immune to noise than an injection-locked oscillator or a first-order inertialless PLL. The primary difference lies in the variance of the
mean residual frequency detuning, $\tilde{\phi}$. The larger $\tau$ is the smaller the variance $\sigma_{\tilde{\phi}}^2$. It should also be borne in mind that the presence of an RC filter narrows the acquisition range. Therefore, in practice the tendency is to choose filters that ensure the required noise immunity for a specified acquisition range. This can be accomplished with the integral-control proportional plus type or a nonlinear loop filter.

Regardless of the filter time constants, the mean residual frequency detuning $\tilde{\phi}$ is always equal to zero when $\Lambda_0 = 0$, while its variance is greater than zero. In fact, the mean and variance of the residual frequency detuning increases as the loop SNR decreases and the initial frequency detuning increases. The mean value of the residual detuning will always be lower for the integral-control proportional plus (ICPP) loop filter than for the RC filter with the same time constant. It follows that the ICPP filter, which provides a larger acquisition range and smaller residual detuning, has advantages over the RC filter and inertialess first-order system when noise immunity is concerned. In section (II-15) we shall show that a third-order system provides better noise immunity in the synchronous mode than any of the above mentioned filters. This alone serves to make the approach of choosing the loop filter by Wiener techniques suspect.

10.9 Related Studies

The contributions made to the analysis, design, and optimization of second-order tracking loops with a sinusoidal phase-detector characteristic are too numerous to mention; Gardner (Ref. 11) lists more than 150 contributions in a bibliography. In what follows, we present a list of reference material that is typical of what the author has found useful in studying the subject of second-order loops. Notable contributions for the no-noise case have been made by Richman (Ref. 3), Preston and Tellier (Ref. 12), Gruen (Ref. 13), Jelonek and Cowan (Ref. 14), Viterbi (Ref. 15), and Fleishmann and Leonhardt (Ref. 16).

Although the sinusoidal function has been studied extensively, other functions have received considerably less attention. The triangular-shaped function has been analyzed by Spilker (Ref. 17) and Zegers (Ref. 18), and a slight modification of it be Springett (Ref. 19). The sawtooth function has been studied by Byrne (Ref. 20), and a modified tangent function appearing in a device known as the Tanlock receiver has been analyzed by Robinson (Ref. 21). Gill (Ref. 22) placed square-law detectors in the correlators of a delay-locked loop and has studied the new nonlinearity that resulted.

Some work has been done on the comparison of different phase-detector characteristics. Long and Rutledge (Ref. 23) have obtained digital computer programs for phase-plane analysis and compared the sinusoidal phase-locked characteristic with sawtooth and Tanlock characteristics. Uhran and Linden-
laub (Ref. 10) have generalized the Tanlock characteristic and compared performance with variations of the nonlinearity. Optimization of performance through design of the phase-detector characteristic has been carried out by Shaft and Dorf (Refs. 24, 25, 26). Other work on optimization of the performance was published by Stiffler (Ref. 27) and Layland (Ref. 28).

In the Soviet Union, Kapranov (Refs. 29, 30, 31), Bakayev (Ref. 32), Pervachev (Ref. 33), and Shakhtarin (Ref. 34) have compared the locking range of several phase-detector characteristics, including the sinusoidal, triangular, and square-wave characteristics.

The contradiction between the requirements of a large acquisition range and the narrowband filtering capability of the loop can be partially eliminated by proper choice of the type of parameters in the loop filter. It is conjectured that the acquisition range can be increased by designing a loop filter in which the resistance \( R \) and the capacitance \( C \) are nonlinear. It would be interesting to investigate this approach. The author (Ref. 35) presented the analysis of second-order loops with an arbitrary, periodic, phase-detector characteristic when noise was present at the input of the system.

Finally, various studies (Refs. 36, 37) have been concerned with the solution to (10-85) when \( F_0 = 0 \), i.e., \( F(p) = 1/(1 + \tau p) \). When \( \Lambda_0 \neq 0 \), obtaining a solution to (10-85) appears to be a formidable problem; however, Tikhonov (Ref. 36) was able to obtain the approximate solution

\[
\rho(\phi, \dot{\phi}) = \tau \sqrt{\frac{\tau \rho}{2\pi A K}} \frac{\sin \pi \beta}{2\pi |A p(\alpha)|^2} \exp \left( -\frac{\tau \rho}{2\pi A K \phi^2} \right) \cdot \\
\left\{ \phi + \frac{\exp(\pi \beta)}{2\pi \sin \pi \beta} \int_0^{\phi + 2\pi} \exp \left[ \beta \phi + \alpha \cos \phi \right] \exp \left[ -\beta x - \alpha \cos x \right] dx \right\}
\]

\[ (10-148) \]

where \( \rho = 4A/N_0 K \) and \( \beta = \gamma \rho \) and \( \phi \in (0, \infty) \). It is easy to show that the ratio of the residual detuning to the initial detuning is given by

\[
\frac{\dot{\phi}}{\Lambda_0} \approx \frac{\sin \pi \beta}{\pi \beta} \frac{1}{|A p(\alpha)|^2}
\]

and

\[ (10-149) \]

\[
\sigma_\phi \approx \frac{4K}{\tau \rho}
\]

Figure 9-16 represents a plot of the ratio given in (10-149). We conclude, as in the case of a first-order loop and \( F(p) \) of Fig. 10-2, that the mean frequency of the VCO does not coincide with the synchronizing frequency when \( \Lambda_0 \neq 0 \).
APPENDIX I

DERIVATION OF THE ACQUISITION RANGE

The acquisition range of a PLL generally defined by the condition that the loop, for signal frequencies within this range, becomes stable synchronized with the applied signal regardless of the initial conditions of the system. For all other signal frequencies, the system will operate in a nonsynchronized state; that is, the limit cycle occurs. Thus the defining condition for the acquisition range is the nonexistence of a limit cycle. No clear mathematical proof of this criterion can be given at this point—that is, this is a necessary but not sufficient condition.

We wish to outline the derivation of (10-56). Introducing the new variables \( v = 1/\sqrt{AK\tau_1}, \eta = AK\tau_2/\sqrt{AK\tau_1}, \) and \( x = vAKt = v\tau \) into loop equation gives

\[
\frac{d^2\varphi}{dx^2} + v \frac{d\varphi}{dx} + \eta T' \frac{d(\sin \varphi)}{dx} + T' \sin \varphi = \frac{\Lambda_0}{AK} = \gamma
\]  

(I-1)

where, in operational form, \( T'(d/dx) = T(vAK \, d/dx) \). Since the loop phase error is periodic in time in the nonsynchronized state, we can write

\[
\varphi = \Omega x + S(x)
\]

where \( \Omega = \omega_f/vAK \) and

\[
S(x) \triangleq \sum_{n=1}^{\infty} (a_n \sin n\Omega x + b_n \cos n\Omega x)
\]  

(I-2)
Now
\[
\sin \varphi = \sin \Omega x \cos S + \cos \Omega x \sin S
\]
and when \(S \ll 1\),
\[
\sin \varphi \approx \sin \Omega x + \cos \Omega x [S(x)]
\]

This equation is then multiplied by \(T(\omega)\), given in (10-57), and by \((d/dx)T' = T' d/dx\). These results and the time periodic for \(\varphi\) can be used in (I-1) to determine conditions which periodic solutions. Defining \(a_n \triangleq a(n\omega)\) and \(\epsilon_n \triangleq \epsilon(n\omega)\), one obtains for the dc component of (I-1)
\[
v\Omega + \frac{a_0 \beta_1}{2} = \frac{\Lambda_0}{AK}
\]  
(I-3)

For a time-periodic solution to exist, the coefficient of \(\sin \Omega x\) yields
\[
-\Omega^2 \alpha_1 - v\Omega \beta_1 - \eta \Omega \alpha_1 \left[ \beta_2 \cos \frac{\epsilon_1}{2} - \left( 1 + \frac{\alpha_2 \sin \epsilon_1}{2} \right) \right]
\]
\[
+ a_1 \left[ \left( 1 + \frac{\alpha_2}{2} \right) \cos \epsilon_1 + \frac{\beta_1 \sin \epsilon_1}{2} \right] = 0
\]  
(I-4)

while the coefficient of \(\cos \Omega x\) produces
\[
-\Omega^2 \beta_1 + v\Omega \alpha_1 + \eta \Omega \alpha_1 \left[ \left( 1 + \frac{\alpha_2 \cos \epsilon_1}{2} \right) + \frac{\beta_2 \sin \epsilon_1}{2} \right]
\]
\[
+ a_1 \left[ \frac{\beta_2 \cos \epsilon_1}{2} - \left( 1 + \frac{\alpha_2}{2} \right) \sin \epsilon_1 \right] = 0
\]  
(I-5)

In general, we have that the coefficient of \(\sin n\Omega x\) \((n \geq 2)\) gives
\[
-n^2 \Omega^2 \alpha_n - v\Omega n \beta_n - \eta n \Omega \alpha_n \left[ \left( \frac{\beta_{n-1} + \beta_{n+1}}{2} \right) \cos \epsilon_n - \left( \frac{\alpha_{n-1} + \alpha_{n+1}}{2} \right) \sin \epsilon_n \right]
\]
\[
+ a_n \left[ \left( \frac{\alpha_{n-1} + \alpha_{n+1}}{2} \right) \cos \epsilon_n + \left( \frac{\beta_{n-1} + \beta_{n+1}}{2} \right) \sin \epsilon_n \right] = 0
\]  
(I-6)

and from the coefficient of \(\cos n\Omega x\) \((n \geq 2)\) we obtain
\[
-n^2 \Omega^2 \alpha_n - v\Omega n \beta_n - \eta n \Omega \alpha_n \left[ \left( \frac{\beta_{n-1} + \beta_{n+1}}{2} \right) \sin \epsilon_n + \left( \frac{\alpha_{n-1} + \alpha_{n+1}}{2} \right) \cos \epsilon_n \right]
\]
\[
+ a_n \left[ \left( \frac{\alpha_{n-1} + \alpha_{n+1}}{2} \right) \sin \epsilon_n - \left( \frac{\beta_{n-1} + \beta_{n+1}}{2} \right) \cos \epsilon_n \right] = 0
\]  
(I-7)
From (I-4) and (I-5) the coefficients \( \alpha_2 \) and \( \beta_2 \) can be evaluated for a given \( \Omega \) as a function of \( \alpha_1 \) and \( \beta_1 \) and the various constants. From (I-6) and (I-7) the higher coefficients can be derived. Neglecting (I-3) for the moment, all coefficients \( \alpha_n \) and \( \beta_n \) \((n \geq 2)\) are defined only as a function of \( \alpha_1, \beta_1, v, \eta, \) and \( \Lambda_0/AK \). Therefore, even after introducing (I-3), the two variables \( \Omega \) and \( \alpha_1 \) remain to be determined. In principle, we would have to check for which values of \( \Omega \) and \( \alpha_1 \) the resulting set of coefficients \( \alpha_n \) and \( \beta_n \) would represent the coefficient of a converging Fourier series. However, when \( S \ll 1 \), it follows from (I-1) that \( \alpha_{n+1}/\alpha_n \approx \beta_{n+1}/\beta_n \approx (v\Omega + 1)/\Omega^2 \) for large \( n \). Thus a good approximation should generally be obtained by setting \( \alpha_n = \beta_n = 0 \) for \( n \geq N \). Letting \( N = 2 \) and eliminating \( \alpha_1 \) and \( \beta_1 \) in (I-3), (I-4), and (I-5), one obtains

\[
\frac{\Lambda_0}{AK} = v\Omega + \frac{\eta}{2\Omega} \frac{a_1}{a_0} \cos \epsilon_1 - \frac{a_1}{2a_0\Omega^2} (1 - v\eta) \sin \epsilon_1
\]

\[
\approx v\Omega + \frac{a_1\eta}{2\Omega} \cos \epsilon_1 - \frac{a_1}{2\Omega} \sin \epsilon_1
\]

(I-8)

since \( a_0 = 1 \) because \( F(0) = 1 \). This is (10-56) in the main text and is largely due to Fleischmann and Leonhardt (Ref. 16). In this respect, the earlier work of Preston and Tellier (Ref. 12), Gruen (Ref. 13), Jelonek and Cowan (Ref. 14), Richman (Ref. 4), and Viterbi (Ref. 15) can be deduced from (I-8).

In the regions where the approximations and assumptions hold, (I-8) relates the detuning \( \Lambda_0 \) to the asymptotic beat frequency \( \omega_f \) in a nonsynchronized asymptotic state of the loop. It follows that the absolute minimum of (I-8) as a function of \( \omega_f \) determines the minimum \( \Lambda_0 \) for which a limit cycle exists.
APPENDIX II

APPROXIMATING
THE CONDITIONAL
EXPECTATIONS

The worth of a theory, or mathematical model, is its ability to predict the results of future experiments (or possibly computer simulations) that can be carried out in the laboratory. If the predictions made by the mathematical equation are not justified by experiments or simulation, a new mathematical model must be sought. Phase-locked loop theory is certainly no exception to these statements.

Historically speaking, the development of a working theory for use in the design and testing of phase-tracking systems has been one of developing an approximate theory. As the demand for better system performance has increased, it has been necessary to refine the theory and seek more accurate solutions. The first attempt to refine the theory was that of extending the linear theory by quasi-linear methods. Unfortunately, this theory had its limitations in that the time to loss of lock (among other things) was not considered, nor did the theory extend the results predicted by the linear theory much beyond 4 or 5 dB signal-to-noise ratio in the loop bandwidth. The Volterra method, discussed in Section 9-7.3, of approximating loop response was plagued by the fact that it was difficult to apply. On the other hand, the linear spectral theory, discussed in Section 9-7.2, being the most accurate of all the above approximate theories for predicting the variance of the phase error, did not allow for the determination of ρ(φ), nor account for cycle slipping. As is the case in most nonlinear problems, it is difficult to give a general theoretical analysis.
Some problems are characteristically nonlinear, while others may be linearized. Thus one quickly adopts the philosophy, "Don’t hesitate to approximate."

All mathematical theories are, to some extent, approximations; nature has been unkind enough to see to that. It is not unreasonable, then, that different mathematical theories are devised to describe one set of physical observations. If the theories are different, the degree of approximation to nature will be different. This does not mean that the theory with the poorer degree of approximation is not a useful one. Often more accurate theories are not used because their accuracy is not warranted or their application is too difficult. In general, the more closely the mathematical theory approximates nature, the less tractable and susceptible the equations of that mathematical theory are to solution.

From a tracking point of view, the road to travel is clear; the solutions to the partial differential equations and integro-differential equations presented are not at all simple. It has been necessary to make use of the orthogonality principle to obtain certain answers. The validity of certain approximations must be justified only by the plausibility of the solution. The tests of their validity must await the performance of experiments and/or computer simulations. At times we might not even be this fortunate. With these remarks, as preambles to this appendix, we proceed with the mathematical development.

Here we derive (10-94), starting with (10-2) and $y_1$ replaced by $y_k$. First, multiply both sides of (10-2) by $v/\tau_k$ to obtain

$$\frac{d}{dv} \left[ y_k(v) \exp \left( \frac{v}{\tau_k} \right) \right] = -\left(1 - \frac{F_k}{\tau_k}\right) \left[ AKg[\phi(v)] + Kn(v) \right] \exp \left( \frac{v}{\tau_k} \right)$$

(II-1)

Taking expectations of both sides, conditioned on $\phi(t)$, we obtain

$$E \left[ \frac{d}{dv} \left[ y_k(v) \exp \left( \frac{v}{\tau_k} \right) \right] \right] =$$

$$\left(1 - \frac{F_k}{\tau_k}\right) E \left[ \left[ AKg[\phi(v)] + Kn(v) \right] \exp \left( \frac{v}{\tau_k} \right) \right]$$

(II-2)

Assuming that

$$\frac{\partial^2 R_{yx}(t_1, t_2)}{\partial t_1 \partial t_2}$$

exists, we can interchange the order of expectation with differentiation and write

$$\frac{d}{dv} \left[ E[y_k(v)|\phi(t)] \exp \left( \frac{v}{\tau_k} \right) \right] = -\left(1 - \frac{F_k}{\tau_k}\right) AK E[g[\phi(v)]|\phi(t)] \exp \left( \frac{v}{\tau_k} \right)$$

$$-\left(1 - \frac{F_k}{\tau_k}\right) K E[n(v)|\phi(t)] \exp \left( \frac{v}{\tau_k} \right)$$

(II-3)
Integrating both sides of (II-3) from $t$ to infinity gives

$$\int_t^\infty \frac{d}{dv} \left\{ E[y_k(v)|\phi(t)] \exp\left( \frac{v}{\tau_k} \right) \right\} \, dv$$

$$= -\frac{(1 - F_k)AK}{\tau_k} \int_t^\infty E[g(\phi(v))|\phi(t)] \exp\left( \frac{v}{\tau_k} \right) \, dv \quad (II-4)$$

The reason the integration is taken from $t$ to infinity (not $-\infty$ to $t$) is that $\phi(t)$ depends on the noise in the past and

$$\int_{-\infty}^t E[n(v)|\phi(t)] \exp\left( \frac{v}{\tau_k} \right) \, dv$$

is not readily evaluated; however,

$$\int_t^\infty E[n(v)|\phi(t)] \exp\left( \frac{v}{\tau_k} \right) \, dv = 0 \quad (II-5)$$

Carrying out the integration in (II-4) and rearranging gives

$$\exp\left( \frac{t}{\tau_k} \right) E[y_k(t)|\phi(t)] = \frac{AK(1 - F_k)}{\tau_k} \int_t^\infty E[g(\phi(v))|\phi(t)] \exp\left( \frac{v}{\tau_k} \right) \, dv$$

$$+ \lim_{v \to \infty} \left\{ E[y_k(v)|\phi(t)] \exp\left( \frac{v}{\tau_k} \right) \right\} \quad (II-6)$$

Introducing the change of variables $v = t + \tau$ in (II-7) yields (this assumes that $t$ is fixed)

$$\exp\left( \frac{t}{\tau_k} \right) E[y_k(t)|\phi(t)] = \frac{AK(1 - F_k)}{\tau_k} \int_0^\infty E[g(\phi(t + \tau))|\phi(t)] \exp\left( \frac{t + \tau}{\tau_k} \right) d\tau$$

$$+ \lim_{(t + \tau) \to \infty} \left\{ \exp\left( \frac{t + \tau}{\tau_k} \right) E[y_k(t + \tau)|\phi(t)] \right\} \quad (II-8)$$

Multiplying both sides of (II-8) by $\exp(-t/\tau_k)$ and adding and subtracting

$$\frac{AK(1 - F_k)}{\tau_k} \int_0^\infty g(\phi(t + \tau)) \exp\left( \frac{t + \tau}{\tau_k} \right) \, d\tau$$

(II-9)

to the right-hand side of (II-8) gives
\[ E[y_k(t)|\phi(t)] = \]
\[ \frac{AK(1 - F_k)}{\tau_k} \int_0^\infty \{ E[g[\phi(t + \tau)]|\phi(t)] - g[\phi(t + \tau)] \} \exp \left( \frac{\tau}{\tau_k} \right) d\tau \]
\[ + \frac{AK(1 - F_k)}{\tau_k} \int_0^\infty g[\phi(t + \tau)] \exp \left( \frac{\tau}{\tau_k} \right) d\tau \]
\[ + \lim_{\tau \to -\infty} \exp \left( \frac{\tau}{\tau_k} \right) E[y_k(t + \tau)|\phi(t)] \]  \hspace{1cm} (II-10)

since \( t \) is fixed. In (II-10) the overbar denotes statistical average. In the steady state, the process \([\phi(t), y_k(t)]\) becomes strictly stationary so that we can write

\[ \lim_{t \to \infty} E[y_k(t)|\phi(t)] = E_s[y_k(t)|\phi(t)] = \frac{AK(1 - F_k)}{\tau_k} \]
\[ \times \int_0^\infty E_k[g[\phi(t + \tau)] - g[\phi(t + \tau)]|\phi(t)] \exp \left( \frac{\tau}{\tau_k} \right) d\tau \]
\[ - AK(1 - F_k)\bar{g} + \lim_{\tau \to -\infty} \exp \left( \frac{\tau}{\tau_k} \right) \left[ E_s[y_k(t + \tau)|\phi(t)] + AK(1 - F_k)\bar{g} \right] \]  \hspace{1cm} (II-11)

where \( \bar{g} \) is the average \( g[\phi(t + \tau)] \) in the steady state and \( E_s[\cdot] \) denotes that the expectation is taken with respect to the steady-state probability density function of the bracketed quantity. Equation (II-11) reduces to (10-94) if, in the steady state,

\[ Q_k = \lim_{\tau \to -\infty} \exp \left( \frac{\tau}{\tau_k} \right) \left[ E_s[y_k(t + \tau)|\phi(t)] + AK(1 - F_k)\bar{g} \right] \]  \hspace{1cm} (II-12)

equals zero. Now

\[ \lim_{y \to \infty} \{ E_s[y_k(v)|\phi(t)] + AK(1 - F_k)\bar{g} \} = 0 \]  \hspace{1cm} (II-13)

since

\[ \lim_{y \to \infty} E_s[y_k(v)|\phi(t)] = E_s[y_k(v)] = -AK(1 - F_k)\bar{g} \]  \hspace{1cm} (II-14)

However, this fact does not help in the evaluation of (II-12), for we obtain in the limit, the indeterminant form \( 0 \cdot \infty \). Any attempt to use L’Hôpital’s rule to evaluate (II-12) by placing the indeterminant form in the form \( 0/0 \) is futile. One can evaluate (II-12) in the linear range of operation of the loop, and it can be shown that \( Q_k = 0 \) when \( F_k < 1 \). On the other hand, when \( A = 0\)
that is, zero loop signal-to-noise ratio—it is indeed true that $Q_k \equiv 0$. As is typically the case in nonlinear problems of communication theory, one can often effect an analysis in the low and high signal-to-noise ratio region; however, analysis in the intermediate range is frequently formidable. In order to proceed with the analysis based on the Fokker-Planck method, we are forced to produce solutions on the basis that $Q_k = 0$ for all signal-to-noise ratios. The validity of the "general" solutions produced under this assumption must await justification based on either computer simulations and/or hardware experiments. Special cases have been accounted for in Section 10-5.

Problems

10-1  A color-carrier reference signal for color television is to be provided by an imperfect second-order PLL sync system. Phase errors in the reference signal are visible as variations of hue in the reproduced picture. Dynamic as well as static phase errors can occur; dynamic errors result in fluctuating horizontal color streaks. The peak tolerable, steady-state phase error is 5 deg. A loop bandwidth of $W_L = 200$ Hz appears to be optimum while designing the loop such that $r = 2$ gives best pull-in performance. The maximum pull-in range is taken to be 2.5 kHz. With this given data, determine,
(a) The required loop gain.
(b) The time constant $\tau_2$.
(c) The constants $F_0$ and $\tau_1$.
(d) Find the maximum pull-in range of the circuit. Does this meet the functional requirements?
(e) What is the acquisition time?
(f) Determine the minimum acceptable $N_0/2P_e$.

10-2  For the imperfect second-order sinusoidal PLL with VCO rest frequency $\omega_0$ and input signal phase $\dot{\theta}(t) = \Omega_0 t + \theta_0$:
(a) Find $\lim_{t \to \infty} \theta(t)$ exactly. Do not linearize!
(b) Explain why your answer in (a) does not depend on $\tau_1$ and $\tau_2$.
(c) Compare your answer with the results obtained when you linearize the loop.

10-3  Determine where the singularities of an ideal second-order sinusoidal PLL with input $\dot{\theta}(t) = \theta_0 + \Omega_0 t + \frac{1}{2} \Omega_1 t^2$ occur. [Hint: Verify and use (10-76).]
By multiplying (10-76) by $\nu(\phi)$ and integrating from $-\pi$ to $\pi$ on $\phi$, determine a necessary condition that is met when a limit cycle occurs in the loop.

10-4  By multiplying both sides of (10-21) by $\nu(\phi)$ and integrating from $-\pi$ to $\pi$, show that the condition

$$
\frac{2\pi \Lambda_0}{(AK)^2 \tau_1} = \int_{-\pi}^{\pi} \nu(\phi) \left[ \frac{1}{AK \tau_1} + F_0 \frac{d\nu(\phi)}{d\phi} \right] d\phi
$$
must be satisfied in order for a limit cycle to occur in an imperfect second-order PLL. Remember that \( v(\phi) \) is periodic in \( \phi \) when a limit cycle occurs.

10-5 A sinusoidal PLL is mechanized with the loop filter

\[
F(s) = 1 + \frac{a}{s} + \frac{b}{s^2}
\]

(a) Show that the equation of operation is given by (\( p = d/dt \))

\[
\left[ p^3 + AK(p^2 + ap + b) \frac{\sin \phi}{\phi} \right] \phi = 0
\]

(b) Treat \( \sin \phi/\phi \) as a part of the coefficients of the last three terms of the polynomial in \( p \) and perform a Routh-Hurwitz test on the polynomial to show that a condition for stability is given by

\[
b \leq (AK)^2 \frac{\sin \phi}{2\phi}
\]

10-6 By multiplying both sides of (10-74) by \( v(\phi) \) and integrating from \(-\pi\) to \( \pi \), show that the acquisition range for a perfect second-order sinusoidal PLL is infinite.

10-7 For an ideal (i.e., perfect second-order) sinusoidal PLL, it is of interest to determine the normalized frequency error decay, \( \Delta v = v(\pi) - v(-\pi) \), per cycle slipped during acquisition. It can be assumed that for large enough offsets the rate of frequency decay is constant; that is, \( v(\phi) \) can be approximated by

\[
v(\phi) = v(0) \left[ 1 - \frac{\sin \phi}{v(0)} + \frac{\Delta v}{2\pi v(0)} \right]
\]

(a) If \( v(0) \gg 0 \), derive an expression for \( \Delta v \) in terms of \( v(0) \) and \( AK\tau_1 \). [Hint: Substitute the approximation for \( v(\phi) \) in (10-74) and integrate from \(-\pi\) to \( \pi \).]

10-8 An imperfect second-order PLL is implemented with one of the two phase-detector (PD) characteristic illustrated in the Fig. P10-8.

(a) Assuming that \( AK\tau_1 \gg 1 \), determine for each of the PD characteristics the conditions under which the loop will lock; that is, find the acquisition range in the absence of noise.

(b) Find an analytic expression for the mean time to acquire frequency lock in the absence of noise; that is, find the time to slip a cycle and multiply by \( (\Lambda N/2\pi)^2 \).

(c) Compare the answers obtained in (a) and (b) with those given in the text for a sinusoidal PLL.
10-9 Find the value of $r$ that minimizes the acquisition time $T_f$ in an imperfect second-order sinusoidal PLL. Discuss the meaning of your answer.

10-10 With $r = 1, 2, 4, 10$, compute the critical values of delay for which an imperfect, second-order sinusoidal PLL will not achieve phase lock.

10-11 Compute the acquisition time and the asymptotic acquisition range for an imperfect second-order sinusoidal PLL when $\tau_1 = 2630$ sec, $\tau_2 = 0.0833$ sec, for $r = 1, 2, 4, 10$. Assume that $\Lambda_0/2\pi = 200, 300, 400, 500$, and 600 Hz. Use (10-53) to make your computations.

10-12 A sinusoidal PLL is implemented with an imperfect integrating loop filter.
(a) Find the tuning or sweep voltage equation that renders zero phase error in the steady state if

$$\theta(t) = \theta_0 + \Omega_0 t + \frac{\Omega_1 t^2}{2}$$

(b) Repeat (a) when the loop contains a perfect integrator.

10-13 A 1 MHz clean-up loop is to be used to provide spectrally pure references from a local or a remote rubidium standard. The loop employs a precision quartz oscillator and is designed to phase lock automatically to a 1 MHz ± 60 milli-Hz signal. The loop is designed such that $\tau_1 = 7600, r = 4$, and $AK = 2$. Experimental results indicate that a $W_L = 0.02$ Hz is required in order to yield a pure reference signal.
(a) Compute the frequency acquisition time when an imperfect, integrating loop filter is mechanized into the loop. Assume that \( \Lambda_0/2\pi = 10, 30, 60, 90 \) milli-Hertz and that \( g(\phi) = \sin \phi \). [Use (10-20) since the loop bandwidth is less than one hertz].

(b) From (a) we note that this narrow bandwidth gives rise to large acquisition times. To overcome this problem, \( W_L \) is increased to 0.08 Hz. Compute the shortened acquisition time and compare with (a).

10-14 When a bandpass limiter precedes a narrowband sinusoidal PLL with an imperfect integrating loop filter, show that the formula (10-52) for the time to acquire frequency lock converts to

\[
T_f \approx \sqrt{2\pi^2 F_0 \left( \frac{r_0/\mu + 1}{r_0/\mu} \right)} \frac{\Lambda_0}{(2\pi^2)W_L}
\]

where \( W_L \) is defined by (4-75) of Chapter 4. Notice that \( T_f \) is inversely proportional to the loop bandwidth.

10-15 An S-band receiver of the PLL type is implemented such that \( \tau_1 = 2630 \) sec, \( \tau_2 = 0.0834 \) sec, \( K_m = 3 \) volt/deg, \( K_i = 110.5 \), \( K_v = 120 \) Hz/volt, \( \alpha_{10} \sqrt{P_{10}} = 0.053 \), \( W_{L0} = 2B_{L0} = 18 \) Hz. Using the formula for \( T_f \) developed in Prob. 10-14:

(a) Evaluate the time to acquire frequency lock when an imperfect integrating loop filter is placed in the loop. Assume that \( \Lambda_0/2\pi = 500, 600, 700 \) Hz and that \( 2P_c/N_0 W_{L0} = 16 \) dB.

(b) Based on experimental measurements, it has been found that \( T_f = 32 \) sec when \( \Lambda_0/2\pi = 500 \) Hz, \( T_f = 51 \) sec when \( \Lambda_0/2\pi = 600 \) Hz, \( T_f = 76 \) sec when \( \Lambda_0/2\pi = 700 \) Hz. Plot these results and compare them with those obtained in (a).

10-16 Compute the maximum acquisition sweep rate for the S-band receiver in Prob. 10-15 when \( W_{L0} = 18 \) Hz, \( \tau_1 = 2630 \) sec, and \( \tau_2 = 0.0834 \). Assume no IF limiting and take values of \( \rho = 5, 10, 100 \).

10-17 An imperfect second-order PLL is designed such that \( R_1 = 206.5 \) k\( \Omega \), \( R_2 = 998 \) \( \Omega \), \( C = 250 \mu F \). Find

(a) \( \tau_1 \) and \( \tau_2 \)

(b) The loop damping.

(c) Loop natural frequency.

(d) Loop bandwidth.

(e) Determine the loop acquisition range.

(f) Find the acquisition time if \( \Lambda_0/2\pi = 5, 10, \) and 20 Hz. Assume that \( AK = 3328 \).

10-18 An imperfect second-order sinusoidal PLL is designed with \( R_2 = 34 \) k\( \Omega \), \( R_1 = 40 \) k\( \Omega \), and \( C = 250 \mu F \). The open-loop gain is adjusted such that \( AK = 2 \) and the loop is to be used as a clean-up loop. Consequently, the loop bandwidth must be small. Find
(a) $\tau_1$ and $\tau_2$, using the theory developed in Chapter 4.
(b) $r$ and the loop damping.
(c) The loop bandwidth.
(d) The loop acquisition range.
(e) What is the time to acquire frequency lock when $\Lambda_0/2\pi = 10, 20, 30, 40, 50, 60$, and $70$ milli-Hertz? The loop is to be used as an element in a frequency synthesizer.
(f) Determine the maximum sweep rate if $\rho = 5, 10, \text{and } 100$.

10-19 A second-order sinusoidal PLL is designed to operate with $r = 4$ and $F_0 = 0.002$.
(a) Find the variance of the phase error when $\rho = 1$ and $\rho = 2$. (Use both the linear and nonlinear PLL theory.)
(b) Determine the mean time to first slip if $\phi_0 = 0$, $W_L = 10 \text{ Hz}$, and $\gamma = \Lambda_0/\alpha K = 0$ and $0.25$.

10-20 It is of interest to determine $\overline{\cos \phi}$, $\overline{\sin \phi}$, $\sigma^2_{\cos \phi}$, and $\sigma^2_{\sin \phi}$ in an imperfect second-order sinusoidal PLL designed such that $r = 4$ and $F_0 = 0.002$.
(a) If $\rho = 3.2$ and $\gamma = 0.2$, use Fig. 10-19 to determine the corresponding values of $\alpha$ and $\beta$.
(b) From these values of $\alpha$ and $\beta$, determine the circular moments $\overline{\sin \phi}$, $\overline{\cos \phi}$, $\sigma^2_{\cos \phi}$, and $\sigma^2_{\sin \phi}$ using the graphs given in Chapter 9.

10-21 An imperfect second-order sinusoidal PLL is designed such that $\tau_1 = 65.5$, $\tau_2 = \frac{1}{s}$, and $W_L = 12 \text{ Hz}$.
(a) Find the required loop gain.
(b) Determine the loop acquisition range and acquisition time.
(c) A capacitor $C_1$ is placed in parallel with $R_2$ such that the loop filter now assumes the form

$$F(s) = \frac{1 + \tau_3 s}{1 + \tau_1 s} \left( \frac{1}{1 + \tau_3 s} \right)$$

Find the new acquisition range if $\tau_3 = \tau_2/10 = 1/80$.
(b) Compare this result with that found in (b).

10-22 An imperfect second-order PLL is implemented such that the PD characteristic is rectangular; that is, $g(\phi) = \text{sgn} [\sin \phi]$. See Fig. P10-8.
(a) Find an expression for the steady-state p.d.f.
(b) From (a) determine the variance $\sigma^2_\phi$ when $\beta = 0$.
(c) Show that for large $\alpha$,

$$\sigma^2_\phi \approx \frac{\pi^2 d}{3} \frac{1}{\exp (\pi \alpha) - 1}$$

(d) Determine the probability currents $\mathcal{J}$, $N_+$, and $N_-$.
(e) Develop a formula for the average number of slips per second.
Develop expressions for the p.d.f. \( p(\phi) \) for all loops given in Table 3-1 when \( \theta(t) = \Omega_0 t + \theta_0 \) and an imperfect integrator is placed in the loop.

Evaluate, and thereby justify, the intensity coefficients given in (10-82) and (10-83), starting with (10-81).

Starting with (10-84) and (10-85), apply the appropriate boundary conditions and verify that the reduced FP equation given is (10-86) and (10-87) is correct.

Based on the theory of the bandpass limiter given in Chapter 4, develop expressions for \( p(\phi), \bar{S}, \tau(2\pi|\varphi_0), \sigma_\phi^2, J_s, P[N = n] \), when a bandpass limiter precedes the loop. [Hint: Show that \( W_L \) must be replaced by \( W_L = W_{LO}(1 + r_0/\mu)/(1 + r_0) \) and \( r = r_0\alpha_t/\alpha_{10} \) in all formulas.]  

(a) Develop an expression for the probability of loss of phase lock in an imperfect second-order PLL in \( t \) seconds. Assume that \( g(\varphi) = \sin \varphi \) and that the cycle-slipping activity is characterized by the Poisson process specified in (10-141).  
(b) Repeat (a) when a bandpass limiter precedes the loop.

Develop the expressions for the parameters \( \alpha(1) \) and \( \beta(1) \) of (10-96) which serve to characterize \( p(\phi) \) given in (10-102). [Hint: Start with (10-88), go to steady state, and then use (10-94).]

Using (10-102), determine the steady-state probability current \( J_0 \). Also justify (10-136) and (10-137) for an imperfect second-order sinusoidal PLL.

Verify the recursive formula (10-125) for the \( n \)th moment of the first-slip time.

Develop an approximate expression for the steady-state variance of the phase error rate (i.e., \( \sigma_\phi^2 \)) for an imperfect second-order loop. [Hint: Use the conditional expectation estimate \( E(y_k|\phi) \) to obtain a stochastically equivalent differential equation of operation in the phase error and phase error rate. Assume that \( \theta(t) = \Omega_0 t + \theta_0 \).]

Show that the conditional expectation \( E(y_k|\phi) \) is periodic in \( \phi \). Justify whether it possesses odd symmetry, even symmetry, or neither.

If the differenced oscillator instability process \( \{\Delta \psi(t)\} \) is characterized by the stochastic differential equation

\[
\Delta \psi(t) = n_\phi(t) + G(p)n_s(t)
\]

where

\[
G(p) = \frac{\omega_\phi^2 + \omega_\tau^2}{p + \omega_\tau}
\]

and \( \{n_\phi(t)\} \) and \( \{n_s(t)\} \) are white Gaussian noise processes of single-sided spectral density \( N_\phi \) and \( N_\tau \) respectively,
References

(a) Find the spectral density of the process \( \Delta \psi \).
(b) Find the spectral density of \( \Delta \psi \) when \( N_w = 0, \omega_w = 0 \).
(c) Find the spectral density of \( \Delta \psi \) when \( N_w = 0, \omega_w = 0 \).
(d) Repeat (c) when \( N_w = 0 \) and \( \omega_w \) approaches infinity.

10-34 Oscillator instabilities in an imperfect second-order sinusoidal PLL system are to be characterized as in Prob. 10-33.
(a) Find the canonical form for the steady-state p.d.f. \( p(\phi) \) when the loop equation is given by

\[
\dot{\phi} = \Omega_0 - KF(p)[A \sin \phi + n_s(t)] + \Delta \psi
\]

and \( F(p) \) is defined in Fig. 10-2. Assume that \( \{n_s\} \) is white Gaussian noise \( \{N_0/2\} \).
(b) Devise methods for estimating the required conditional expectations.
(c) Apply the methods in (c) to the answer obtained in part (a).

10-35 Develop expressions for the figures of merit \( (F.M.)_{IR} \) and \( (F.M.)_{III} \) of an imperfect second-order sinusoidal PLL.

10-36 Using (10-72), find the maximum rate at which the VCO should be swept in order to maintain frequency lock. Assume that at \( t = 0, e = 0 \).

10-37 Derive (10-119) and (10-121).

References


References


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11

NONLINEAR THEORY OF HIGHER-ORDER SYNCHRONOUS CONTROL SYSTEMS

11-1 Introduction

This chapter sets forth a rather general theory pertaining to the performance and synthesis of higher-order synchronous control systems. In fact, a large number of the following results generalize much of the material presented in Chapters 9 and 10. It may well be asked, why not treat the more general problem at the outset? There are at least two reasons. It is the author's opinion that one loses much of the insight gained from the thorough treatment of first- and second-order SCSs just given. Moreover, as we shall presently see, the more general theory is too massive to be handled by the beginning student. If, on the other hand, the advanced reader has skipped the detailed study given in Chapters 9 and 10, he will find that the present chapter preserves the continuity from Chapters 3 and 4.

The analysis that follows is based on the theory of continuous Markov processes, in particular, the Fokker-Planck techniques presented and discussed in detail in Chapters 7 and 8. In this chapter we consider \((N+1)\)-order synchronous control systems and show that the transition p.d.f. \(p(y, t|y_0, t_0)\) is the solution to an \((N+1)\)-dimensional Fokker-Planck equation. Here the
vector \((y, t) = (\phi, y_1, y_2, \ldots, y_N, t)\) is Markov and \(\phi\) is the modulo-2\(\pi\) version of \(\varphi\).*

According to the theory, the transition p.d.f.'s \(p(\phi, t|\phi_0, t_0), p(y_k, t|y_{k_0}, t_0); k = 1, 2, \ldots, N\) of the state variables \(\phi, y_1, y_2, \ldots, y_N\) satisfy a set of second-order partial differential equations (PDEs) that represent equations of flow taking place in each direction of \((N + 1)\)-space. Each equation (and solution) is characterized by a potential function \(U_k(y_k; t)\) that is related to the nonlinear restoring force \(h_k(y_k; t) = -\nabla U_k(y_k; t); k = 0, 1, \ldots, N\). Note that \(y_0 \triangleq \phi\). In turn, the potential functions are completely determined by the set of conditional expectations \(E(y_k, t|\phi, y_0), E(g(\phi), t|y_k, y_0); k = 1, 2, \ldots, N\).

It is the author's conjecture that the potential functions represent the projections of the system "Liapunov function" that characterize system stability.

A canonical form for the steady-state p.d.f. of the phase error is presented. Unfortunately, this canonical form depends on being able to evaluate the set of conditional expectations \(\mathcal{S}_1 = \{E(y_k|\phi), k = 1, 2, \ldots, N\}\). This prohibits the exact determination of the steady-state p.d.f. \(p(\phi)\) and leads to the search for approximate solutions and procedures for estimating the conditional expectations. Two approaches to approximating \(\mathcal{S}_1\) are taken. One is based on least-squares estimation theory, while a second is based on performing averages on the stochastic differential equations of the state coordinates, \(y_1, y_2, \ldots, y_N\).

A theory for predicting the average number of cycles slipped per unit time is given. This theory is obtained by using the methods introduced in Chapter 7. In addition, general formulas for determining the transition p.d.f.'s \(p(y_k; t)\) of the state variables \(y_k, k = 1, 2, \ldots, N\), are given. It is shown that the transition p.d.f.'s \(p(y_k; t), k = 1, 2, \ldots, N\) are solutions to second-order partial differential equations of flow. The transition p.d.f.'s are shown to be characterized by the set of conditional expectations \(\mathcal{S}_2 = \{E(g(\phi), t|y_k, y_0), k = 1, 2, \ldots, N\}\). In the steady state these second-order PDEs become first-order ordinary differential equations for which the solutions are shown. Recursive formulas for the moments of the first-passage time of the projections \(y_{k+1}, k = 0, 1, 2, \ldots, N\), are also derived. Finally, an application of the theory is made to determine the performance of a third-order loop.

### 11-2 Synchronous Control System Representation and Equivalent Model

Figure 11-1 illustrates the synchronous control system (SCS) under consideration. As discussed in Chapter 3, we distinguish between carrier tracking systems, in which the modulation lies outside the loop bandwidth, and modulation tracking systems, in which the modulation spectrum is primarily

*Here we use the vector \(y\) to denote a process which is approximately Markov. In essence, we are invoking the diffusion approximation discussed in Chapter 7, Section 7-14.*
inside the loop bandwidth. The first type of loop is used for Doppler and carrier tracking, synchronization and phase demodulation, whereas the second is used for phase and frequency demodulation of analog signals. In this chapter we present the theory of SCSS in the absence of random modulation, while the next chapter is concerned with the nonlinear theory of modulation tracking.

The equivalent system (loop) model of Fig. 11-1 is illustrated in Fig. 11-2, whereas the stochastic differential equation of operation, as determined in Chapter 3, Section 3-11, has the form

\[
\varphi = \theta - \frac{KF(p)}{p} [Ag(\varphi) + n_\varepsilon] - \frac{K_v e}{p} - \psi_2 \tag{11-1}
\]

when the phase modulation \(\theta(t)\) lies inside the loop bandwidth. If one wishes to track the carrier component, then the stochastic differential equation of interest is given by (see Chapter 3)

\[
\varphi_c = d + \varphi_1 - \frac{KF(p)}{p} [A \cos \theta g(\varphi_c - \psi_2) + n_\varepsilon] - \frac{K_v e}{p} \tag{11-2}
\]

and

\[
\varphi_d = d - \frac{KF(p)}{p} [A \cos \theta g(\varphi_d + \Delta \psi) + n_\varepsilon] - \frac{K_v e}{p} \tag{11-3}
\]
when the Doppler signal is being tracked. We shall be concerned with the
problem of obtaining the statistical dynamics produced by (11-2) and (11-3)
when $\psi_1 = \psi_2 = 0$. This says that $\varphi_e = \varphi_d$. The next chapter considers the
case of modulation tracking described in (11-1). Furthermore, since $\cos M$
can be combined with $A$, we shall consider the stochastically equivalent, integro-
differential equation

$$\varphi = d - \frac{KF(p)}{p} [Ag(\varphi) + n_z] - \frac{K_r e}{p}$$ (11-4)

in what follows. As discussed in Chapter 3, the noise process $[n_g(t)]$ has the
approximate correlation function $N \delta(t)/2$. Physically speaking, $g(\varphi)$ is the
normalized cross-correlation function (double-frequency terms neglected) be-
tween the input signal component $s(t, \Phi)$ and the control signal $r(t, \hat{\Phi})$. It will
be of interest to select the waveforms $s(t, \Phi)$ and $r(t, \hat{\Phi})$, under appropriate
constraints, and select that $F(p)$ which will optimize loop performance; that is,
we first seek the loop filter $F(p)$ which will optimize loop performance subject
to some performance criterion and then optimize further by a choice of the
nonlinearity $g(\varphi)$.

If we write the loop filter transfer function in the partial fraction ex-
pansion

$$F(p) = F_0 + \sum_{k=1}^{N} \frac{1}{1 + \tau_k p}$$ (11-5)

we have $F(p) = 1$ if $F_0 = 1$ and $F_k = 1$ for all $k = 1, 2, \ldots, N$. For $N = 1,$
$F_0 = F_1 = \tau_2/\tau_1$ reduces to the proportional-plus-integral control filter studied
in the preceding chapter. Equation (11-5) describes a class of linear filters with
poles, $p_k = 1/\tau_k$, along the negative real axis of the complex $p$ plane if $F_0$ and
$F_k$ are less than or equal to unity and if $\tau_k$ is positive and real for all $k$.* Later
in the chapter we shall consider a general expansion for any realizable linear
filter. We shall also assume that the Doppler signal is the frequency offset de-
defined by $d(t) = \Omega_0 t + \theta_0$. If we substitute $d(t)$ and (11-5) in (11-4), we can write

$$\dot{\varphi} = \Omega_0 - F_0 K [Ag(\varphi) + n_z] - K_r e$$

$$- \sum_{k=1}^{N} \frac{1}{1 + \tau_k p} [AKg(\varphi) + Kn_z]$$ (11-6)

where $\Omega_0 \Delta = \omega - \omega_0$ is the initial loop detuning. Introducing the state variable

*Strictly speaking, we mean the $s = j \omega$ vs $\sigma$ plane; however, in what follows we
interchange the role of $p$ and $s$ without any elaboration.
The \((N + 1)\)-Dimensional Fokker-Planck Equation

\[
y_k \triangleq -K \left( \frac{1 - F_k}{1 + \tau_k P} \right) [Ag(\phi) + n_k]
\]  

(11-7)

for \(k = 1, 2, \ldots, N\), we can replace (11-6) by the equivalent system of \((N + 1)\) first-order stochastic differential equations; that is,

\[
\begin{align*}
y_0 &= \dot{\phi} = \Lambda_0 - F_0K[Ag(\phi) + n] + \sum_{k=1}^{N} y_k \\
y_1 &= \frac{y_1}{\tau_1} - \frac{(1 - F_1)K[Ag(\phi) + n]}{\tau_1} \\
&\quad \vdots \\
y_N &= \frac{y_N}{\tau_N} - \frac{(1 - F_N)K[Ag(\phi) + n]}{\tau_N}
\end{align*}
\]  

(11-8)

where \(y_0 = \phi\) and \(\Lambda_0 = \Omega_0 - K_p\). Written this way, it is clear from (11-8) that the coordinates \(y_0, y_1, \ldots, y_N\) form components of a \((N + 1)\)-dimensional Markov vector \((y, t) = (\phi, y_1, \ldots, y_N, t)\), since each component, \(\dot{y}_k\), depends only on the present values of \((y, t)\) and a white Gaussian noise process. For convenience, we define the vector \(y_0 \triangleq (y_1, \ldots, y_N)\). Also, in what follows, we shall refer to a specific component of \(y\) as the projection of \(\dot{y}\). By defining the Markov extension such that \(\phi\) is one component of the vector process allows one to proceed with the analysis. Heretofore the vector process has been defined such that \(\phi\) is a weighted sum of the projections, thus leading to formidable mathematical difficulties if \(N > 1\). Furthermore, if the \(\tau_k\)'s are much greater than one (narrowband loop), then it is clear that the \(y_k\) variables are slowly varying random processes.

The most complete characterization of the state \(y\) is its statistical description by means of the transition p.d.f.—namely, \(P(y, t|y_0, y_0)\). In writing transition (conditional) p.d.f.'s, the conditioning variables will be written to the right of the vertical bar. The transition p.d.f. \(P(y; t) \triangleq P(y, t|y_0, t_0)\), of course, can be formally determined by using the theory of Markov processes given in Chapters 7 and 8. In the next section we describe the procedure.

11-3 The \((N + 1)\)-Dimensional Fokker-Planck Equation

Given the fact that the components of \(y\) form a vector Markov process, \(P(y; t)\) satisfies the \((N + 1)\)-dimensional Fokker-Planck equation of Chapter 7, that is,

\[
\frac{\partial P(y; t)}{\partial t} + \sum_{k=0}^{N} \frac{\partial}{\partial y_k} \left\{ \left[ K_k(y, t) - \frac{1}{2} \sum_{i=0}^{N} \frac{\partial}{\partial y_i} K_{ik}(y, t) \right] P(y; t) \right\} = 0
\]  

(11-9)
where the intensity coefficients $K_k(y, t)$ and $K_{ik}(y, t)$ are defined by the formulas

\[
K_k(y, t) \triangleq \lim_{\Delta t \to 0} \frac{E[\Delta y_k|y]}{\Delta t}
\]

\[
K_{ik}(y, t) \triangleq \lim_{\Delta t \to 0} \frac{E[\Delta y_i \Delta y_k|y]}{\Delta t}
\] (11-10)

and the $E[\cdot|y]$ denotes mathematical expectation of the enclosed quantity given $y$. In the case of a stationary system, the coefficients $K_k(y, t)$ and $K_{ik}(y, t)$ do not depend explicitly on $t$. We also note that we have started all sums at zero. In what follows, the concept of probability current density

\[
\mathcal{J}_k(y; t) \triangleq \left[ K_k(y, t) - \frac{1}{2} \sum_{i=0}^{N} \frac{\partial}{\partial y_i} K_{ik}(y, t) \right] P(y; t) \quad k = 0, 1, \ldots, N
\] (11-11)

will be of use when evaluating the average number of cycles slipped per unit of time. Thus, we can write the Fokker-Planck equation in the form of the equation of flow,

\[
\nabla \cdot \mathcal{J}(y; t) + \frac{\partial P(y; t)}{\partial t} = 0
\] (11-12)

where the vector $\mathcal{J}(y; t) \triangleq [\mathcal{J}_0(y; t), \ldots, \mathcal{J}_N(y; t)]$ may be interpreted as a probability current density vector and $\nabla$ is the differential operator for space, the del operator,

\[
\nabla \triangleq \left[ \frac{\partial}{\partial y_0}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_N} \right]
\] (11-13)

We will also need to consider the probability current of the $k$th projection of $\mathcal{J}(y; t)$; that is,

\[
\mathcal{J}_k(y_k; t) \triangleq \int \cdots \int \mathcal{J}_k(y; t) \, dy_k
\] (11-14)

where $dy'_k \triangleq dy_0, dy_1, \ldots, dy_{k-1}, dy_{k+1}, \ldots, dy_N$. The probability current of the $k$th projection describes the amount of probability crossing the hyperplane $y_k = y'_k$ in the positive direction per unit time. A geometric interpretation of (11-14) is possible. Define $n_k$ to be a unit vector pointed along the positive direction of the $y_k$ axis. Then $\mathcal{J}(y; t) \cdot n_k = \mathcal{J}_k(y; t)$ represents the probability current density flowing in the positive $y_k$ direction, and $\mathcal{J}_k(y; t) \, dy'_k$ represents
the amount of probability current flowing through the differential surface area $dy_k'$. Integrating over the surface gives the total probability current flowing through the hyperplane $y_k = y_k'$. Just as the equation of heat conduction involves a flow of heat, (11-9) involves a flow of probability.

Before proceeding we will give a graphic physical interpretation of the Fokker-Planck equation that will prove useful and somewhat picturesque. For every simple function of a vector Markov process, the vector trajectory $y(t)$ can be thought of as the path of a point starting from $y(t_0)$ in an $(N + 1)$-dimensional space $y = (\phi, y_1, y_2, \ldots, y_N)$. The position of this point at time $t$—that is, $[y_0(t), \ldots, y_N(t)]$—may be envisioned as a Brownian particle undergoing diffusion in an $(N + 1)$-space as a function of time. The set of sample functions of the process $\{y(t)\}$ is the ensemble of trajectories that move about in a random manner. The fraction of time that the particle spends in any region of the probability space is proportional to the total probability in that region.

Since the noise is a stationary process and the system is time invariant, the Markov process $y$ is temporally homogeneous (see Chapter 7) and its statistical incremental moments defined in (11-10) are time independent. The coefficients required for (11-9) can be straightforwardly evaluated by using (11-8) and (11-10). The differential equation whose solution describes the p.d.f. $P(y; t)$ can then be written by mere substitution of these coefficients $K_k$ and $K_{ik}$ into (11-9). Such a computation yields

\[
\frac{\partial P(y; t)}{\partial t} = -\frac{\partial}{\partial \phi} \left[ \Lambda_0 - F_0 AKg(\phi) + \sum_{k=1}^{N} y_k \right] P(y; t)
\]

\[
+ \frac{F_0^2 N_0 K^2}{4\phi^2} P(y; t)
\]

\[
+ \sum_{k=1}^{N} \left[ \frac{\partial}{\partial y_k} y_k - \frac{1}{\tau_k} AKg(\phi) \right] P(y; t)
\]

\[
+ \left[ \frac{(1 - F_k)^2 N_0 K^2}{4\tau_k^2} \frac{\partial^2}{\partial y_k^2} + F_0 (1 - F_k) N_0 K^2 \frac{\partial^2}{\partial \phi \partial y_k} \right] P(y; t)
\]

\[
+ \sum_{k=1}^{N} \sum_{l=0}^{N} \frac{(1 - F_k)(1 - F_l) N_0 K^2}{4\tau_k \tau_l} \frac{\partial^2}{\partial y_k \partial y_l} P(y; t)
\]

(11-15)

With $N = 1$ and $F_0 = F_1$, (11-14) becomes the FP equation for the SCS system with the proportional-plus integral-control loop filter treated in Chapter 10.

11-4 Initial Conditions and the Periodic Extension Method

In order to obtain solutions to the FP equation, we have to supplement it with initial conditions and boundary conditions. For our purposes, we specify the initial p.d.f. $P(y; t)$ as time $t = t_0$ to be
\[ \lim_{t \to t_0} P(y; t) = \prod_{k=0}^{N} \delta[y_k - y_{k0}] \]  

which says that \( y_0 \triangleq (\varphi_0, y_{10}, \ldots, y_{N0}) \) is the initial value of \( y \) at \( t = t_0 \). The subsequent evolution of this p.d.f. is found from the normalization condition

\[ \int \cdots \int P(y, t|y_0, t_0) \, dy = 1 \]  

(11-17)

which must also be satisfied for all \( t \geq t_0 \).

It is evident that if \( g(\varphi) \) is periodic with period \( 2\pi \), then the coefficients of the FP equation are also periodic in \( \varphi \). Thus if \( P(y; t) \) is the principal solution of (11-15) for the initial condition (11-16), then \( P(\varphi \pm 2n\pi, y'; t) \) is the principal solution for the initial condition

\[ \lim_{t \to t_0} P(\varphi \pm 2n\pi, y', t|y_0, t_0) = \delta[\varphi \pm 2n\pi - \varphi(t_0)] \prod_{k=1}^{N} \delta[y_k - y_{k0}] \]  

(11-18)

where \( y'_0 = (y_1, \ldots, y_N) \) and \( n \) is any integer.

Because noise is present, the loop slips cycles at a certain mean rate, executing a random nonstationary motion between “lock-in” points, much like the discrete random walk discussed in Chapter 7. This cycle slipping phenomenon gives rise to the diffusion of the phase error process from its initial distribution to a steady-state distribution with an infinite variance. As a consequence of the cycle slipping phenomenon, it follows that

\[ \lim_{t \to \infty} P(\varphi, y'_0, t|y_0, t_0) = \lim_{t \to \infty} P(\varphi \pm 2n\pi, y'_0, t|y_0, t_0) = 0 \]  

(11-19)

which says that the steady-state variance of \( \varphi \) is infinite.

In order to obtain a mathematically tractable transition p.d.f. that diffuses to a steady-state p.d.f. with a finite variance, we define the periodic extension

\[ \tilde{p}(\varphi, y'_0, t|y_0, t_0) \triangleq \sum_{n=-\infty}^{\infty} P(\varphi \pm 2n\pi, y'_0, t|y_0, t_0) \]  

(11-20)

for all \( |\varphi| \leq \infty \). This function satisfies the FP equation, for (11-15) is linear in \( P(y; t) \); that is, since each term of the series in (11-20) is a principal solution of (11-15), with initial conditions given by (11-18), the infinite sum (11-20) satisfies (11-15) with the initial condition.
\[
\lim_{t \to t_0} \tilde{p}(y; t) = \prod_{k=1}^{N} \delta(y_k - y_{k0}) \sum_{n=-\infty}^{\infty} \delta(\varphi - \varphi_0 + 2n\pi) \tag{11-21}
\]

From (11-21) we note that \(\tilde{p}\) is periodic in \(\varphi\) and that \(\tilde{p}\) is not a transition p.d.f., for it is an infinite sum of transition p.d.f.'s, each having unit area. More important is the fact that the initial condition (11-21) is not of the form for \(\tilde{p}\) to be a transition p.d.f. Therefore, in order to obtain a mathematically tractable transition p.d.f., further considerations are required. First we note that the phase error trajectories \(\varphi(t)\) can be written as \(\{\varphi(t)\} = \{\phi(t)\} + 2\pi [J(t)]\), where \(\{\phi(t)\}\) represents the modulo \(2\pi\) version of \(\{\varphi(t)\}\) about the axes \(\varphi = (2n-1)\pi\) and \(\varphi = (2n+1)\pi\). The process \([J(t)]\) accounts for the net number of phase-jumps that have occurred since \(t = t_0\). In fact, at any instant of time \(J(t)\) is a discrete r.v. that takes on values \(j = 0, \pm 1, \pm 2, \ldots\). At \(t = t_0\) we note that \(\varphi = \varphi_0 = \phi_0\), for \(J(t_0) = 0\). Notice also that \(\tilde{p}(\phi, y_0, t) = \tilde{p}(\phi + 2n\pi, y_0, t, t_0)\) if \(\phi \in I(n)\). If we define the conditional (on \(n\)) transition p.d.f.

\[
p(\phi, y_0, t_0, n) \triangleq \begin{cases} 
\tilde{p}(\phi + 2n\pi, y_0, t|y_0, t_0) & \phi \in I(n) \\
0 & \text{elsewhere}
\end{cases} \tag{11-22}
\]

where \(I(n) \triangleq [(2n-1)\pi, (2n+1)\pi]\), \(n\) any fixed integer, then we see that the conditional transitional p.d.f. is defined in the hyperslab (probability space)

\[
\Omega \triangleq \begin{cases} 
(\phi, y_0, t), & \phi \in I(n) \\
|y_k| \leq \infty, & k = 1, 2, \ldots, N
\end{cases} \tag{11-23}
\]

We note that \(p(\phi, y_0, t|y_0, t, n)\) satisfies (11-15) with the initial condition

\[
\lim_{t \to t_0} p(\phi, y_0|y_0, t_0, n) = \delta(\phi - \phi_0 - 2n\pi) \prod_{k=1}^{N} \delta(y_k - y_{k0}) \tag{11-24}
\]

and the normalization condition

\[
\int_{-\infty}^{\infty} \cdots \int_{(2n+1)\pi}^{(2n+1)\pi} p(\phi, y_0, t|y_0, t_0, n) \, dy = 1 \tag{11-25}
\]

for all \(t \geq t_0\). In passing, we also note that \(p(\phi, y_0, t) \equiv p(\phi, y_0, t|y_0, t_0, n)\) is equivalent to the solution to the FP equation (11-15) when the phase error trajectories \(\varphi(t)\) are folded modulo-\(2\pi\) about the axes \(\varphi = (2n-1)\pi\) and \(\varphi = (2n+1)\pi\). In the following sections we shall be concerned with the problem of determining the steady-state marginal p.d.f.'s of the variables \(\phi\) and \(y_k, k = 1, 2, \ldots, N\). We note that at any time
\[ \varphi = \phi + 2j\pi \]  

(11-26)

where \( j \) represents the net number of \( 2\pi \) phase-jumps appearing in the modulo-
\( 2\pi \) phase error process.

11-5 Boundary Conditions

The FP equation—that is, (11-15) with \( P(y; t) \) replaced by \( p(y; t) \)—is a second-order, parabolic, partial differential equation with respect to each of the \( N + 1 \) random variables \( \phi, y_k; k = 1, 2, \ldots, N \). Therefore in order to determine its principal solution, we must specify \( 2(N + 1) \) independent boundary conditions. In general, these conditions are determined by the physics of the problem. Moreover, they must be chosen such that the normalization condition (11-25) is satisfied for all \( t \).

Since the \( y_k \)'s can take on all possible values from \(-\infty \) to \(+\infty \), the boundary conditions for these r.v.'s take the form of conditions at \( y_k = \pm \infty \). Due to (11-25), the function \( p \), viewed as a function of \( y_k \) with all other variables kept fixed, must approach zero faster than \( |y_k|^{-(1+\epsilon)} \), \( \epsilon > 0 \), as \( |y_k| \to \infty \), \( k \neq 0 \). Therefore along any edge of the hyperslab \( \Omega \)—that is, the edges \( |y_k| = \pm \infty \)—we obtain the \( N \) boundary conditions

\[
\lim_{{|y_k| \to \infty}} y_k p(\phi, y_0', y_0, t | y_0, t_0, n) = 0, \quad k = 1, \ldots, N
\]

(11-27)

In addition, since \( p \) is bounded above by \( |y_k|^{-(1+\epsilon)} \) as \( y_k \) approaches infinity, we note that

\[
\lim_{{|y_k| \to \infty}} p(\phi, y_0', t | y_0, t_0, n) \leq \lim_{{|y_k| \to \infty}} |y_k|^{-(1+\epsilon)}
\]

(11-28)

By differentiating this with respect to \( y_k \) and assuming that one can interchange the order of differentiation and limit, we obtain the \( N \) additional independent boundary conditions.

\[
\lim_{{|y_k| \to \infty}} \frac{\partial}{\partial y_k} p(\phi, y_0', t | y_0, t_0, n) = 0, \quad k = 1, \ldots, N
\]

(11-29)

Now \( \hat{p}(\phi, y_0'; t) = \hat{p}(\phi, y_0', t | y_0, t_0) \) is periodic in \( \phi \) because it is the sum of periodic functions; therefore

\[
p[(2n + 1)\pi, y_0'; t] = p[(2n - 1)\pi, y_0'; t]
\]

(11-30)

It then follows from (11-30) that
\[
\frac{\partial p[(2n + 1)\pi, y_0'; t]}{\partial y_k} = \frac{\partial p[(2n - 1)\pi, y_0'; t]}{\partial y_k}
\]  
(11-31)

for \(k = 1, 2, \ldots, N\). Finally, if probability flow is to be conserved in all directions of the coordinate axes, and since \(\rho(\phi, y_0'; t)\) is periodic, we have

\[
\left.\frac{\partial p(\phi, y_0'; t)}{\partial \phi}\right|_{\phi = (2n + 1)\pi} = \left.\frac{\partial p(\phi, y_0'; t)}{\partial \phi}\right|_{\phi = (2n - 1)\pi}
\]  
(11-32)

Equations (11-27), (11-29), (11-30), and (11-31) define \(2(N + 1)\) independent boundary conditions. From (11-31) and (11-32) we can write

\[
\nabla p(y; t)|_{\phi = (2n + 1)\pi} = \nabla p(y; t)|_{\phi = (2n - 1)\pi}
\]  
(11-33)

In passing, we point out the fact that if \(\Lambda_0 = 0\), the symmetry of (11-8) indicates that \(p(y; t) = p(-y; t)\). Using the definition of \(J_0(\phi, y_0'; t)\), it is easy to show that

\[
J_0[(2n + 1)\pi, y_0'; t] = J_0[(2n - 1)\pi, y_0'; t]
\]  
(11-34)

which says that the probability current density in the \(\phi\) direction evaluated at the boundaries \((2n + 1)\pi\) and \((2n - 1)\pi\) are equal. Consequently, the boundary conditions of \(p(y; t)\) at \(\phi = (2n + 1)\pi\) or \(\phi = (2n - 1)\pi\) are such that the normalization condition (conservation of probability) is satisfied.

It is interesting to note the merger of Maxwell's equations (Chapter 6) for stochastic fields with the theory of Markov processes, in particular, the Fokker-Planck equation. Using the Gauss theorem, we note

\[
\oint_{\Omega} \nabla \cdot J dy = \oint_{\Gamma} n \cdot J d\Gamma = -\frac{\partial}{\partial t} \oint_{\Omega} p(y; t) dy = 0
\]  
(11-35)

where \(n\) is the unit vector normal to the surface \(\Gamma\) of \(\Omega\) and directed positively outward. From Maxwell's field equations we know that the divergence of the probability current density \(J\) is just the time rate of change of the probability density \(p(y; t)\). Also, the divergence of the probability flux density \(D\) is equal to \(p\). Hence if we interpret \(\rho\) as the transition p.d.f. \(p(y; t)\) and \(D\) as a probability flux density, then we write \(\nabla \cdot D = p(y; t)\); that is, the net probability flux flowing out of a volume \(dy\) at time \(t\) is just equal to the probability of being in that volume at time \(t\).

Interestingly enough, if we integrate both sides of (11-15) with respect to \(y_j\) for all \(j \neq k \neq 0\) and make use of the boundary conditions (11-27) and (11-29), we arrive at

\[
\frac{\partial p}{\partial t} + \frac{\partial}{\partial \phi} [J_0(\phi, y_k; t)] + \frac{\partial}{\partial y_k} [J_k(\phi, y_k; t)] = 0
\]  
(11-36)
where \( p = p(\phi, y_k; t) = p(\phi, y_k, t|y_0, t_0, n) \),

\[
K_0(\phi, y_k, t) = \Lambda_0 + \sum_{j \neq k \neq 0}^N E(y_j, t|\phi, y_k, y_0) + y_k - AKF_0 g(\phi) \\
K_k(\phi, y_k, t) = -\frac{1}{\tau_k} [y_k + (1 - F_k)AKg(\phi)] \\
K_{00} = \frac{N_0 F_0^2 K^2}{2} \\
K_{0k} = K_{k0} = \frac{(1 - F_k)F_0 N_0 K^2}{2 \tau_k} \\
K_{kk} = \frac{(1 - F_k)^2 N_0 K^2}{2 \tau_k^2} \quad k \neq 0 \tag{11-38}
\]

for \( k = 1, 2, \ldots, N \) and the probability currents are defined by

\[
\mathcal{J}_0(\phi, y_k; t) = \left\{ \left[ K_0(\phi, y_k, t) - \frac{K_{00}}{2} \frac{\partial}{\partial \phi} - \frac{K_{k0}}{2} \frac{\partial}{\partial y_k} \right] p \right\} \\
\mathcal{J}_k(\phi, y_k; t) = \left\{ \left[ K_k(\phi, y_k, t) - \frac{K_{kk}}{2} \frac{\partial}{\partial y_k} - \frac{K_{0k}}{2} \frac{\partial}{\partial \phi} \right] p \right\} \tag{11-39}
\]

In (11-37) the quantity \( E(y_j, t|\phi, y_k, y_0) \) denotes a conditional expectation.

### 11.6 Differential Equations for the Transition Probability Density Functions

To find \( p(y_k; t), k = 0, 1, \ldots, N \), we first need a differential equation whose solution is indeed \( p(y_k; t) \). This equation is easily found by integrating both sides of (11-36) with respect to \( y_k \) and applying the appropriate boundary conditions. Without belaboring the details, we obtain the partial differential equation of flow in the \( k \)th direction

\[
\nabla \cdot \mathcal{J}_k(y_k; t) + \frac{\partial p(y_k; t)}{\partial t} = 0 \tag{11-40}
\]

with probability current

\[
\mathcal{J}_k(y_k; t) = \left\{ \left[ K_k(y_k, t) - \frac{K_{kk}}{2} \frac{\partial}{\partial y_k} \right] p(y_k; t) \right\} \tag{11-41}
\]

where, for \( k = 0 \),

\[
K_0(\phi, t) = \Lambda_0 - AKF_0 g(\phi) + \sum_{k=1}^N E(y_k, t|\phi, y_0) \tag{11-42}
\]
and for all $k \neq 0$,

$$K_k(y_k, t) = - \left[ y_k + AK(1 - F_k) E(g(\phi), t | y_k, y_0) \right] \tau_k$$  \hspace{1cm} (11-43)

where $E(y_k, t | \phi, y_0)$ is the conditional expectation of $y_k$ given $\phi$ at time $t$ and the initial value $y_0$ of the states at $t = t_0$. Similarly, $E(g(\phi), t | y_k, y_0)$ is the conditional expectation of $g(\phi)$ given $y_k$ at time $t$ and $y_0$ at time $t_0$.

It is convenient to introduce into (11-41) the nonlinear restoring force

$$h_k(y_k, t) \triangleq \frac{2K_k(y_k, t)}{K_{kk}}$$  \hspace{1cm} (11-44)

and the potential function

$$U_k(y_k; t) = - \int_{y_k}^{y_k} h_k(x; t) \, dx \hspace{0.5cm} \text{or} \hspace{0.5cm} h_k(y_k; t) = - \frac{\partial U_k(y_k; t)}{\partial y_k}$$  \hspace{1cm} (11-45)

so that (11-41) becomes

$$\mathcal{F}_k(y_k; t) = - \frac{K_{kk}}{2} \exp \left[ -U_k(y_k; t) \right] \times \frac{\partial}{\partial y_k} \left[ p(y_k; t) \exp \left[ U_k(y_k; t) \right] \right]$$  \hspace{1cm} (11-46)

since the $K_{kk}$'s are constants. Assuming in the limit as $t$ approaches infinity that $p(y_k; t)$ approaches the steady-state p.d.f. $p(y_k)$, the stationary diffusion current is constant and obeys the law

$$\mathcal{F}_k = - \frac{K_{kk}}{2} \exp \left[ -U_k(y_k) \right] \frac{\partial}{\partial y_k} \left[ p(y_k) \exp \left[ U_k(y_k) \right] \right]$$  \hspace{1cm} (11-47)

and we have suppressed the dependence of $p(y_k)$ on $n$. Solving (11-47) for $p(y_k)$ yields

$$p(y_k) = C_k \exp \left[ -U_k(y_k) \right] \left\{ 1 + D_k \int_{l_k}^{y_k} \exp \left[ U_k(x) \right] \, dx \right\}$$  \hspace{1cm} (11-48)

where $D_k = -2 \mathcal{F}_k / C_k K_{kk}$, and the lower limit $l_k = (2n - 1)\pi$ if $k = 0$ and $l_k = -\infty$ if $k \neq 0$.

To evaluate the constants $C_k$ and $D_k$ for $k = 0$, we make use of the boundary conditions. For $n = 0$, $p(\pi) = p(-\pi)$ from (11-30); and from (11-48), with $y_0 = \phi$, we can write
\begin{equation}
D_0 = \frac{\exp [-U_0(-\pi)] - \exp [-U_0(\pi)]}{\exp [-U_0(\pi)] \int_{-\pi}^{\pi} \exp [U_0(x)] \, dx} \tag{11-49}
\end{equation}

The constant \( C_0 \) is easily determined by means of the normalization condition. Furthermore, since \( g(\phi) \) is periodic and continuous, we may also write (see Appendix I, Chapter 9)

\begin{equation}
p(\phi|n) = C_0' \exp [-U_0(\phi)] \int_{\phi}^{\phi+2\pi} \exp [U_0(x)] \, dx \tag{11-50}
\end{equation}

for any \( \phi \) belonging to an interval of width \( 2\pi \) as defined in (11-22), and \( U_0(\phi) \) is defined in (11-45) with \( k = 0 \). This p.d.f. represents the canonical form for the phase error distribution. We note that (11-50) is independent of \( n \) so that \( p(\phi|n) = p(\phi) \).

Equations (11-48) and (11-50) are remarkable in that they hold for all order loops and a broad class of nonlinearities. In fact, it is clear from (11-42) and (11-50) that the p.d.f. of the phase error of an \( (N + 1) \)-order loop is completely determined by the set of conditional expectations \( \mathcal{J}_1 = \{E(y_k|\phi)\} \) for all \( k \). Interestingly enough, \( E(y_k|\phi) \) is the minimum mean-square-error estimate of \( y_k \) given \( \phi \).

For \( k \neq 0 \), the constant \( C_k \) is a normalization constant for \( p(y_k) \), while the constant \( D_k \) is determined from the boundary condition \( p(y_k) = 0 \) at \( y_k = \pm \infty \). Thus from (11-48)

\begin{equation}
D_k = -\left\{ \int_{-\infty}^{\infty} \exp [U_k(x)] \, dx \right\}^{-1} \tag{11-51}
\end{equation}

When the \( D_k \)'s are zero (i.e., \( \mathcal{J}_k = 0 \)), the marginal p.d.f.'s reduce to

\begin{equation}
p(y_k) = C_k \exp [-U_k(y_k)] \tag{11-52}
\end{equation}

for all \( k = 0, 1, \ldots, N \). When the conditions are such that (11-52) is true, the \( k \)th projection is not allowed to penetrate the region of infinitely large positive or negative values, \( k > 0 \).

In order to obtain explicit solutions for \( p(y_k) \), it appears that the conditional expectations must either be approximated or measured by the method of computer simulation. It is remarkable that the marginal p.d.f.'s \( p(y_k) \) are determined by the set of conditional expectations \( \{E(y_k|\phi), E(g(\phi)|y_k)\} \) for all \( k \).

From (11-48) it is clear that, in the steady state, the means of the coordinates \( y_k, k \neq 0 \), are
\[ \ddot{y}_k = -(1 - F_k)AKg(\phi), \quad k = 1, 2, \ldots, N \quad (11-53) \]

where the overbar denotes statistical average. Furthermore, for small signal strengths (i.e., small \( AK \)), it is seen that the \( y_k \)'s are zero-mean Gaussian random variables having variance

\[ \sigma_k^2 = (1 - F_k)^2 \frac{K^2 N_0}{4\tau_k} \quad k = 1, 2, \ldots, N \quad (11-54) \]

For \( k = 0 \), it is clear from (11-8) and (11-53) that the mean of the phase error rate \( \phi \), that is, the mean residual frequency detuning, is given by

\[ \bar{\phi} = \Lambda_0 - AKF_0 g(\phi) + \sum_{k=1}^{N} \ddot{y}_k = \Lambda_0 - AKF(0) g(\phi) \]

\[ = 2\pi \mathcal{J}_0 \quad (11-55) \]

where \( F(0) \) is the value of \( F(s) \) at \( s = 0 \).

To this end we have assumed that the transition p.d.f. \( \rho(y; t) \) exists for all \( t > t_0 \). It is clear, however, that for certain loop designs the steady-state p.d.f.'s may not exist, for the loop may go unstable because of a large loop gain or the form of \( F(s) \) itself. It is the conjecture of this author that the set of potential functions \( \{ U_k(y_k, t), k = 0, 1, \ldots, N \} \) form the basis from which one may construct a "Liapunov function," say \( V(y, t) \), where \( V(y, t) \) represents the total potential at the point \( y(t) \). If for all \( y \in \Omega, t > t_0 \), this function behaves as required by Liapunov's asymptotic stability theorem (Ref. 1), which says that if there exist three positive definite functions \( W(y, t), W_1(y, t), W_2(y, t) \) such that in \( \Omega \) and for \( t \geq t_0 \) we can demonstrate that \( W_1(y, t) \geq V(y, t) \geq W(y, t); V(y, t) \leq W_2(y, t) \) and, in addition, that in \( \Omega \) all the partial derivatives \( \partial V/\partial y_k \) are bounded for \( t \geq t_0 \)—that is, \( |\partial V/\partial y_k| \leq M, k = 0, 1, \ldots, N \)—then \( \rho(y; t) \) exists for all \( t \geq t_0 \). Obviously the ability to demonstrate that \( U_k(y_k, t) \) is bounded for all \( y_k \) and \( t \geq t_0 \) is directly related to the problem of showing that the conditional expectations \( E(y_k, t|\phi, y_0) \) and \( E(g(\phi), t|y_k, y_0) \) are bounded. Considerable work needs to be devoted to this problem.

11-7 Synchronous Control Systems with \( F_0 = 0 \)

Going back to (11-36) and (11-39), we see that when \( F_0 = 0 \), the reduced FP equation degenerates and the technique previously used fails. There is then no other alternative than to solve (11-15) with \( F_0 = 0 \), although it is possible to reduce it somewhat by integrating with respect to \( y', \ldots, y_N \) and by using the boundary conditions (11-27) and (11-29).
This procedure yields the reduced FP equation
\[
\frac{\partial}{\partial y_j} \left\{ \left[ \frac{y_j}{\tau_j} + \frac{AK(1 - F_j)g(\phi)}{\tau_j} \right] p(\phi, y_j; t) \right\} + \frac{K_{vL}}{2} \frac{\partial^2}{\partial y_j^2} p(\phi, y_j; t) \\
- \frac{\partial}{\partial \phi} \left\{ \left[ \Lambda_0 + y_j + \sum_{k \neq j \neq 0} E(y_k, t|\phi, y_j, y_0) \right] p(\phi, y_j; t) \right\} = \frac{\partial p(\phi, y_j; t)}{\partial t}
\]
(11-56)

For the case \( N = 1, \Lambda_0 = 0 \), we obtain in the steady state
\[
p(\phi, \dot{\phi}) = C \exp \left[ -\frac{\tau_p}{2AK} \dot{\phi}^2 - \rho \int_0^\phi g(x) \, dx \right]
\]
(11-57)

where \( |\dot{\phi}| < \infty, \phi \in [(2n - 1)\pi, (2n + 1)\pi] \) for \( n \) any fixed integer, \( \dot{\phi} = y_1 \), and \( \rho \) is the signal-to-noise ratio existing in the loop bandwidth \( W_L = 2B_L = AK/2 \); that is, \( \rho = A^2/N_0B_L = 2A^2/N_0W_L \). The parameter \( C \) is a normalization constant. (See Chapter 10, Section 10-8 for an approximate solution when \( \Lambda_0 \neq 0 \).)

Note that the density \( p(\phi, \dot{\phi}) \) can be written as \( p(\phi)p(\dot{\phi}) \); \( \phi \) and \( \dot{\phi} \) are therefore statistically independent r.v.'s. Furthermore, we note that for \( \Lambda_0 = 0, p(\phi) \) is identical to the expression for a first-order loop and that \( \dot{\phi} \) is a Gaussian r.v. with variance \( N_0K^2/4\tau \). For this case it is interesting to note that the equation of loop operation is identical to that arising in an induction or synchronous motor operating under a randomly time-varying load. Here \( \phi \) is the power angle between the rotor and armature fields, \( \tau \dot{\phi} \) represents the inertia torque due to the mass of the rotor and its connected load, the term \( \dot{\phi} \) arises due to slip of the squirrel cage winding relative to the rotating field, \( AK \sin \phi \) is due to the angle between the fields of the rotor and armature, and \( n_s(t) \) represent the random external load on the motor.

11-8 Evaluating the Steady-State Conditional Expectations \( E(y_k|\phi) \) and the Steady-State Density \( p(\phi) \) for an \( (N + 1) \)-Order SCS

In the analysis of most problems of a nonlinear nature, it is usually the case that somewhere along the line a suitable approximation must be introduced in order to proceed with the analysis. We present, in this section, two rather general methods for approximating conditional expectations. The first method, which is by far the simplest, is based on the orthogonality principle to produce the least-squares estimate of the conditional expectation \( E(s|\phi) \), where \( s \) is any state variable. The second method modifies and generalizes a method due to Viterbi (Ref. 2) and Holmes (Ref. 3). This method, as we shall see, is based on the differential equations of the state variables that relate the state variables \( y_k; k = 1, 2, \ldots, N \) to the phase error process.
11-8.1 The Method of Least-Squares Estimation

Since the periodic extension of \( p(\phi) \) is periodic, the implication is that
the conditional expectation \( E(s|\phi) \) is periodic in \( \phi \). Thus we can write

\[
E(s|\phi) = E(s|\phi) - \bar{s} + \bar{s} = E(s - \bar{s}|\phi) + \bar{s}
\]

(11-58)

In applying the method, we wish to estimate \( E(s - \bar{s}|\phi) \) by \( \kappa[g(\phi) - \bar{g}] \), where
\( \kappa \) is chosen so that the mean square error

\[
\bar{e}^2 = E_\phi[E[(s - \bar{s})|\phi] - \kappa[g(\phi) - \bar{g}]]^2
\]

is minimized. Here we make use of the fact that \( E(s|\phi) \) is periodic in \( \phi \) and we
approximate it by two terms. Using the orthogonality principal, it is an easy
matter to show that a minimum is met by choosing \( \kappa \) such that

\[
\kappa = \frac{E[(s - \bar{s})(g(\phi) - \bar{g})]}{\sigma^2_G} = \frac{R_{SG}(0)}{\sigma^2_G}
\]

(11-59)

where \( S \triangleq s - \bar{s}, G = g(\phi) - \bar{g}, \sigma^2_G \) is the variance of \( G \), and \( R_{SG}(0) \) is the
cross-correlation function of \( S \) and \( G \) evaluated at zero shift. Thus the least-squares estimate \( \hat{E}(s|\phi) \) of \( E(s|\phi) \) is

\[
\hat{E}(s|\phi) = \frac{R_{SG}(0)}{\sigma^2_G} G(\phi) + \bar{s}
\]

(11-60)

where \( s \) is any state variable in \( y_0'. \) The minimum error is easily evaluated.

As a result of this approximation, we have that \( U_0(\phi) \) in (11-45) reduces to

\[
\hat{U}_0(\phi) = -\beta(N)\phi - \alpha(N) \int^{\phi} g(\chi) d\chi
\]

(11-61)

when we make use of (11-42), (11-44), and (11-60). Here

\[
\alpha(N) = \frac{2}{K_{00}} \left[ AKF_0 - \frac{1}{\sigma^2_G} \sum_{k=1}^{N} R_{\phi G}(0) \right]
\]

\[
\beta(N) = \frac{2}{K_{00}} \left[ \Lambda_0 - \frac{\bar{g}}{\sigma^2_G} \sum_{k=1}^{N} \left( R_{\phi G}(0) + \frac{\sigma^2_{\bar{g}} y_k}{\bar{g}} \right) \right]
\]

(11-62)

Use of (11-61) in (11-50) produces an approximate result for \( p(\phi) \). Obviously
\( p(\phi) \) will be symmetric only if \( \beta(N) = 0 \)—that is,
\[ \Lambda_0 = \frac{\bar{g}}{\sigma_0^2} \left[ \sum_{k=1}^{N} \left( R_{yy}(0) + \frac{\sigma_y^2 \bar{y}_k}{\bar{g}} \right) \right] \quad (11-63) \]

where \( Y_k = y_k - \bar{y}_k \).

**11-8.2 Estimation of \( E(y_k|\phi) \) via the Stochastic Differential Equation of State**

Even though this method lacks in strict mathematical rigor, it does produce results that render and warrant its application. This fact is borne out by those who have handled the experimental side of the problem.

We begin by multiplying both sides of (11-8) by \( \exp(v/\tau_k) \) to obtain (see Appendix II of Chapter 10)

\[
\frac{d}{dv} \left[ y_k(v) \exp\left( \frac{v}{\tau_k} \right) \right] = -\frac{(1 - F_k)K}{\tau_k} \times [Ag(\phi) + n(v)] \exp\left( \frac{v}{\tau_k} \right) \quad (11-64)
\]

and we set \( n_x = n \) for convenience. If we now take expectations of both sides conditioned upon \( \phi(t) \) and interchange the order of expectation with differentiation, we have

\[
\frac{d}{dv} \left[ E[y_k(v)|\phi(t)] \exp\left( \frac{v}{\tau_k} \right) \right] = -\frac{(1 - F_k)K}{\tau_k} \times [AE[g(\phi(v))|\phi(t)] + E[n(v)|\phi(t)] \exp\left( \frac{v}{\tau_k} \right)] \\
(11-65)
\]

Integrating both sides of (11-65) from \( t \) to infinity [the noise term drops out, since it is independent of \( \phi(t) \) in the future] and introducing the change of variables \( v = t + \tau \) yields, in the steady state,

\[
E(y_k|\phi) = \frac{AK(1 - F_k)}{\tau_k} \int_0^\infty \exp\left( \frac{\tau}{\tau_k} \right) E[(g(\phi(t + \tau))] \\
- \frac{g(\phi(t + \tau))|\phi(t)]}{g(\phi_1)} d\tau - AK(1 - F_k)g(\phi(t + \tau) \quad (11-66)
\]

for all \( k = 1, 2, \ldots, N \). We note that \( g(\phi(t + \tau)] \) is strictly stationary in the steady state, for \( \phi(t) \) reduced modulo-\( 2\pi \) is strictly stationary in the steady state. The expectation under the integral sign of (11-66) may be estimated by using the orthogonality principle (OP) to find the best \( \rho_0(\tau) \) such that \( E[[g(\phi_2) - g(\phi_2)]|\phi_1] \) is estimated by \( \rho_0(\tau)[g(\phi_1) - g(\phi_1)] \) in the best linear mean square
sense. Here $\phi_2 = \phi(t + \tau), \phi_1 = \phi(t), \text{ and } \rho_o(\tau) \text{ approaches zero as } \tau \text{ approaches minus infinity. First we define the error } \epsilon$

$$
\epsilon \triangleq E_{\phi_1}[[E_{\phi_2}([g(\phi_2) - \bar{g}(\phi_2)]\phi_1) - \rho_o(\tau)[g(\phi_1) - \bar{g}(\phi_1)]^2]] 
$$

(11-67)

Let

$$
y(\phi_1) \triangleq E_{\phi_1}[g(\phi_2) - \bar{g}(\phi_2)]\phi_1 
$$

(11-68)

$$
x(\phi_1) \triangleq [g(\phi_1) - \bar{g}(\phi_1)]
$$

and write

$$
\epsilon = E_{\phi_1}[[y(\phi_1) - \rho_o(\tau)x(\phi_1)]^2] 
$$

(11-69)

Thus the function $\rho_o(\tau)$, which produces the mean square estimate $E[g(\phi(t + \tau) - g(\phi(t + \tau))|\phi_1]$ is easily shown, using the OP, to be given by

$$
\rho_o(\tau) = \frac{R_x(\tau) - (\bar{g})^2}{\sigma_o^2} = \frac{R_o(\tau)}{\sigma_o^2} 
$$

(11-70)

where for a stationary $\phi$ process, $\bar{g}(\phi_1) = g(\phi_2) = \bar{g}$ and $G = g(\phi) - \bar{g}$. The minimum mean square error is easily evaluated. Also, we note that a random process $\{x(t)\}$ is said to be of the separable class if it is second-order stationary and satisfies $E[x(t + \tau)|x(t)] = \rho_x(\tau)x$ for all $\tau$. The correlation function $\rho_x(\tau) = E[x(t + \tau)x(t)]/E(x^2)$.

Replacing the expectation $E[(g(\phi_2) - \bar{g})|\phi_1]$ by the mean square estimate $\rho_o(\tau)[g(\phi) - \bar{g}]$ in (11-66), we have

$$
\hat{E}[y_k|\phi] = \frac{AK(1 - F_k)G(\phi)}{\tau_k} \int_0^\infty \rho_o(\tau) \exp\left(\frac{\tau}{\tau_k}\right) d\tau
$$

$$
- AK(1 - F_k)\bar{g} \left(1 + \frac{S_o(0)}{2\tau_k\sigma_o^2}\right) 
$$

(11-71)

and the caret is used to denote the fact that we have used (11-70). If the loop is designed such that the correlation time of $\rho_o(\tau)$ in much less than $\tau_k$, then, to a good approximation (see Appendix II of Chapter 10),

$$
\hat{E}[y_k|\phi] \approx \frac{AK(1 - F_k)G(\phi)}{2\tau_k\sigma_o^2} S_o(0) - AK(1 - F_k)\bar{g} \left(1 + \frac{S_o(0)}{2\tau_k\sigma_o^2}\right) 
$$

(11-72)

for all $k = 1, 2, \ldots, N$. In (11-72), $S_o(0)$ is the spectral density of $G(\phi) = g(\phi) - \bar{g}$ at the origin.
At this point in the development of a working theory it appears that the validity of the assumptions that lead one from (11-66) to (11-72) must be justified by direct measurement of \( E(y_k|\phi) \). The measurement of \( E(y_k|\phi) \) can readily be adapted for simulation on a digital computer. The fact that \( E(y_i|\phi) \) is sinusoidal in the steady state for a sinusoidal PLL when \( \Lambda_0 = 0, N = 1 \) has been verified by computer-simulation techniques. Typical results from the simulation for a low signal-to-noise ratio are shown in Fig. 10-9 of Chapter 10 along with a plot of \( \dot{E}(y_i|\phi) \) to accentuate the agreement.

From (11-42), (11-44), and (11-72), the steady-state restoring force, for an \((N + 1)\)-order SCS, becomes

\[
\hat{h}_o(\phi) = -\beta(N) - \alpha(N)g(\phi) \tag{11-73}
\]

where

\[
\beta(N) = \frac{2}{K_{oo}} \left\{ \Lambda_0 - AKg \sum_{k=1}^{N} (1 - F_k) \left[ 1 + \frac{S_\phi(0)}{2\tau_k \sigma_{\phi}} \right] \right\} = \frac{2\dot{\phi}}{K_{oo}} + \alpha(N)\bar{g}
\]

\[
\alpha(N) = \frac{2}{K_{oo}} \left[ AKF_0 - \frac{AKS_\phi(0)}{2\sigma_{\phi}^2} \sum_{k=1}^{N} \frac{1 - F_k}{\tau_k} \right] \tag{11-74}
\]

The use of (11-73) in (11-45) and (11-50) produces \( p(\phi) \) for an \((N + 1)\)-order SCS. The factor \( \beta(N) > 0 \) is responsible for the asymmetry in \( p(\phi) \); hence \( p(\phi) \) will be symmetric if the loop is operating such that \( \beta(N) = 0 \).

\[
\gamma = \frac{\Lambda_0}{AK} = \bar{g} \sum_{k=1}^{N} (1 - F_k) \left[ 1 + \frac{S_\phi(0)}{2\tau_k \sigma_{\phi}} \right] \tag{11-75}
\]

It is clear that this equation can never be satisfied for \( \Lambda_0 \neq 0 \); however, (11-75) can hold in an approximate sense. We note that \( S_\phi(0)/\sigma_{\phi}^2 \) is just the correlation time \( \tau_\phi \) of \( G \).

For the special case of a sinusoidal PLL, the p.d.f. of the phase error is easily shown to be given by (9-38), where \( \beta \) and \( \alpha \) are approximated by (11-62) or (11-74). Consequently, we do not elaborate on the details of this p.d.f., for all the results derived in Section 8-3.3 hold with \( \alpha(N) \) replacing \( \alpha \) and \( \beta(N) \) replacing \( \beta \). In fact, the result for \( p(\phi) \) is a generalization of those given in Chapters 9 and 10.

In some applications, the variance of the phase error rate is of interest. According to this theory, and the argument of stochastically equivalent processes discussed in Chapter 7, we can write

\[
\sigma_{\phi}^2 = \Omega^2 + \Delta^2 \sigma_{\phi}^2 + (F_0K\sigma_{\phi})^2 \tag{11-76}
\]

where
\[
\Omega = \Lambda_0 - AK \bar{g} \sum_{k=1}^{N} (1 - F_k) \left[ 1 + \frac{S_0(0)}{2\tau_k \sigma_G^2} \right] \\
\Delta = AKF_0 - \frac{AKS_0(0)}{2\sigma_G^2} \left[ \sum_{k=1}^{N} \left( \frac{1 - F_k}{\tau_k} \right) \right] 
\]
(11-77)

When \( \beta = 0 \), then \( \bar{g}(\phi) = 0 \) and from (11-55) we see that the mean phase error rate \( \bar{\dot{\phi}} = \bar{\phi} - \omega_0 = \Lambda_0 = 0 \). This says that the average frequency of the reference controlled oscillator equals \( \omega_0 \).

11-9 The First-Passage Time Model and Boundary Conditions, on \( P(y; t) \)

As discussed earlier, the variance of \( \varphi \) as determined from \( P(y; t) \) will be unbounded in the steady state. However, for \( t < \infty \), the variance of \( \varphi \) as determined from (11-15) is bounded. In what follows, we work with \( P(y; t) \) when computing the moments of the first-passage time of \( \varphi \). However, when computing the moments of the projections of \( y_0 = (y_1, \ldots, y_N) \), one may work with the solution \( P(y; t) \). The domain \( R' \) spanned by the variables \( y_k \) is \( R' = [|y_0| \leq \infty, |y_1| \leq \infty, \ldots, |y_N| \leq \infty] \), and if \( P(y; t) \) is to be a probability density in \( R' \), we must have

\[
\int \cdots \int P(y; t) \, dy = 1 
\]
(11-78)

for all \( t \). Moreover, the transition p.d.f. of the \( k \)th projection of \( y \) is given by

\[
P(y_k; t) = \int \cdots \int P(y; t) \, dy_k 
\]
(11-79)

The flow of probability from the probability space \( R' \) through the surface \( \Gamma' \) surrounding \( R' \) into the infinite medium is characterized by the facts that along any edge of \( \Gamma' \)

\[
y_j P(y; t)|_{y_j = \pm \infty} = P(y; t)|_{y_j = \pm \infty} = 0 
\]
(11-80)

for all \( j = 0, 1, \ldots, N \). The fact that \( y_j P(y; t) \) in (11-80) approaches zero as \( y_j \) approaches infinity requires that \( P(y; t) \) approach zero faster than \( |y_j|^{-\frac{1+\epsilon}{\epsilon}} \), \( \epsilon > 0 \). This is required in order for the law of conservation of probability to hold. Furthermore, along any edge of \( \Gamma' \),

\[
\frac{\partial}{\partial y_j} P(y; t)|_{y_j = \pm \infty} = 0, \quad j = 0, 1, \ldots, N 
\]
(11-81)
In addition, when considering the first-passage time of the $k$th projection $y_k$, we must specify boundary conditions at the barriers $y_k = \pm y_{kl}$ for all $k = 0, 1, \ldots, N$. By this we mean that we create a new process $q_k(t)$ such that for all $k = 0, 1, \ldots, N$,

$$q_k(t) = \begin{cases} y_k(t) & \text{if } y_k(\tau) < |y_{kl}| \text{ for every } \tau < t \\ \pm y_{kl} & \text{if } |y_k(\tau)| = y_{kl} \text{ for some } \tau \leq t \end{cases} \quad (11-82)$$

Such restrictions on the $y(t)$ process place a restriction on the solution $P(y; t)$ such that the normalization condition (11-78) no longer holds. (We discuss this point more fully in the next section.) We refer to the solution of (11-15) for the new process $[q_k(t), k = 0, \ldots, N]$ as the restricted solution and denote it by $Q(y; t) = Q(y, t|y_o, t_0)$. The boundary conditions (11-80) and (11-81) still hold if we replace $P(y; t)$ by $Q(y; t)$. In the next section we shall discuss further boundary conditions that must be imposed on the solutions $Q(y_k; t)$ for all $k = 0, 1, \ldots, N$. The function $Q(y_k; t)$ is determined from (11-79) with $P(y; t)$ replaced by $Q(y; t)$. The reader unfamiliar with the first-passage time problem would do well to return to Chapter 7 for a rather detailed account.

11-10 Moments of the First-Passage Time of the $k$th Projection, $k = 1, 2, \ldots, N$

In this section we shall be concerned with the problem of evaluating the moments of the random time $T_{fp}(y_k)$ it takes for the $k$th projection of $y(t)$ to exceed either of the barriers $b_1 = -y_{kl}, b_2 = y_{kl}$ for the first time while the initial position at time $t = t_0$ is $y_o$. Stated another way, we wish to derive expressions for the moments of the expected time $E[T_{fp}(y_k)]$ before one of the barriers is crossed for the first time.

We now show how to calculate the moments of the first-passage time $\tau^n(y_{kl}) = E[T_{fp}^n(y_k)]$ by confining ourselves to the case where coefficients $K_k$ and $K_{ik}$ in the Fokker-Planck equation are time independent. For convenience, we suppress the dependence of $\tau^n(y_{kl})$ on $y_o$. Excluding from consideration any trajectory of the projection $y_k(t)$ as soon as it reaches the barrier values $-y_{kl}$ or $y_{kl}$ for the first time, we can describe the remaining trajectories by a probability density $Q(y_k; t)$ such that $\Delta P = Q(y_k; t) \Delta y_k + O((\Delta y_k)^2)$. Here $\Delta P$ is the probability that the projection $y_k(t)$ assumes a value in the interval $[y_k, y_k + \Delta y_k]$ without ever having reached the barrier during the entire time interval $(t_0, t)$. Then the integral

$$\psi_k(t) = \int_{-y_{kl}}^{y_{kl}} Q(y_k; t) \, dy_k, \quad k = 0, 1, \ldots, N \quad (11-83)$$
gives the probability that \( y_k(t) \) never reaches the barrier during the time interval \((t_0, t)\). Initially, when no realization of \( y_k(t) \) has yet managed to reach the barrier, the probability density is the same as the initial density so that

\[
\psi_k(t_0) = 1 \quad (11-84)
\]

At subsequent times the normalization condition

\[
\int_{-\infty}^{\infty} Q(y_k; t) \, dy_k = 1 \quad (11-85)
\]

is no longer valid, since with the passage of time more and more trajectories of the projection \( y_k(t) \) are excluded from consideration as a result of having reached the barriers. Sooner or later all possible trajectories arrive at the barriers; hence

\[
\psi_k(\infty) = 0 \quad (11-86)
\]

Inside the interval \([-y_k, y_k]\), the behavior of the probability density \( Q(y_k; t) \) is described by the reduced Fokker-Planck equation, for the \( k \)th projection of the trajectory \( y(t) \) cannot end inside \([-y_k, y_k]\). In fact, trajectories of the \( k \)th projection are excluded from consideration only when the barrier is reached. Furthermore, there is a nonzero probability current accumulating at the barrier that corresponds to those trajectories of the \( k \)th projection, which are absorbed. The value of this probability current at either barrier \( y_k = y_{ki} \) or \( y_k = -y_{ki} \) will, when used as the weight of a delta function, be that amount of probability required to normalize \( Q(y_k; t) \). Therefore the boundary conditions for the \( k \)th projection have to be altered in the following way. At time \( t \) slightly greater than \( t_0 \) there are practically no trajectories near the barriers—that is, no trajectory has yet touched the barrier. But it is just these trajectories that are described by the probability density \( Q(y_k; t) \); hence we have the absorption conditions (see Chapter 7)

\[
Q(\pm y_{ki}; t) = 0 \quad \text{for all } t > t_0 = 0 \quad (11-87)
\]

We define

\[
Q(\pm y_{ki}) \triangleq \int_{t_0}^{\infty} Q(y_{ki}; t) \, dt = 0. \quad (11-88)
\]

After calculating \( Q(y_k; t) \) from (11-15), using (11-80) and (11-81), we can find the probability
\[ \psi_k(t_0) - \psi_k(t) = 1 - \psi_k(t) \]  \hspace{1cm} (11-89)

that the barrier is first reached during the time interval \((t_0, t)\). Differentiating (11-89) with respect to \(t\), we obtain the p.d.f. of the \(k\)th projection of the first-passage time.

\[ p_k(t|y_0) = -\frac{\partial \psi_k(t)}{\partial t} \]  \hspace{1cm} (11-90)

The \(n\)th moment of the first-passage time of \(y_k\) is therefore

\[ \tau^n(y_{k1}) = \int_{-y_{k1}}^{y_{k1}} [t^n \psi_k(t)]_0^\infty - n \int_0^\infty t^{n-1} Q(y_k; t) \, dy_k \, dt \]  \hspace{1cm} (11-91)

To simplify notation here, we have suppressed the initial condition \(y_0\), for it is clear that \(\tau^n(y_{k1})\) depends on the initial positions of the state variables. Integrating (11-91) by parts yields

\[ \tau^n(y_{k1}) = -\int_{-y_{k1}}^{y_{k1}} \left[ t^n \psi_k(t)\right]_0^\infty - n \int_0^\infty t^{n-1} Q(y_k; t) \, dt \, dy_k \]  \hspace{1cm} (11-92)

But from (11-84) we have that \(\psi_k(t_0) = 1\) and from (11-86) we have \(\psi_k(\infty) = 0\), so that (11-92) reduces to

\[ \tau^n(y_{k1}) = n \int_{-y_{k1}}^{y_{k1}} \int_0^\infty t^{n-1} Q(y_k; t) \, dt \, dy_k \]  \hspace{1cm} (11-93)

if we assume that \(\psi_k(t)\) goes to zero faster that \(t^{-n}\) as \(t\) approaches infinity. Define

\[ Q_{n-1}(y_k) \triangleq \int_0^\infty t^{n-1} Q(y_k; t) \, dt \]  \hspace{1cm} (11-94)

Then the \(n\)th moment of the first-passage time to \(y_k\) is given by

\[ \tau^n(y_k) = n \int_{-y_{k1}}^{y_{k1}} Q_{n-1}(z) \, dz \]  \hspace{1cm} (11-95)

for all \(k = 0, 1, \ldots, N\).

The functions \(Q_{n-1}(y_k)\) in (11-94) may be determined from (11-40) and (11-41) by noting that each projection of \(y_k\) satisfies a partial differential equation of the flow from

\[ \nabla \cdot \mathcal{F}_k(y_k; t) + \frac{\partial Q(y_k; t)}{\partial t} = 0 \]  \hspace{1cm} (11-96)
If we substitute for $J_k(y_k; t)$ from (11-41) into the preceding equation, we easily write

$$\frac{K_{kk}}{2} \frac{\partial}{\partial y_k} \left[ h_k(y_k; t) - \frac{\partial}{\partial y_k} \right] Q(y_k; t) + \frac{\partial Q(y_k; t)}{\partial t} = 0 \quad (11-97)$$

where we have replaced $p(y_k; t)$ by the restricted form $Q(y_k; t)$. The conditional expectations that define $h_k(y_k; t)$ are now taken with respect to the restricted p.d.f. $Q(y_k, t|\varphi, y_o)$. Multiplying both sides of (11-97) by $t^n$ and integrating between $t_0 = 0$ and infinity, we obtain the differential equation

$$\frac{d^2 Q_n(y_k)}{dy_k^2} - \frac{d}{dy_k} \left[ h_k(y_k; \bar{t})Q_n(y_k) \right] = \frac{2}{K_{kk}} \int_0^\infty t^n \frac{\partial Q(y_k; t)}{\partial t} \, dt \quad (11-98)$$

where $Q_n(y_k)$ is defined in (11-94). In arriving at (11-98) we have made use of the mean value theorem (Ref. 4), which says that

$$E(y_k, \bar{t}|\varphi, y_o)Q_n(\varphi) = \int_0^\infty t^n E(y_k, t|\varphi, y_o)Q(\varphi; t) \, dt \quad (11-99)$$

$$E(g(\varphi), \bar{t}|\varphi, y_o)Q_n(y_k) = \int_0^\infty t^n E(g(\varphi), t|y_k, y_o)Q(y_k; t) \, dt$$

and $\bar{t}$ is a point such that $\bar{t} \in [0, \infty]$. Integrating by parts on the right-hand side of (11-98) yields

$$\frac{d^2 Q_n(y_k)}{dy_k^2} - \frac{d}{dy_k} \left[ h_k(y_k; \bar{t})Q_n(y_k) \right] = -\frac{2nQ_{n-1}(y_k)}{K_{kk}} \quad (11-100)$$

This result may be integrated once so that

$$\frac{dQ_n(y_k)}{dy_k} - \left[ h_k(y_k; \bar{t})Q_n(y_k) \right]$$

$$= -\frac{2n}{K_{kk}} \int_{y_k}^{\bar{t}} Q_{n-1}(x) \, dx + \frac{2C(n)}{K_{kk}} \quad (11-101)$$

where $C(n)$ is a constant of integration to be determined. From (11-95) we write

$$\frac{dQ_n(y_k)}{dy_k} - h_k(y_k; \bar{t})Q_n(y_k) = \frac{2}{K_{kk}} \left[ C(n) - \tau^*(y_k) \right] \quad (11-102)$$
where $\tau^0(y_k) = u(y_k - y_{k_0})$ is the unit step occurring at $y_k = y_{k_0}$. Solving (11-102) for $Q_n(y_k)$ yields

$$Q_n(y_k) = \exp \left[ -U_k(y_k; \bar{t}) \right] \left\{ E_k + \int_{-y_k}^{y_k} \frac{2}{K_{kk}} [C_k(n) - \tau^n(x)] \times \exp \left[ U_k(x; \bar{t}) \right] dx \right\}$$

(11-103)

and

$$\tau^0(x) = u(x - y_{k_0})$$

where $E_k$ is a constant of integration. Since $Q(-y_k) = 0$ from (11-88), we have that $E_k = 0$; and since $Q(y_k) = 0$, we find that

$$C_k(n) = \int_{-y_k}^{y_k} \frac{\tau^n(x) \exp \left[ U_k(x; \bar{t}) \right] dx}{\int_{-y_k}^{y_k} \exp \left[ U_k(x; \bar{t}) \right] dx}$$

(11-104)

Using these values of $C_k(n)$ and $E_k$ in (11-103) and making use of (11-95) gives the recursive formula

$$\tau^n(y_{k_1}) = \frac{2n}{K_{kk}} \int_{-y_{k_1}}^{y_{k_1}} \int_{-y_{k_1}}^{y_{k_1}} [C_k(n - 1) - \tau^{n-1}(x)] \times \exp \left[ U_k(x; \bar{t}) - U_k(y_k; \bar{t}) \right] dx dy_k$$

(11-105)

and

$$\tau^0(x) = u(x - y_{k_0})$$

for the $n$th moment of the first-passage time of the state variable $y_k$, $k = 0, 1, 2, \ldots, N$. The $n$th moment of the first-slip time and frequency acquisition time can be obtained from (11-105). This equation generalizes earlier work due to Darling and Siegert (Ref. 5) and Siegert (Ref. 6), as well as the results given in Chapters 7, 9, and 10 pertaining to first-passage time problems.

11-11 Net Flow of Probability per Unit of Time and the Average Number of Cycles Slipped per Unit of Time

Because of the additive noise, discontinuities of oscillator synchronization arise. The locally controlled reference oscillator may slip or gain a cycle relative to the oscillations of the external signal $s(t, \Phi)$; that is, diffusional spreading of the number of oscillations produced in the reference signal takes place. In tracking applications, the average number of cycles slipped per unit time is
an important parameter, for it is indicative of the error introduced into any Doppler measurement made to obtain velocity and changes in range.

To calculate the average number of cycles slipped per unit of time in the steady state, we make use of the concept of probability current (introduced earlier) in the φ direction. The average flow of probability through the hyperplane $\phi = \phi'$ in the positive φ direction per unit of time is easily found from (11-47), (11-50), and (11-61) to be given exactly by

$$\mathcal{J}_0 = C_0'K_{00}\exp\left(-\Delta U/2\right) \sinh\left(\Delta U/2\right) = \bar{\phi}/2\pi$$  \hspace{1cm} (11-106)

where $\Delta U \triangleq U_o(\phi) - U_o(\phi + 2\pi)$. Using (11-73) and (11-45) to approximate $p(\phi)$ in (11-50) we can write, using (11-47),

$$\mathcal{J}_0 \approx \frac{K_{00}\sinh\pi\beta}{4\pi^2|I_{j\beta}(\alpha)|^2}$$  \hspace{1cm} (11-107)

for all φ. This says that the flow of probability per unit time through the hyperplane $\phi = \phi'$ is constant. From Maxwell's equations for stochastic fields (Chapter 6), we know that the curl of the stochastic magnetic field intensity $\mathcal{H}$ is equal to the probability current density $\mathcal{J}$ in the steady state. This curl is zero when the detuning is zero; that is, if $\Lambda_0 = 0$, then $\beta = 0$, $\mathcal{J}_0(\phi) = 0$ and there is no net flow of probability through the hyperplane $\phi = \phi'$.

In practice, it is sometimes convenient to know the average number of cycles slipped per unit of time independent of direction. Denote the average number of cycles slipped to the right (positive φ direction) per unit of time by $N_+$ and the average number of cycles slipped to the left (negative φ direction) per unit of time by $N_-$. Using (11-55) and following the argument given in Section 9-3.5 we can relate $\mathcal{J}_0$ to the residual detuning $\bar{\phi}/2\pi$ through $\mathcal{J}_0 = N_+ - N_-$. Thus $\mathcal{J}_0$ represents the (net) average number of cycles slipped per unit time. The ratio of the currents can be obtained by using results from the theory of statistical mechanics, which specifies the rate of escape of Brownian particles over a potential barrier (see Chapter 7). For a cycle-slip to occur it is necessary for the particle normally remaining in the plane of potential minimum at the position $\phi(t)$ to overcome the potential barrier represented by $\bar{U}_0(\phi)$. As a result of $\Lambda_0 \neq 0$, the height of the barrier to the right is not equal to the height of the barrier to the left. Consequently, (7-149) can be used to specify the ratio $N_+/N_-; \text{ thus}$

$$\frac{N_+}{N_-} = \frac{[C_0(0) - u(b_2 - \phi_0)]}{[C_0(0) - u(b_1 - \phi_0)]}$$  \hspace{1cm} (11-108)

where $C_0(0)$ is defined in (11-104) and $b_2 = y_{ol} = 2\pi + \phi_0$ and $b_1 = -y_{ol} = \phi_0 - 2\pi$. The total number of cycles slipped per unit time independent of di-
rection is \( \tilde{S} = N_+ + N_- \). Using this fact, plus (11-108), and noting that \( J_0 = N_+ - N_- \), gives

\[
\tilde{S} = \frac{[(N_+/N_-) + 1]J_0}{[(N_+/N_-) - 1]}
\]

which can be used to characterize the threshold of a SCS. The quantity \( D_c = (2\pi)^2 \tilde{S} \) is the phase error diffusion coefficient representing the rate at which \( \phi(t) \) is undergoing diffusion, while \( D_c t \) accounts for the fact that (11-19) is true. The expected value of the time intervals between cycle-slip events is well approximated by \( \Delta T = 1/\tilde{S} \). In general, this parameter is not at all related to the mean time to first loss of phase synchronization even though it is much more important in the design of a practical communication system.

The probability currents \( J_k, k = 1, 2, \ldots, N, \) may be easily computed from (11-47) and (11-48) in a manner similar to that described in this section. Thus the probability current flowing through the hyperplane \( y_k = y_k' \) per unit of time in the positive direction of the \( y_k \) axis is \( J_k = -C_k D_k K_{kk}/2 \) for all \( k = 1, 2, \ldots, N \). Again we find that the net flow of probability is constant and the field is irrotational when \( D_k = 0 \). This says that there is no net flow of probability in the \( y_k \) direction.

Finally, we point out that the probability that the net, \( N = n \), number of cycles slipped in \( t \) seconds is characterized by (10-140) with \( N_+/N_- \) and \( \tilde{S} \) determined from (11-108) and (11-109) respectively. Moreover, the probability of losing phase lock in \( t \) seconds can be found from (11-142) with \( \tilde{S} \) characterized by (11-109).

11-12 Synthesis of Optimum Synchronous Control Systems

11-12.1 Coherent Communications

The problem of choosing the "best" loop filter \( F(p) \) as well as the "best" nonlinearity so as to provide for an optimum tracking loop depends on what is meant by optimum. For example, during the signal acquisition mode, the performance index is acquisition time. After the signal has been acquired, the problem becomes one of either tracking or data demodulation, or both, and the performance index changes. Hence after acquisition a design based on minimum acquisition time becomes suboptimum. For the case of phase-coherent communications one would want to minimize the mean-squared phase error—that is, minimize the functional

\[
\min_{F(p), g(\phi)} G[F(p), g(\phi)] = \sigma_{\phi}^2
\]
subject to the linearity constraint on $F(p)$ and a gain constraint on the class of nonlinearities \{g(\phi)\}. In the case of tracking, one desires to maximize the expected time to loss of phase synchronization. In general, however, it is conceivable that such an optimization technique may be formidable or a solution to (11-110) may not even exist if $F(p)$ is restricted to being linear. Be that as it may, however, a few results are presently available for the case where one constrains the loop filter to be of the form $F(p) = 1$ or $F(p) = 1/(1 + \tau_s p)$ and then selects that nonlinearity $g(\phi)$ such that the mean square error is minimum. This turns out to be equivalent (Refs. 7, 8) to minimizing the area under the tail of the density $p(\phi)$. For these cases it can be shown (Refs. 7, 8) that the optimum nonlinearity is

$$g(\phi) = \text{sgn} \left[ \sin \phi \right] \quad (11-111)$$

when $F(p) = 1$ and it is assumed that $\Lambda_0 = 0$. The restoring force $h_0(\phi)$ becomes rectangular for this $g(\phi)$, since $\text{sgn} x = 1$ for $x \geq 0$ and $-1$ for $x < 0$.

In practice, it is desirable to design the loop such that $p(\phi)$ is symmetric so that no bias is introduced in the phase measurement. From (11-49) and (11-61) it can be seen that any asymmetry in $p(\phi)$ is due to $\beta(N)$. By proper design of the loop filter, $\beta(N)$ can be made arbitrarily small for reasonable frequency offsets. This requires $N \geq 1$ so that

$$p(\phi) \approx C_0 \exp \left[ -\alpha(N) \int_{\phi}^{g(x)} \right] \quad (11-112)$$

if $\beta(N) \approx 0$. Paralleling the arguments due to Stiffler and Shaft (Refs. 7, 8), we have, for the $(N + 1)$-order SCS with $\rho(\phi)$ defined by (11-112), that the nonlinearity which minimizes the mean-squared phase error is also given by (11-112) for all $N$. Since $[\alpha(N)]^{-1}$ can be interpreted as a variance, then $\sigma_\phi^2$ is minimized over the choice of $F(p)$ when $\alpha(N)$ is maximized; that is, the potential wells are deepest. This is accomplished with $N = 1$—that is, a second-order system. Hence we are lead to believe that, according to this theory, a second-order loop operating such that $\beta(1) \approx 0$ with $g(\phi)$ given by (11-111) is, for all practical purposes, that SCS which minimizes the variance of the phase error when $d(t) = \theta_0 + \Omega_0 t$.

For the second-order SCS the minimum mean square error obtainable is, from (11-111) and (11-112),

$$\left( \sigma_\phi^2 \right)_{\text{min}} = \frac{2\left[ 1 - \{1 + \pi \alpha + (\pi \alpha)^2 / 2 \} \exp (-\pi \alpha) \right]}{\alpha^2 \left[ 1 - \exp (-\pi \alpha) \right]} \quad (11-113)$$

where $\alpha = \alpha(1)$ is determined from (11-62) or (11-74). For high signal-to-noise ratios, $\alpha \approx \rho$ and
\[(\sigma^2_{\phi})_{min} \approx \frac{2}{\rho^2}, \quad \rho \gg 1 \quad (11-114)\]

At low values of \(\rho\),

\[(\sigma^2_{\phi})_{min} \approx \frac{\alpha \pi^3}{3} \frac{1}{\exp(\pi \alpha) - 1} \quad (11-115)\]

Thus for large signal-to-noise ratios, the improvement in \(\sigma^2_{\phi}\) offered by the second-order SCS with optimum nonlinearity is \(I = 10 \log_{10}(\rho/2)\) dB better than a second-order phase-locked loop with \(g(\phi) = \sin \phi\). For equal phase variances and large signal-to-noise ratios, the second-order SCS with optimum

---

**Fig. 11-3.** Comparison of the Variance of the Phase Error versus \(\rho\) for Various \(r\).
nonlinearity requires that $\rho_{\text{PLL}} = \rho^2/2$, where $\rho$ is the loop signal-to-noise ratio (see Fig. 11-3).

If for an $(N + 1)$-order SCS, with $\Lambda_0 = 0$, we desire to minimize the variance of $\phi$; then the factor $\beta$ is zero and $a(N)$ is maximized when $N = 0$. The minimum mean square error is still given by (11-113) with $a = \rho = 4A/N_0K$. Thus we are led to the conclusion that the optimum SCS is one for which the nonlinearity is given by (11-111) and $F(p) = 1$ when $\Lambda_0 = 0$. Hence a first-order system is optimum from the point of view of producing the minimum phase error variance when $\Lambda_0 = 0$. Other approaches may be used, aside from signal design, to vary $g(\phi)$. These were discussed in Chapter 3.

\textbf{11-12.2 Tracking and Synchronization Systems}

In tracking applications, as opposed to phase-coherent communications, a more significant performance criterion is the mean time to first loss of phase synchronization. For data demodulation with a SCS, the mean square error assumes priority as a measure of performance. Furthermore, the definition of loss of phase synchronization is somewhat arbitrary in that the location of the barriers has to be selected.

Any attempt to perform stochastic optimization of the loop in the non-linear region of operation for $N > 0$ by selecting that $F(p)$ and $g(\phi)$ such that

$$
\tau(y_k) = \max_{F(p), g(\phi)} E[T(y_k)]
$$

is plagued by inexact knowledge of the expectations $E(y_k, \bar{r}|\phi, y_0)$. For the case where $N = 0$, $\Lambda_0 = 0$, and $F_0 = 1$, it is possible to select that $g(\phi)$ for which $\tau(\phi_i)$ is a maximum. For this first-order SCS it is easy to show (Ref. 8) that the nonlinearity which maximizes $\tau(\phi_i)$ is given by (11-111). The corresponding first-passage time, as computed from (11-105) with $N = 0$, $n = 1$, $y_{kt} = \phi_t$ and $g(\phi) = \text{sgn} \left[ \sin \phi \right]$, is given by

$$
\tau(\phi_i) = \frac{1}{2W} \left\{ \frac{1}{\rho} \left[ \exp (\phi_i \rho) - 1 \right] - \phi_i \right\}
$$

(11-117)

where $\rho = 4A/N_0K$. For small $\rho$, (11-117) approaches

$$
\tau(\phi_i) \approx \frac{\phi_i^2 \rho}{4W_L}
$$

(11-118)

while for large $\rho$, (11-117) simplifies to

$$
\tau(\phi_i) \approx \frac{1}{2W_L} \left[ \frac{1}{\rho} \exp (\rho \phi_i) - \phi_i \right]
$$

(11-119)
Fig. 11-4. Comparison of the Second-Order SCS with a First-Order SCS for \( \Lambda_0 \neq 0 \).
The Optimum Reference Signal for a Square-Wave Input

Using the steady-state, linear tracking theory for an \((N + 1)\)-order loop, it is easy to show that

\[
\sum_{k=1}^{N} E(y_k, i|\varphi, y_o) = \Lambda K F_0 \left(1 - \frac{AKF_0}{2W_L}\right) \varphi
\]  

(11-120)

when \(\Lambda_0 = 0\). Shaft (Ref. 8) has shown that the optimum nonlinearity is again \(\text{sgn}[\sin \varphi]\) for \(|\varphi| \leq \varphi_i\). The mean time to first loss of sync may be computed from (11-105) with \(n = 1\). Omitting the details, we have for \(\Lambda_0 = 0\) that

\[
\tau(\varphi_i) = \left(\frac{r + 1}{r}\right)^2 \frac{\rho}{2W_L} \int_{\varphi_i}^{\varphi} \exp \left[ \left(\frac{r + 1}{r}\right) \rho(|\varphi| - |x|) + \frac{\rho}{2r} (\varphi^2 - x^2) \right] d\varphi \, dx
\]

(11-121)

Numerical results comparing the performance of various SCSs have been made in Fig. 11-4.

11-13 The Optimum Reference Signal for a Square-Wave Input

Since a square-wave cross-correlation function is difficult to mechanize, various other approaches can be used to vary \(g(\varphi)\). For example, for a first-order loop (Ref. 9), Layland constrained the signal \(s[t, \varphi(t)]\) to be a square wave (the case of digital communications) and maximized \(p(0)\) with respect to \(r(t, \varphi)\) under a unit power constraint. If the same methodology is applied here for the second-order SCS, then one can show that (Ref. 9) the optimum reference control signal is given by

\[
r(\tau, \varphi) = D' \left\{ \int_{\tau}^{\varphi/2} (\varphi - \tau)w(\varphi) \, d\varphi + \left(\frac{\pi}{2} - \tau\right)w\left(\frac{\pi}{2}\right) \right. \\
\times \left. \int_{0}^{\varphi/2} \exp \left[ -\frac{2\alpha}{\pi} \int_{0}^{\varphi/2} \left(\frac{\pi}{2} - \max[\zeta, \eta]\right) r(\eta) \, d\eta \, d\zeta \right] \right. \\
+ \left. w\left(\frac{\pi}{2}\right) \int_{0}^{\varphi/2} \left(\frac{\pi}{2} - \max[\zeta, \tau]\right) \right. \\
\times \left. \exp \left[ -\frac{2\alpha}{\pi} \int_{0}^{\varphi} (\varphi - \eta)r(\eta) \, d\eta \right] \, d\zeta \right\}
\]

(11-122)

where \(D'\) is a normalizing constant for \(\tau \in [0, \pi/2]\) and

\[
w(\varphi) = \exp \left[ -\frac{2\alpha}{\pi} \int_{0}^{\varphi} (\varphi - \eta)r(\eta) \, d\eta \right]
\]
which is independent of $\Lambda_0$ for the second-order SCS operating such that $\beta \approx 0$. [Note that $r(\tau, \Phi)$ is implicitly a function of itself through $w(\varphi)$.] Even though (11-122) is recursive and does not admit to a solution in closed form, it can be evaluated by employing an iterative numerical procedure on a digital computer. It is readily shown that the variance of the phase error becomes inversely proportional to $\sqrt{\alpha^{-3/2}}$ for large signal-to-noise ratios. Figure 11-5 illustrates $r(\tau, \Phi)$ for various values of $\alpha$. In the limit as $\alpha$ approaches infinity, $r(\tau, \Phi)$ becomes a delta function train and $g(\varphi)$ is given by (11-111).

![Fig. 11-5. Optimum Reference Waveforms for Various Values of $\alpha$. (Courtesy of J. Layland)](image)

11-14 Extensions to an Arbitrary Loop Filter and Further Studies

The loop filter defined in (11-5) admits only those filters that have poles along the real axis in the left half-plane provided that the $E_k$'s are less than one and the $\tau_k$'s are positive and real. If one allows for complex $\tau_k$'s and $E_k$'s in (11-5), then a broader class of filter functions is admitted; however, the differential equations for the $y_k$'s become complex and occur in conjugate pairs. On the basis of (11-6), the solution for $p(\phi)$ or $Q_n(\phi)$ is real, for the sum of the conditional expectations involving $E(y_k|\phi)$ or $E(y_k, \tilde{r} | \phi, y_0)$ is real. However, the solutions for $p(y_k)$ or $Q_n(y_k)$, $k \geq 1$, are no longer valid, for they become complex and occur in conjugate pairs. It is possible to return to (11-8) and produce a new vector Markov process with real components by adding and
subtracting the conjugate pairs of the differential equations for the \( y_k \)'s. The solution to the revised FP equation will then render solutions for the marginal densities that are real.

It is more interesting, although more difficult, to consider the general loop filter \( F(s) = F_0 + F'(s) \), where

\[
F'(s) = \frac{N(s)}{D(s)} = \frac{\sum_{k=1}^{N} p_k s^{k-1}}{\sum_{k=1}^{N+1} q_k s^{k-1}}
\]

(11-123)

where the \( q_k \)'s and \( p_k \)'s are real.

If one defines the vector Markov process

\[
x_1 = -\frac{K[A_g(\phi) + n_g]}{D(s)}
\]

\[
x_2 = \dot{x}_1
\]

\[
\vdots
\]

\[
x_N = \dot{x}_{N-1}
\]

then from (11-4) we have

\[
\dot{\phi}(t) = \Lambda_0 - KF_0[A_g(\phi) + n_g] + \sum_{k=1}^{N} p_k x_k
\]

\[
\dot{x}_1 = x_2
\]

\[
\dot{x}_2 = x_3
\]

\[
\vdots
\]

\[
\dot{x}_N = -\frac{1}{q_{N+1}} \sum_{k=1}^{N} q_k x_k - \frac{K}{q_{N+1}} [A_g(\phi) + n_g]
\]

(11-124)

where \( x = (\phi, x_1, \ldots, x_N) \) is a vector Markov process. The FP equation for \( Q(x; t) \) or \( p(x; t) \) is easily set up and may be integrated, as before, over \( x' \) giving

\[
\frac{\partial H}{\partial t} + \frac{K_{00}}{2} \frac{\partial}{\partial \phi} \left\{ h(\phi; t) - \frac{\partial}{\partial \phi} H \right\} = 0
\]

(11-125)

when the boundary conditions are applied. Here \( H = p(\phi; t) \) or \( Q(\phi; t) \). For \( N = 1 \), the foregoing equation agrees with (11-40) and (11-101). However, for \( N > 1 \), the expectations \( E(x_k, t|\phi, y_0) \) are, in general, different from those defined in (11-42) and (11-43). The function \( h_0(\phi; t) \) is given by

\[
h_0(\phi; t) = \frac{2}{K_{00}} \left[ \Lambda_0 - KF_0 A_g(\phi) + \sum_{k=1}^{N} p_k E(x_k, t|\phi, y_0) \right]
\]

(11-126)
Assuming that the steady-state p.d.f. exists, then we can write (11-126) as

$$h_0(\phi) = \frac{2}{K_{00}} \left[ \Lambda_0 - AKF_0 g(\phi) + \sum_{k=1}^{N} p_k E(x_k|\phi) \right]$$

(11-127)

The conditional expectations $E(x_k|\phi)$ can be approximated by using the orthogonality principle.

In fact, using (11-62), which is valid for any state variable, we can write

$$\hat{h}_0(\phi) = \beta(N) - \alpha(N) g(\phi)$$

(11-128)

with $X_k \triangleq (x_k - \bar{x}_k)$ and

$$\alpha(N) = \frac{2}{K_{00}} \left[ AKF_0 - \frac{1}{\sigma_G^2} \sum_{k=1}^{N} p_k R_{X_k G}(0) \right]$$

$$\beta(N) = \frac{2}{K_{00}} \left[ \Lambda_0 - \frac{\bar{g}}{\sigma_G^2} \sum_{k=1}^{N} p_k \left( R_{X_k G}(0) + \frac{\sigma_G^2}{\bar{g}} \bar{x}_k \right) \right]$$

(11-129)

so that the potential function characterizing $p(\phi)$ in (11-50) can be determined when a steady-state solution exists. It would be interesting to study these general solutions in more detail and attempt to determine and characterize the stability of a SCS in the presence of noise.

11-15 Application of the Nonlinear Theory to Obtain Performance of a Third-Order Loop

In considering an application of the above theory to the design of a third-order loop, we ask the question: Is there an equivalent imperfect, two-pole loop filter that results in the same expression for steady-state performance when $\Lambda_0 = 0$ as the second-order loop, but significantly reduces the residual frequency detuning $\bar{\phi}$ when $\Lambda_0 \neq 0$? In other words can we design a filter for use in a third-order loop that will produce $\alpha(1) \approx \alpha(2)$ and $\beta(2) \ll \beta(1)$?

The answer to the above question is yes if we consider the loop filter

$$F(p) = \frac{1 + \tau_2 p}{1 + \tau_1 p} + \frac{1}{(1 + \tau_1 p)(\delta + \tau_2 p)}$$

(11-130)

from which we identify
\[ F_0 = \frac{\tau_3}{\tau_1}; \quad 1 - F_1 = \frac{\tau_1}{\delta \tau_2 - \tau_3} \]

\[ 1 - F_2 = \frac{(\tau_3/\delta)^2}{\tau_2 - \tau_1} \]  

(11-131)

using (11-5) and (11-130).

The first part of the filter in (11-130) is the imperfect second-order loop filter. Based upon this fact one might consider acquisition by means of the second-order loop so as to avoid any problems a third-order system out of lock might have and subsequent addition of the second filter term in (11-130) to reduce the steady-state phase error. A particular realization of (11-130), suggested in the detailed study of Ref. 11-12, is illustrated in Fig. 11-6. The iso-amplifiers (ISO) are high-input impedance devices considered to have unity gain. Here \( \tau_1 = (R_1 + R_2)C_1, \tau_2 = R_2C_1, \tau_3 = R_3C_2, 1/\delta = R_4/R_3, k = \tau_2/\tau_3. \)

![Diagram](image)

Fig. 11-6. Third-Order Loop Filter Mechanization by Extension of a Second-Order Loop Filter by One Extra Integration.

To determine the improvement in \( \beta(2) \) over \( \beta(1) \) we substitute (11-131) into (11-74) with \( N = 2 \) and get, after simplification,

\[ \beta(2) = \left( \frac{r + 1}{r} \right) \frac{p \Lambda_0}{F_0 AK} \left[ (1 - F_0)\bar{g} \left( 1 + \frac{F_0}{p(r + 1)\sigma_0^2} \right) + \frac{1}{\delta} \right] \]  

(11-132)

Comparing with (10-101) we have satisfied our initial goal provided that \( 1/\delta \) can be chosen large enough. It should be pointed out that \( \delta = 0.01 \) is quite reasonable and that \( p \) and \( W_L \) in the above equations are specified relative to the imperfect second-order loop. The bandwidth \( W_{L3} \) for this third-order loop is related to \( W_L \) of a second-order loop through

\[ W_{L3} \approx \frac{rW_L}{r + 1} \left[ \frac{r + 1 - k}{r - k} \right] \]  

(11-133)

so that \( W_{L3} \approx W_L \) if \( k \ll 1 \).
If \( F_0 \ll 1 \) and \( \delta \approx 0 \), then the loop filter \( (11-130) \) is of the form given in \( (4-66) \) which minimizes the Wiener error when \( d(t) = \Omega_1 t^2 / 2 \), see Section 4-5.3; however, it is also interesting to note that the filter \( (11-130) \) gives better noise immunity than a second-order system even though the optimized linear theory suggests its use in optimally tracking \( d(t) = \theta_0 + \Omega_0 t \). We may also conclude that, when \( \Lambda_0 = 0 \), the third-order loop has essentially the same noise performance as the second-order loop. On the other hand, the reduction in the loop stress \( \beta(2) \) over \( \beta(1) \) is on the order of \( AKg / \delta \) when \( \Lambda_0 \neq 0 \).

**11-15.1 Signal Acquisition Time and Acquisition Range in a Third-Order Loop**

As in the case of first- and second-order loops, a beat note appears at the phase detector output whose dc value tends to force the loop toward lock. The extra integration in the loop accumulates this force, accelerating the loop toward lock. If the loop damping is not properly set, the force due to two integrations may carry the loop frequency error past the lock point so rapidly that recovery is not possible. Proper setting of the factor \( r \) and \( k \) can reduce this acceleration through the zero-beat frequency region enough to prevent any frequency overshoot or irrecoverable loss of lock. In fact, a third-order differential equation in the frequency acquisition time and voltages on the capacitors \( C_1 \) and \( C_2 \) can be developed in a manner analogous to that given in Section 10-2.2. To date the solution to the equation has not been investigated by the author.

Finally, the maximum normalized input frequency offset which the loop will acquire unaided is given approximately by

\[
\gamma_m = \left| \frac{\Lambda_0}{AK_{m}} \right| = \sqrt{2F_0[\delta + 1/\delta]} \quad (11-134)
\]

so that the enhancement in acquisition range is approximately \( \sqrt{\delta^{-1}} \) over the imperfect second-order loop. If \( \delta = 0.01 \) this improvement is 10 dB.

**Problems**

11-1 In the measurement of the rotation of the plane of polarization of the Pioneer VI spacecraft as its telemetry signal passed through the Sun's corona, a polarization tracker was built. The dynamical equations of operation for the polarization \( \theta \) and the carrier phase error are well approximated by

\[
\dot{\theta} = - (A_1 \cos \varphi) \sin \theta + n_1(t)
\]

\[
\dot{\varphi} = - (A_2 \cos \theta) \sin \varphi + n_2(t)
\]
where \( \{n_1(t)\} \) and \( \{n_2(t)\} \) are independent white Gaussian noise processes with single-sided spectral densities \( N_1 \) and \( N_2 \) respectively.

(a) Find the FP equation whose solution is the p.d.f. \( p(\phi, \theta) \).
(b) Find the reduced FP equation whose solution is \( p(\phi) \).
(c) What is the form of the solution to the equation obtained in (b)?
(d) Devise methods for estimating the unknown conditional expectation.

11-2 When \( F_0 = 0 \) in (11-5) and (11-15), show that the FP equation is given by (11-56). Let \( N = 1 \) and \( \Lambda_0 = 0 \), and show that (11-57) solves (11-56).

11-3 Consider a SCS system in which \( g(\phi) = \text{sgn} \ (\sin \phi) \).

(a) Find the steady-state p.d.f. \( p(\phi) \).
(b) Develop asymptotic expressions for the variance of the phase error defined in (11-113) when \( \rho \gg 1 \) and \( \rho \ll 1 \).
(c) Repeat (b) for the time to first slip by starting with (11-117).

11-4 The loop filter in a SCS has the form

\[
F(s) = \frac{\sum_{k=1}^{N} p_k s^{k-1}}{\sum_{k=1}^{N-1} q_k s^{k-1}} + F_0
\]

where the \( p_k \)'s and \( q_k \)'s are real and \( F_0 \) is constant.

(a) Define an appropriate Markov vector \( x = (\phi, x_1, \ldots, x_N) \) such that \( \phi \) is one component.
(b) What are the intensity coefficients that characterize the FP equation?
(c) Find the reduced FP equation whose solution is \( p(\phi, t|x_0, t_0) \). State the boundary and initial conditions.
(d) Find the reduced FP equation whose solutions are the transition p.d.f.'s \( p(x_k, t|x_0, t_0) \).
(e) What is the form of the steady-state solution \( p(\phi) \) assuming that it exists?

11-5 Develop an expression for the \( n \)th moment of the first-slip time for the phase error process generated by the loop defined in Prob. 11-4.

11-6 For the SCS defined in (11-4) suggest two methods for approximating the steady-state conditional expectation \( E(g(\phi)|y_k); k = 1, 2, \ldots, N \). Apply one of these methods and determine approximate formulas for the steady-state p.d.f.'s \( p(y_k), k = 1, \ldots, N \). Assume, of course, that these p.d.f.'s exist.

11-7 On the basis of the approximation introduced in (11-60) and (11-72), develop the formulas, (11-62) and (11-74), for the parameters \( a(N) \) and \( b(N) \) characterizing the steady-state p.d.f. \( p(\phi) \).

11-8 Consider the stochastic differential equation

\[
\dot{\phi} = \Lambda_0 - AKF_0 g(\phi) + \sum_{k=1}^{N} E(y_k, t|\phi, \phi_0) + KF_0 n_k(t)
\]
where \( \{n_s(t)\} \) has a finite variance. By using the approximations given in (11-72) for the conditional expectations, develop expressions for the steady-state variance of the phase error rate \( \sigma_2^2 \). Your answer should agree with (11-76). If \( g(\phi) = \sin \phi \), use the circular moments defined in Chapter 9 to simplify your answer.

11-9 Using the stochastic differential equations (11-8) for \( y_k, k = 1, 2, \ldots, N \), show that the steady-state mean and variance of \( y_k \) are given by (11-53) and (11-54) respectively. Using these results, verify (11-55).

11-10 Show that the conditional expectations, \( E(y_k|\phi) \), \( k = 1, 2, \ldots, N \), are periodic in \( \phi \). Justify whether they possess odd symmetry, even symmetry, or neither in \( \phi \).

11-11 When the loop cross-correlation function \( g(\phi) \) is defined by (11-111), show that the steady-state p.d.f. of the phase error is given by

\[
p(\phi) = \frac{\alpha \exp[\alpha(\pi - |\phi|)]}{2[\exp(\pi\alpha) - 1]}
\]

for \( |\phi| \leq \pi \).

References


NONLINEAR THEORY OF SYNCHRONOUS CONTROL SYSTEMS WITH RANDOM MODULATION INPUTS

12-1 Introduction

In this chapter we begin our study of the nonlinear problem of angle demodulation posed in Chapter 4, Section 4-7. Our approach is first to obtain the Fokker-Planck equation that describes the nonlinear behavior of the analog system illustrated in Fig. 4-11 when an arbitrary periodic nonlinearity is present in the system. By introducing the periodic extension of its solution, we will then be able to obtain the canonical form for the steady-state p.d.f. of the phase error when it exists. This will be accomplished for an arbitrary loop filter when the angle modulation (FM or PM) is a stationary random process. Next we show that the nonlinear behavior of the system is determined (exactly) to within a knowledge of a set of conditional expectations. In order to obtain a theory that can be used to specify system performance, we proceed by estimating these expectations based on the methods introduced in Chapter 11. In
Transmitter-Receiver Characterization

fact, a great many of the results in this chapter represents slight generalizations of those presented in Chapter 11—which accounts for its length.

The theory presented provides the motivation that leads to the development of more general methods of solving the Fokker-Planck equation—in particular, the sequence method discussed in Chapter 13 and the conditional expectation method discussed in Chapter 14. These methods are required when attempting to obtain performance analysis and numerical evaluations for the performance of frequency demodulators in the nonlinear region of operation.

12-2 Transmitter-Receiver Characterization

In this chapter we are interested in exploring the statistical dynamics of the stochastic integro-differential equation

$$\phi_M = \theta - \frac{KF(p)}{p} [Ag(\phi_M) + n_e] - \frac{K_v e}{p} \quad (12-1)$$

developed in Chapter 3 when

$$\theta(t) = \Omega_0 t + \theta_0 + M(t) \quad (12-2)$$

and oscillator instabilities are neglected. We assume that the transmitter is angle modulated with a zero-mean, stationary, Gaussian process \{m(t}\). As noted in Chapter 7, Section 7-9.2, a particular state representation for random process \{m(t)\} is the Markov process generated by the stochastic differential equation

$$\frac{dm(t)}{dt} = -am(t) + n_i(t) \quad (12-3)$$

where \{n_i(t)\} is a white Gaussian noise process with spectral density \(K_i(1)\). The spectral density of \{m(t)\} is given in (2-54) with \(k = 1\). The angle modulation present at the transmitter is conveniently characterized by

$$M(t) = K_i \int m(\tau) \, d\tau \quad (12-4)$$

where \(K_i\) is taken to be the VCO gain associated with the transmitter of Fig. 4-11.

Introducing the loop filter expansion defined in (11-5) and the state variable defined in (11-7) into (12-1), we obtain the equations
\[
\dot{\phi} = \Lambda_0 + \dot{M}(t) - K F_0 [A g(\phi) + n_e] + \sum_{k=1}^{N} y_k
\]

\[
\dot{y}_1 = -\frac{y_1}{\tau_1} - \frac{(1 - F_1)K[A g(\phi) + n_e]}{\tau_1}
\]

\[
\vdots
\]

\[
\dot{y}_N = -\frac{y_N}{\tau_N} - \frac{(1 - F_N)K[A g(\phi) + n_e]}{\tau_N}
\]

where, for convenience, we have dropped the subscript \( M \) on \( \phi_M \).

According to the theory presented in Chapter 7, the vector \( Y \triangleq y \oplus m \) represents a vector Markov process in \( L = N + 2 \) dimensions. Thus the Fokker-Planck theory given in Chapters 7 and 8 can be used to determine the transition p.d.f. \( P(Y, t|Y_0, t_0) \). Here \( Y_0 = (\phi_0, y_{10}, \ldots, y_{N0}, m_0) \) represents the initial value of \( Y \).

12-3 The Periodic Extension of \( P(Y; t) = P(Y, t|Y_0, t_0) \)

The same procedure used in Chapter 11 may be applied here to obtain the periodic extension

\[
\tilde{\rho}(\phi, Y'_0; t) \triangleq \sum_{n=-\infty}^{\infty} P[\phi + 2n\pi, Y'_0; t] \tag{12-6}
\]

where \( Y'_0 \triangleq (y'_0, m_0), y'_0 \) is defined in Chapter 11 and \( P[Y; t] \triangleq P[Y, t|Y_0, t_0] \).

As before, to obtain a solution having the properties of a transition p.d.f., we define the conditional transition p.d.f.

\[
p(\phi, Y'_0, t|Y_0, t_0, n) = \begin{cases} 
\tilde{\rho}(\phi + 2n\pi, Y'_0, t|Y_0, t_0) & \phi \in I(n) \\
0 & \text{elsewhere} 
\end{cases} \tag{12-7}
\]

where \( \{\phi(t)\} = \{\phi(t)\} + 2\pi J(t) \) and \( I(n) = [(2n - 1)\pi, (2n + 1)\pi], n \) any integer. The jump process \( J(t) \) is characterized in Section 11-4. The conditional transition p.d.f. satisfies the initial condition

\[
\lim_{t \to t_0} p(\phi, Y'_0; t) \triangleq \lim_{t \to t_0} p(\phi, Y'_0, t|Y_0, t_0, n)
\]

\[
= \delta(\phi + 2n\pi - \phi_0) \prod_{k=0}^{L-1} \delta(Y_k - Y_{k0}) \tag{12-8}
\]

and the normalization condition
The Periodic Extension of \( P(Y; t) = P(Y, t|Y_0, t_0) \)

\[
\int_{-\infty}^{\infty} \cdots \int_{(2n-1)\pi}^{(2n+1)\pi} p(\phi, Y'_0; t) \, d\phi dY'_0 = 1 \tag{12-9}
\]

We also note that the conditional transition p.d.f. must satisfy the following boundary conditions: Along any edge of the surface \( \Gamma \) of the probability space—that is, the edges \( Y_k = \pm \infty \) for all \( k = 1, 2, \ldots, L - 1 \)—we have the \( L - 1 \) boundary conditions

\[
Y_k p(\phi, Y'_0; t)|_{Y_k = \pm \infty} = 0, \quad k = 1, 2, \ldots, L - 1 \tag{12-10}
\]

since \( p(Y; t) \) approaches zero faster that \( Y_k^{(1+\epsilon)} \), \( \epsilon > 0 \), as \( Y_k \) approaches infinity. As a consequence of (12-10), we also have

\[
p(\phi, Y'_0; t)|_{Y_k = \pm \infty} = 0, \quad k = 1, \ldots, L - 1 \tag{12-11}
\]

Since (12-9) holds for all \( t \), we have \( L - 1 \) other independent boundary conditions,

\[
\frac{\partial}{\partial Y_k} [p(\phi, Y'_0; t)]|_{Y_k = \pm \infty} = 0. \quad k = 1, \ldots, L - 1 \tag{12-12}
\]

Now \( \tilde{p}(\phi, Y'_0; t) \) is periodic in \( \phi \); therefore

\[
p[(2n - 1)\pi, Y'_0; t] = p[(2n + 1)\pi, Y'_0; t] \tag{12-13}
\]

It also follows from (12-6) and (12-7) that

\[
\frac{\partial p[(2n - 1)\pi, Y'_0; t]}{\partial Y_k} = \frac{\partial p[(2n + 1)\pi, Y'_0; t]}{\partial Y_k} \quad k = 1, 2, \ldots, L - 1 \tag{12-14}
\]

which are not independent from the condition (12-13). Since \( \tilde{p}(\phi, Y'_0; t) \) is periodic in \( \phi \), we have

\[
\left. \frac{\partial p(\phi, Y'_0; t)}{\partial \phi} \right|_{\phi = (2n-1)\pi} = \left. \frac{\partial p(\phi, Y'_0; t)}{\partial \phi} \right|_{\phi = (2n+1)\pi} \tag{12-15}
\]

Equations (12-10), (12-11), (12-13), and (12-15) define \( 2L \) independent boundary conditions. From (12-14) and (12-15) we can write in vector notation,

\[
\nabla p(Y; t)|_{\phi = (2n-1)\pi} = \nabla p(Y; t)|_{\phi = (2n+1)\pi} \tag{12-16}
\]
In passing, we note that if \( \Lambda_0 = 0 \), the symmetry of (12-5) indicates that \( p(Y; t) = p(-Y; t) \).

12-4 Reduction of the Fokker-Planck Equation for the Case of Angle Modulation

If one evaluates the intensity coefficients using (7-186) and (7-187), substitutes these into (7-185), and the integrates both sides of the result with respect to \( Y_j \) for all \( j \neq k \neq 0 \), one arrives at, using (12-10) and (12-12),

\[
\frac{\partial p}{\partial t} + \frac{\partial}{\partial \phi} \left[ \mathcal{J}_0(\phi, Y_k; t) \right] + \frac{\partial}{\partial Y_k} \left[ \mathcal{J}_k(\phi, Y_k; t) \right] = 0 \tag{12-17}
\]

for all \( k = 1, \ldots, N \). In (12-17), \( p \triangleq p(\phi, Y_k; t) \) and the probability currents are given by

\[
\mathcal{J}_0(\phi, Y_k; t) = \left\{ \left[ K_0(\phi, Y_k, t) - \frac{K_{00}}{2} \frac{\partial}{\partial \phi} - \frac{K_{0k}}{2} \frac{\partial}{\partial Y_k} \right] p \right\} \tag{12-18}
\]

\[
\mathcal{J}_k(\phi, Y_k; t) = \left\{ \left[ K_k(\phi, Y_k, t) - \frac{K_{kk}}{2} \frac{\partial}{\partial Y_k} - \frac{K_{k0}}{2} \frac{\partial}{\partial \phi} \right] p \right\}
\]

where

\[
K_{00} = \frac{N_0 F_0^2 K^2}{2} \quad K_{0k} = K_{k0} = \frac{N_0 K^2 (1 - F_k) F_0}{2 \tau_k} \quad \text{for } 0 < k \leq N \tag{12-19}
\]

\[
K_{kk} = \frac{(1 - F_k)^2 N_0 K^2}{2 \tau_k^2} \quad 0 < k \leq N
\]

and

\[
K_0(\phi, Y_k, t) = \Lambda_0 - AKF_0 g(\phi) + Y_k + K_i J(\phi, Y_k, t) + \sum_{j \neq k \neq 0}^N E(Y_j, t|\phi, Y_k, Y_0)
\]

\[
K_k(\phi, Y_k, t) = -\frac{1}{\tau_k} \left[ Y_k + (1 - F_k) AKE(g(\phi), t|Y_k, Y_0) \right] \tag{12-20}
\]

for \( 0 < k \leq N \). In (12-20) we have introduced the quantity

\[
J(\phi, Y_k, t) \triangleq E(m, t|\phi, Y_k, Y_0) \tag{12-21}
\]

Again we note the importance of a knowledge of the conditional expectations.
12-5 Differential Equations for the Conditional Transition Probability Density Functions

In order to find the transition probability density functions \( p(Y_k; t) \); \( k = 0, 1, 2, \ldots, N \), we first need a differential equation whose solution is indeed \( p(Y_k; t) \). This is easily found by integrating both sides of (12-17) with respect to \( Y_k \) and applying the appropriate boundary conditions. Without belaboring the details, we obtain the partial differential equation of flow in the \( k \)th direction

\[
\nabla \cdot \mathcal{J}_k(Y_k; t) + \frac{\partial p(Y_k; t)}{\partial t} = 0 \tag{12-22}
\]

with probability currents

\[
\mathcal{J}_k(Y_k; t) = \left[ K_k(Y_k, t) - \frac{K_{kk}}{2} \frac{\partial}{\partial Y_k} \right] p(Y_k; t) \tag{12-23}
\]

When \( k = 0 \) we note, by averaging over \( Y_k \) in (12-20) conditioned upon \( \phi \), that

\[
K_0(\phi, t) = A_0 - AKF_0 g(\phi) + \sum_{k=1}^{N} E(Y_k, t|\phi, Y_0) + K, J(\phi, t) \tag{12-24}
\]

where \( J(\phi, t) = E(m, t|\phi, Y_0) \). For all \( k \neq 0 \), a similar averaging procedure yields,

\[
K_k(Y_k, t) = -\left[ \frac{Y_k + AK(1 - F_k)E(g(\phi), t|Y_k, Y_0)}{\tau_k} \right] \tag{12-25}
\]

Introducing into (12-23) the nonlinear restoring force

\[
h_k(Y_k; t) = \frac{2K_k(Y_k, t)}{K_{kk}} \tag{12-26}
\]

and the potential function

\[
U_k(Y_k; t) = -\int_{Y_k}^{Y_k} h_k(x; t) \, dx \tag{12-27}
\]

we have
\[ J_k(Y_k; t) = -\frac{K_{kk}}{2} \exp \left[ -U_k(Y_k; t) \right] \frac{\partial}{\partial Y_k} \left[ p(Y_k; t) \exp \left[ U_k(Y_k; t) \right] \right] \]

(12-28)

when \( K_{kk} \) is constant. Assuming in the limit as \( t \) approaches infinity that \( p(Y_k; t) \) approaches the steady-state p.d.f. \( p(Y_k) \), the stationary diffusion current is constant and obeys the law

\[ J_k = -\frac{K_{kk}}{2} \exp \left[ -U_k(Y_k) \right] \frac{\partial}{\partial Y_k} \left[ p(Y_k) \exp \left[ U_k(Y_k) \right] \right] \]

(12-29)

Notice here that we have suppressed the dependence of \( p(Y_k) \) on \( n \) for convenience. Solving (12-29) for \( p(Y_k) \) yields

\[ p(Y_k) = C_k \exp \left[ -U_k(Y_k) \right] \left\{ 1 + D_k \int_{l_k}^{\gamma_k} \exp \left[ U_k(x) \right] dx \right\} \]

(12-30)

where \( D_k = -2J_k/C_k K_{kk} \) and the lower limit \( l_k = (2n - 1)\pi \) if \( k = 0 \) and \( l_k = -\infty \) if \( k \neq 0 \).

To evaluate the constants \( C_k \) and \( D_k \) for \( k = 0 \), we make use of the boundary conditions. For \( n = 0 \), \( p(\pi) = p(-\pi) \) from (12-13); and we have from (12-30), with \( Y_0 = \phi \),

\[ D_0 = \frac{\exp \left[ -U_0(-\pi) \right] - \exp \left[ -U_0(\pi) \right]}{\exp \left[ -U_0(\pi) \right] \int_{-\pi}^{\pi} \exp \left[ U_0(x) \right] dx} \]

(12-31)

The constant \( C_0 \) is easily determined by means of the normalization condition. Furthermore, since \( g(\phi) \) is periodic and continuous, \( h_0(\phi) \) is also periodic, we can write a canonical form for the steady-state p.d.f.

\[ p(\phi|n) = C_0 \exp \left[ -U_0(\phi) \right] \int_{\phi}^{\phi+2\pi} \exp \left[ U_0(x) \right] dx = p(\phi) \]

(12-32)

as was done in Chapters 8, 9, and 10. Here \( \phi \) belongs to an interval of width \( 2\pi \) as defined in (12-7) and \( U_0(\phi) \) is defined in (12-27) with \( k = 0 \).

Equations (12-30) and (12-32) are remarkable in that they hold for all order loops and a broad class of nonlinearities. In fact, it is clear from (12-24) and (12-27) that the steady-state p.d.f. of the phase error of an \((N + 1)\)th-order loop is completely determined by the set of conditional expectations \( J_1 = \{ E(Y_k|\phi), J(\phi), k = 1, 2, \ldots, N \} \). Interestingly enough, \( E(Y_k|\phi) \) is the minimum mean-squared error estimate of \( Y_k \) given \( \phi \).
12-6 The Steady-State Probability Density of the Phase Error 
$p(\phi)$ When the Conditional Expectations Are Approximated

In order to obtain explicit solutions for the p.d.f.'s $p(Y_k)$, it appears, as 
before, that the conditional expectations must be approximated. For the case 
where $k = 0$ we must approximate the conditional expectations \{$E(y_k|\phi); k = 1, \ldots, N$\} and $J(\phi) = E(m|\phi)$. Using the procedures developed in Chapter 11, 
Section 11-7, to approximate these conditional expectations, we can write

\[
\hat{h}_e(\phi) = \beta'(N) - \alpha'(N)g(\phi)
\]  \hspace{1cm} (12-33)

for the restoring force. For FM, $\beta'(N)$ and $\alpha'(N)$ are defined by

\[
\beta'(N) = \beta(N) - \frac{2}{K_{00}} \left[ \frac{K_r R_{mO}(0)g}{\sigma^2} \right]
\]

\[
\alpha'(N) = \alpha(N) - \frac{2}{K_{00}} \left[ \frac{K_r R_{mO}(0)}{\sigma^2} \right]
\]  \hspace{1cm} (12-34)

and $\alpha(N)$ and $\beta(N)$ are given by (11-62) or (11-74). By using (12-33) in (12-27) 
and (12-32), an approximate expression for $p(\phi)$ is produced. Again the factor 
$\beta'(N) > 0$ is responsible for the asymmetry in $p(\phi)$; hence $p(\phi)$ will be symmetric if the loop is operating such that $\beta'(N) \approx 0$; that is,

\[
\frac{\Lambda_0}{AK} \approx g \frac{\sum_{k=1}^N (1 - F_k)}{2\pi \sigma^2} \left[ 1 + \frac{S_o(0)}{2\tau_k \sigma^2} \right] + \frac{K_r R_{mO}(0)g}{AK\sigma^2} \]  \hspace{1cm} (12-35)

For the sinusoidal PLL, $g(\phi) = \sin \phi$ and the steady-state p.d.f. is given 
approximately by

\[
p(\phi) \approx \frac{\exp \left[ \beta' \phi + \alpha' \cos \phi \right]}{4\pi^2 \exp \left( -\pi \beta' \right) I_0(\alpha')^2} \int_{\phi=0}^{\phi+2\pi} \exp \left[ -\beta'x - \alpha' \cos x \right] dx \]  \hspace{1cm} (12-36)

where we have written $\beta'$ for $\beta'(N)$ and $\alpha'$ for $\alpha'(N)$. From this point on, all 
f Formulas of Chapter 11 hold when $\beta'$ is replaced by $\beta$ and $\alpha'$ replaced by $\alpha$. 
For conciseness, we do not repeat them here. Moreover, the moments of the 
mean time to first loss of synchronization and the average number of slips per unit time are given by (11-105), (11-106), (11-108), and (11-109) with 
$U_k(Y_k : t)$ determined from the formulas (12-24), (12-25), (12-26), and (12-27). 
Numerical results will be given in Chapter 15 for a special case of the formulas 
developed in this chapter.

The results presented thus far provide us with a method for obtaining 
the statistical dynamics of the steady-state phase error process; however, our
method does not allow for determining the statistical dynamics of the frequency error. In the case of frequency demodulation by means of a PLL, the mean-squared value of the frequency tracking error $\sigma^2_\text{f}$, defined in Chapter 4, provides a measure of demodulator performance. In order to proceed with the determination of $\sigma^2_\text{f}$ for the FM case, we must sharpen our method of attack. This will be accomplished in the next three chapters.

12-7 Further Studies

When the loop filter is of the form given in (11-123), it is easy to show that

$$h_\text{f}(\phi; t) = \frac{2}{K_{\text{f}0}} \left[ \Lambda_0 - KF_0 Ag(\phi) + KJ(\phi, t) + \sum_{k=1}^{N} p_k E(y_k|t, \phi, Y_0) \right]$$

(12-37)

Using this result, the p.d.f. of the steady-state phase error process can be obtained from (12-27) and (12-32) when it exists. The moments of the mean time to first slip are formally given by determining $U_0(\phi, \bar{t})$ from (12-27) and (12-37) and substituting into (11-105).

The approximations for the conditional expectations may be introduced in order to simplify the final result; however, we postpone further development until we present newer and more general methods of obtaining solutions. We now present reference material to some of the earlier work on the problem of optimum analog demodulation.

Since the appearance of the works of Tikhonov (Refs. 1, 2) there has been considerable interest in applying (Refs. 3, 4, 5) the theory of Markov processes to the analysis of nonlinear demodulators. At the same time there has been considerable effort (Refs. 6, 7, 8, 9, 10) to establish the optimum demodulator for estimating random messages transmitted by a nonlinear modulation over a random channel. Van Trees (Refs. 6, 7, 8, 9) and Thomas and Wong (Ref. 10), among many others, have used maximum a posteriori probability (MAP) or Bayes' criteria to study various analog communication system problems. The MAP approach leads to an integral equation for the message estimate, and the solution to this equation corresponds to a physically unrealizable demodulator. Van Trees (Ref. 8) suggests making an approximation to the unrealizable demodulator for the purpose of implementation. Snyder (Ref. 11), on the other hand, takes the state variable approach to the optimum demodulation problem based on the theory of Markov processes. It turns out that the state variable approach leads directly to a physically realizable demodulator that is "equivalent" to the physically realizable portion of Van Trees' (Ref. 8)
MAP estimator. In fact, Snyder (Ref. 11) establishes quasi-optimum (optimum for large signal-to-noise ratios) phase and frequency demodulators of a stationary Gaussian message corrupted by additive white channel noise. These simplified demodulators reduce to the sinusoidal PLL mechanization discussed in this chapter. Finally, Rhee and Gohain (Ref. 12) consider the evaluation of loop performance when modulation is present. It would be interesting to make numerical evaluations for the average number of slips per second when modulation is present. These results could be used to predict loop threshold, to study the probability of loss of lock, etc., by making use of the definitions of $\alpha', \beta'$, $U_0(\varphi, t)$ given in this chapter and certain results derived in Chapters 10 and 11. These are left as exercises for the reader. Certain numerical evaluations using the theory given in this chapter are presented in Chapter 15.

**Problems**

12-1 Using (12-3) and (12-5), determine the intensity coefficients that specify the multidimensional FP equation for $p(Y, t|Y_0, t_0)$.

12-2 By direct integration of the multidimensional FP equation for $p(Y, t|Y_0, t_0)$, develop (12-17).

12-3 Justify the formulas that specify the reduced intensity coefficients given in (12-24) and (12-25).

12-4 Suggest two methods for approximating the steady-state conditional expectation $E(m|\phi)$.

12-5 Verify the expression (12-34) for $\alpha'(N)$ and $\beta'(N)$, which characterize the steady-state p.d.f. $p(\phi)$.

12-6 Assume that the loop filter has the expansion

$$F(s) = F_0 + \sum_{k=1}^{N} \frac{p_k s^{k-1}}{q_k s^k - 1}$$

where $F_0$, and the $p_k'$s and $q_k$'s are real constants.

(a) If $\theta(t) = \theta_0 + \Omega_0 t + M(t)$, where $M(t)$ is a Markov process defined in (12-3) and (12-4), defined a Markov vector, say $x$, such that $\dot{x}$ is one component of $x$.

(b) From (a) develop expressions for the intensity coefficients.

(c) Develop the reduced FP for $p(\phi, t|x_0, t_0)$ by applying the appropriate boundary conditions and integrating the multidimensional FP equation.

(d) What is the general form of the steady-state p.d.f. $p(\phi)$? What set of conditional expectations characterize its solution? [Assume that $p(\phi)$ exists.]
12-7 For the loop defined in Prob. 12-6, develop a general expression for the $n$th moment of the time to first slip.

12-8 Using the procedure outlined in Chapter 10, Section 10-5.3, develop expressions for the steady-state probability current $f_0, S$, the ratio $N_+/N_-$. Specify explicitly the constant $C_0(0)$ in terms of $U_0(\varphi; t)$.

12-9 Develop an expression for $P(N = n)$ in the same manner as was done in Section 10-5.3.

12-10 Show that the conditional expectation $E(m|\varphi)$ is periodic in $\varphi$. Justify whether it possesses odd symmetry, even symmetry, or neither in $\varphi$.

References


PART FOUR
13

SOLUTIONS TO THE
FOKKER-PLANCK EQUATION
BY THE SEQUENCE METHOD

13-1 Introduction

In the previous chapter it was suggested that solutions to problems by Fokker-Planck (FP) techniques can be approached by two methods. The first method, due to Mayfield (Ref. 1), and the subject of this chapter, is to attack the multidimensional FP equation directly and to approximate solutions. The second method is to write the one-dimensional FP equation for the non-Markov process \( z(t) \) and to approximate the resulting conditional expectations. This method will be presented in Chapter 14 and is essentially a generalization of the methods already used in Chapters 11 and 12.

This chapter begins with the derivation of the solution to the time-varying, multidimensional FP equation in sequence form. This sequence will start from an essentially arbitrary approximation to the transition p.d.f. and each iteration of the sequence will improve on this estimate. The sequence is then simplified by assuming that only steady-state results are required. Next, both forms of the sequence are further developed for the special case when the initial estimate is Gaussian. The steady-state version is also modified in this
case to produce a series converging to the steady-state variance of the process \( \{z(t)\} \). Finally, the first three terms of this series are evaluated for the case where the nonlinearity is sinusoidal and equations suitable for digital computer implementation are presented. From these equations, numerical results given in Chapter 15 are obtained. The matrix notation used in this chapter is summarized in Appendix I of Chapter 8.

13-2 The Sequence Method

The solution to the multidimensional FP equation (7-185) for the statistical response of a nonlinear system can be obtained in closed form only for extremely isolated cases. Even the existence and uniqueness of a solution are difficult to establish, in general. Nevertheless, existence and uniqueness are established by Friedman (Ref. 2) for the case where the matrix formed by the second-order intensity coefficients \( K_{ij} \) is nonsingular; however, this matrix is frequently singular. It is singular in all cases where the number of independent noise inputs is less than the total number of states.

Even though a closed-form solution to this equation normally cannot be found, partial differential equation (PDE) theory (Ref. 2) provides a method that can be modified to produce a sequence solution. In PDE theory this method is called the parametrix method for determining a fundamental solution, and it turns out that the transition p.d.f. satisfies the definition of a fundamental solution. In accordance with these facts, the FP equation (7-185) is first transformed into an integral equation. This integral equation is then solved by a method of successive approximation to yield the solution to the FP equation in the form of a sequence. The first term in the sequence is an arbitrary approximation to the solution and if properly selected will ensure rapid convergence of the sequence.

The dynamical system under consideration in what follows is represented by the stochastic state-space equation

\[
dx(t) = f[t, x(t)] \, dt + B(t) \, dB(t) \tag{13-1}
\]

\[
z(t) = Cx(t) \tag{13-2}
\]

where \( x(t) \) and \( f[t, x(t)] \) are column vectors, \( C \) is a row vector, \( B(t) \) a matrix, and \( dB(t) \) a column vector of Brownian motion. It is assumed without loss of generality that

\[
E[\beta_i(t + \tau) - \beta_i(t)][\beta_j(t + \tau) - \beta_j(t)] = \begin{cases} 
|\tau| & \text{if } i = j \\
0 & \text{if } i \neq j 
\end{cases} \tag{13-3}
\]

so that the matrix \( Q \) defined in (8-12) is the identity matrix.
13-2.1 The Time-Varying Solution to the Fokker-Planck Equation

The derivation begins with the representation of the FP equation (7-185) by the linear parabolic operator $L$ and the solution by $p(x, t|\xi, \tau)$ so that

$$Lp(x, t|\xi, \tau) = 0$$

(13-4)

where

$$L \triangleq \sum_{ij} \frac{K_{ij}(t)}{2} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_i \frac{\partial}{\partial x_i} K_i(x, t) - \frac{\partial}{\partial t}$$

(13-5)

Notice the assumed independence of $K_{ij}$ on $x$. The function $p(x, t|\xi, \tau)$ is the transition p.d.f. of state $x$ at time $t$ given that the system started at time $\tau$ with the initial condition $x(\tau) = \xi(\tau)$ (see Chapter 7).*

In order for $p(x, t|\xi, \tau)$ to be a transition p.d.f., we recall from Chapter 7 that it must satisfy the normalization condition

$$\int_{\Omega} p(x, t|\xi, \tau) dx = 1$$

(13-6)

for all $t > \tau$ and the condition that

$$\lim_{t \to \tau} \int_{\Omega} p(x, t|\xi, \tau)p(\xi, \tau) d\xi = p(x, \tau)$$

(13-7)

for any initial p.d.f. $p(\xi, \tau)$. Condition (13-7) is simply a statement that $p(x, t|\xi, \tau)$ behaves like a Dirac $\delta$ function as $t$ approaches the initial time $\tau$. Condition (13-7) also implies that $p(x, t|\xi, \tau)$ is the fundamental solution in the parametrix method of partial differential equation theory (Ref. 2).

Let $p_0(x, t|\xi, \tau)$ be an initial estimate of $p(x, t|\xi, \tau)$ which is also required to satisfy conditions (13-6) and (13-7). Now, in parallel with the parametrix method discussed in Ref. 2, an effort is made to find a solution $p(x, t|\xi, \tau)$ in the form

$$p(x, t|\xi, \tau) = p_0(x, t|\xi, \tau) + \int_{\tau}^{t} \int_{\Omega} p_0(x, t|\eta, \sigma)\psi(\eta, \sigma, \xi, \tau) d\eta d\sigma$$

(13-8)

for some function $\psi(\eta, \sigma, \xi, \tau)$. Using rules of differentiation of integrals and condition (13-7), it is easily seen that in order for $p(x, t|\xi, \tau)$ to satisfy the FP equation given in (13-4), the function $\psi(x, t, \xi, \tau)$ must satisfy the integral equation

*For convenience in what follows, we set $t_0 = \tau$ and $x(t_0) = x(\tau) = \xi(\tau)$. 

\[
\psi(x, t, \xi, \tau) = Lp_0(x, t|\xi, \tau) \\
+ \int_{\tau}^{t} \int_{\Omega} [Lp_0(x, t|\eta, \sigma)] \psi(\eta, \sigma, \xi, \tau) \, d\eta \, d\sigma \tag{13-9}
\]

The problem of solving a partial differential equation is now transformed into that of solving an integral equation. This equation has flexibility in that \(p_0(x, t|\xi, \tau)\) is arbitrary. In general, if \(p_0(x, t|\xi, \tau)\) can be selected close to \(p(x, t|\xi, \tau)\), (13-8) will not be very sensitive to an error in \(\psi(x, t, \xi, \tau)\) and an approximation to (13-9) can be made fairly crudely.

Now, (13-9) is a Volterra integral equation that is discussed in Friedman (Ref. 2) and Volterra (Ref. 3) and that can be solved by the method of successive approximation. Let the first approximation be given by

\[
\psi_0(x, t, \xi, \tau) = Lp_0(x, t|\xi, \tau) \tag{13-10}
\]

and further approximations by

\[
\psi_{n+1}(x, t, \xi, \tau) = Lp_0(x, t|\xi, \tau) + \int_{\tau}^{t} \int_{\Omega} [Lp_0(x, t|\eta, \sigma)] \psi_n(\eta, \sigma, \xi, \tau) \, d\eta \, d\sigma \tag{13-11}
\]

or, equivalently, by

\[
\psi_{n+1}(x, t, \xi, \tau) = Lp_0(x, t|\xi, \tau) + \int_{\tau}^{t} \int_{\Omega} \psi_n(x, t, \eta, \sigma) \times [Lp_0(\eta, \sigma|\xi, \tau)] \, d\eta \, d\sigma \tag{13-12}
\]

Equations (13-11) and (13-12) can be seen to be equivalent by expanding (13-11) and regrouping factors.

This sequence is shown to converge (Ref. 2) to the correct solution in cases where \(p_0(x, t|\xi, \tau)\) is a solution to the FP equation with \(K(x, t) = 0\). If, instead, \(p_0(x, t|\xi, \tau)\) is a solution to the FP equation where \(K(x, t)\) has been approximated by a linear part, resulting in a Gaussian initial approximation, it is reasonable to expect even faster convergence.*

To complete the derivation of the sequence solution, define

\[
p_{n+1}(x, t|\xi, \tau) = p_0(x, t|\xi, \tau) + \int_{\tau}^{t} \int_{\Omega} p_0(x, t|\eta, \sigma) \times \psi_{n+1}(\eta, \sigma, \xi, \tau) \, d\eta \, d\sigma \tag{13-13}
\]

*An error estimate analysis showing the rate of convergence is not presently available; however, numerical and computer simulation results, to be given in Chapter 15, show that the sequence converges fast enough to be potentially useful for a wide class of problems pertaining to frequency demodulation.
If (13-12) is substituted for \( \psi_{n+1}(\eta, \sigma, \xi, \tau) \), then (13-13) becomes

\[
p_{n+1}(x, t|\xi, \tau) = p_0(x, t|\xi, \tau) + \int_\tau^t \int_\Omega p_0(x, t|\eta, \sigma) \times \left[ Lp_0(\eta, \sigma|\xi, \tau) \right] d\eta d\sigma (13-14)
\]

where this sequence is started by the initial estimate \( p_0(x, t|\xi, \tau) \).

Equation (13-14) defines the sequence method in its most general form. It yields the time-varying, multidimensional transition p.d.f. from the FP equation of a Markov process, or from any such partial differential equation—for example, Pawula's generalized Fokker-Planck equation (Ref. 4). It will be used in Chapter 14, Section 14-4, to refine an estimate of a transition p.d.f. obtained by yet another method.

If the process \( \{x(t)\} \) is stationary or, less restrictively, simply temporally homogeneous, the transition p.d.f. \( p(x, t|\xi, \tau) \) depends only on the difference between the present time \( t \) and the initial time \( \tau \) (see Chapter 7). In this case we note that \( p(x, t|\xi, \tau) = p(x, t - \tau|\xi) \). If this same property also holds for each estimate of \( p(x, t|\xi) \), then (13-13) can be written in the stationary form

\[
p_{n+1}(x, t|\xi) = p_0(x, t|\xi) + \int_0^t \int_\Omega p_0(x, t - \sigma|\eta) [Lp_0(\eta, \sigma|\xi)] d\eta d\sigma (13-15)
\]

This stationary version of the sequence will be useful when we consider the steady-state solution.

In order for (13-13) or (13-14) to be used to produce estimates of \( p(x, t|\xi, \tau) \), it is necessary to establish that each estimate satisfies the properties (13-6) and (13-7). To do so, it is necessary to establish that

\[
\int_\Omega [Lp_0(x, t|\xi, \tau)] dx = 0 (13-16)
\]

This can be done by writing

\[
\int_\Omega [Lp_0(x, t|\xi, \tau)] dx = \sum_i \int_\Omega \frac{\partial}{\partial x_i} \left[ \sum_j K_{ij}(t) \frac{\partial p_0(x, t|\xi, \tau)}{\partial x_j} \right] dx
\]

\[
- K_i(x, t)p_0(x, t|\xi, \tau) + \frac{\partial}{\partial t} \int_\Omega p_0(x, t|\xi, \tau) dx (13-17)
\]

The second term is zero because \( p_0(x, t|\xi, \tau) \) satisfies the normalization condition (13-6), which leaves
\[
\int_\Omega \left[ Lp_0(x, t|\xi, \tau) \right] dx = \sum_i \int_{\Omega_i} \left[ \sum_j \frac{K_{ij}(t)}{2} \frac{\partial p_0(x, t|\xi, \tau)}{\partial x_j} \right. \\
- K_i(x, t)p_0(x, t|\xi, \tau) \left. \right] dx^*_i \bigg|_{x_i = x_i}^{x_i} \quad (13-18)
\]

where \( \Omega^*_i \) is the resulting probability space formed by deleting the \( i \)th component of \( \Omega \) and where \( x_{i_i} \) and \( x_{i_i} \) are the endpoints of \( \Omega_i \). Here \( \Omega_i \) denotes that component of \( \Omega \) in the \( i \)th direction. The quantity inside the sum is the initial approximation to the total probability current defined in Chapter 7. If this probability current is equal at \( x_{i_i} \) and \( x_{i_i} \) for each \( i \), then this term is zero and (13-16) holds. In what follows, it is assumed that this is the case. The fact that (13-6) and (13-7) hold for any \( p_n(x, t|\xi, \tau) \) then follows in a straightforward manner.*

It is interesting to point out that a sequence which converges even faster can be obtained from (13-13). This is done by redefining \( p_0(x, t|\xi \tau) \) after each iteration to be the output of the preceding iteration. If the details of this suggestion are worked out, the result will be

\[
p_{n+1}(x, t|\xi, \tau) = p_n(x, t|\xi, \tau) + \int_\Omega \int_{\mathcal{R}} p_n(x, t|\eta, \sigma) \times \left[ Lp_0(\eta, \sigma|\xi, \tau) \right] d\eta d\sigma 
\]

(13-19)

This sequence actually is a consequence of (13-14); however, it has the disadvantage that \( Lp_n(x, t|\xi, \tau) \) must be recomputed after each iteration.

**13-2.2 The Steady-State Solution to the Fokker-Planck Equation**

A sequence starting with an arbitrary estimate and converging to the general solution of the FP equation was given in (13-15). This equation is simplified in this section for the case where only a steady-state solution is desired.

If such a steady-state solution exists, \( p_n(x, t|\xi) \) will not depend on \( \xi \) for \( t \) large and (13-15) can be written by taking the limit as \( t \) goes to infinity as

\[
p_{n+1}(x) = p_n(x) + \lim_{t \to \infty} \int_\Omega \left[ \int_0^t p_n(x, t - \sigma|\eta)Lp_0(\eta, \sigma|\xi) d\sigma \right] d\eta \quad (13-20)
\]

where

*Note that these conditions are necessary but not sufficient for \( p_0(x, t|\xi, \tau) \) to be a transition p.d.f. It is conjectured that the condition \( p_0(x, t|\xi, \tau) \geq 0 \) (needed for sufficiency) is also satisfied at least for the case where \( p_0(x, t|\xi, \tau) \) satisfies a FP equation with the same \( K_{ij}(t) \) as the original equation. Proof of this condition, however, is left for further study.*
The Sequence Method

\[ p_n(x) \triangleq \lim_{t \to \infty} p_n(x, t|\xi) \quad (13-21) \]

If we define the notation \( Lp_0(x) \) by

\[ Lp_0(x) \triangleq \lim_{t \to \infty} Lp_0(x, t|\xi) \quad (13-22) \]

then the time integral can be expanded by adding and subtracting terms in order to eliminate those terms that will be zero in the limit. Thus

\[
\int_{0}^{t} p_n(x, t - \sigma|\eta)Lp_0(\eta, \sigma|\xi) \, d\sigma = p_n(x)Lp_0(\eta)t \\
+ p_n(x) \int_{0}^{t} [Lp_0(\eta, \sigma|\xi) - Lp_0(\eta)] \, d\sigma \\
+ \int_{0}^{t} [p_n(x, \sigma|\eta) - p_n(x)] \, d\sigma \, Lp_0(\eta) \\
+ \int_{0}^{t} [p_n(x, t - \sigma|\eta) - p_n(x)] [Lp_0(\eta, \sigma|\xi) - Lp_0(\eta)] \, d\sigma 
\quad (13-23) \]

The last term is the convolution of two quantities, each of which has a zero steady-state value. By examining the properties of convolution integrals, it can be seen that this term becomes arbitrarily small as \( t \) gets large. If this term is neglected, if the remaining terms are put back into (13-20), and if (13-16) is considered, (13-20) becomes

\[ p_{n+1}(x) = p_0(x) + \lim_{t \to \infty} \int_{\Omega} \left[ \int_{0}^{t} p_n(x, \tau|\eta) \, d\tau \right] Lp_0(\eta) \, d\eta \quad (13-24) \]

An expression for the integral \( \int_{0}^{t} p_0(x, \tau|\eta) \, d\tau \) must be determined which applies for large \( t \) before (13-24) can be used in a particular application. Again using the sequence equation (13-15) results in

\[
\int_{0}^{t} p_{n+1}(x, \tau|\xi) \, d\tau = \int_{0}^{t} p_0(x, \tau|\xi) \, d\tau \\
+ \int_{\Omega} \left[ \int_{0}^{t} \int_{0}^{t} p_n(x, \tau - \sigma|\eta)Lp_0(\eta, \sigma|\xi) \, d\sigma \, d\tau \right] \, d\eta 
\quad (13-25) \]

The double integral with respect to time can be simplified considerably. This term can be expanded in a manner similar to that for (13-20) to yield
\[
\int_0^t \int_0^\tau p_n(x, \tau - \sigma|\eta) Lp_0(\eta, \sigma|\xi) \, d\sigma \, d\tau \\
= p_n(x) Lp_0(\eta) \frac{\tau^2}{2} + p_n(x) \int_0^\tau \int_0^\tau [Lp_0(\eta, \sigma|\xi) - Lp_0(\eta)] \, d\sigma \, d\tau \\
+ \int_0^\tau \int_0^\tau [p_n(x, \tau - \sigma|\eta) - p_n(x)] \, d\sigma \, d\tau \, Lp_0(\eta) \\
+ \int_0^\tau \int_0^\tau [p_n(x, \tau - \sigma|\eta) - p_n(x)][Lp_0(\eta, \sigma|\xi) - Lp_0(\eta)] \, d\sigma \, d\tau 
\]

(13-26)

Considering (13-16), the first two terms of (13-26) will be zero when the integration over \( \eta \) in (13-25) is taken. The third term will be similarly eliminated by the space integration in (13-24), which leaves only the last term. Since it represents the integral of the convolution of two quantities, each of which has zero steady-state value, the Laplace transform should exist. Let \( \mathcal{L}\{ \ldots \}(s) \) be the Laplace transform operator so that by Laplace transform theory

\[
\mathcal{L}\left\{ \int_0^\tau \int_0^\tau [p_n(x, \tau - \sigma|\eta) - p_n(x)][Lp_0(\eta, \sigma|\xi) - Lp_0(\eta)] \, d\sigma \, d\tau \right\}(s) \\
= \frac{1}{s} \mathcal{L}\{p_n(x, t|\eta) - p_n(x)(s)\mathcal{L}\{[Lp_0(\eta, t|\xi) - Lp_0(\eta)](s) \quad (13-27)\}
\]

The steady-state value of this term can now be determined by the final value theorem of Laplace transform theory, which states that for a function of time \( f(t) \) its final value is given by \( f(\infty) = \lim_{s \to 0} s \mathcal{L}\{f(t)\}(s) \). Using this theorem and the definition of the Laplace transform results in

\[
\int_0^\infty \int_0^\tau [p_n(x, \tau - \sigma|\eta) - p_n(x)][Lp_0(\eta, \sigma|\xi) - Lp_0(\eta)] \, d\sigma \, d\tau \\
= \int_0^\infty [p_n(x, \tau|\eta) - p_n(x)] \, d\tau \int_0^\infty [Lp_0(\eta, \sigma|\xi) - Lp_0(\eta)] \, d\sigma \quad (13-28)
\]

Inserting this equation back into (13-25) and again making use of (13-16) produces the result that

\[
\int_0^\tau p_{n+1}(x, \tau|\xi) \, d\tau = \int_0^\tau p_0(x, \tau|\xi) \, d\tau + \int_\Omega \left[ \int_0^\tau p_n(x, \tau|\eta) \, d\tau \right] \\
\times \left[ \int_0^\tau Lp_0(\eta, \sigma|\xi) \, d\sigma \right] \, d\eta 
\]

(13-29)

where the equality holds only for large \( t \) and terms that are zero when (13-24) is applied have been neglected.
Equations (13-24) and (13-29) can be used to compute a steady-state solution; however, they can be combined into a more convenient form as the single equation

$$p_{n+1}(x, t|\xi) = p_0(x, t|\xi) + \int_0^\infty \int_\Omega p_o(x, \sigma|\eta)Lp_0(\eta, t|\xi) \, d\eta \, d\sigma$$  \hspace{1cm} (13-30)

Equation (13-30) can be seen to hold if (13-24) and (13-29) are formed into a single truncated series; moreover, (13-30) yields the same truncated series whenever $t$ is large.

Equation (13-30) represents the sequence method in cases where steady-state results are desired. Since only the steady-state case is considered in the numerical evaluations that follow, this equation forms the basis for these results. Equation (13-30) affords much simplification over the time-varying version (13-15). As will be seen later, the integral over the probability space $\Omega$ can often be written in closed form. The integration over time may require either approximation or numerical integration. Equation (13-30) ensures that the time integrals appear singly, so that numerical evaluation of multiple integrals is avoided.

13-3 The Gaussian Density Function as an Initial Estimate for the Sequence Method

The general solution to the FP equation is provided in the form of a sequence (13-15), and a simplification is provided (13-30) for use when only the steady-state solution is of interest. Both sequences require an initial estimate $p_0(x, t|\xi)$ to the state transition probability density function. In many cases the sample space $\Omega$ is infinite in each direction, and it is convenient to select $p_0(x, t|\xi)$ as a multidimensional Gaussian p.d.f. that satisfies some portion of the original FP equation.

The state equations for this section are written, for convenience, in their least general form as

$$dx(t) = f(x(t)) \, dt + B \, dB(t)$$

$$x(0) = \xi$$  \hspace{1cm} (13-31)

where both $f$ and $B$ are assumed to be free of a dependence on time. Suppose that the nonlinear function $f(x)$ is separated into a linear part, $Mx$, and a nonlinear part, $l(x)$, so that

$$f(x) = Mx + l(x)$$  \hspace{1cm} (13-32)

Assume that $p_0(x, t|\xi)$ is then selected as the state transition p.d.f. of
\[ dx(t) = Mx(t) \, dt + B \, d\beta(t) \]
\[ x(0) = \xi \]  

(13-33)

Since (13-33) is linear, it is evident that \( p_0(x, t|\xi) \) is Gaussian. Since \( M \) is chosen arbitrarily, \( p_0(x, t|\xi) \) can be a Gaussian transition p.d.f. whose FP equation has the same second partial derivative terms as the original or, equivalently, the same second-order intensity coefficients. If \( M = \mathbf{f}_x(0) \) is chosen, then \( p_0(x, t|\xi) \) is the exact solution for the linear system model and iteration by (13-15) or (13-30) will serve to improve on the linear solution.

In order to determine \( p_0(x, t|\xi) \), it is necessary to compute only the mean of the vector \( x \) and its covariance matrix. The solution to (13-33) is known to be given by (Ref. 5)

\[ x(t) = \Phi(t)\xi + \int_0^t \Phi(t - \tau)B \, d\beta(\tau) \]  

(13-34)

where \( \Phi(t) \) is the state transition matrix for \( M \). Now the mean value of \( x \), given the initial conditions \( \xi \), can be easily determined as

\[ E[x(t)|\xi] = \Phi(t)\xi \]  

(13-35)

If we define the covariance matrix \( R(t) \) by

\[ R(t) \triangleq E[(x(t) - \Phi(t)\xi)(x(t) - \Phi(t)\xi)'|\xi] \]  

(13-36)

then

\[ R(t) = \int_0^t \Phi(\tau)BB'\Phi'(\tau) \, d\tau \]  

(13-37)

and the transition p.d.f. \( p_0(x, t|\xi) \) is given by

\[ p_0(x, t|\xi) = \frac{\exp \left[ -\frac{1}{2} \| x - \Phi(t)\xi \|^2_{\Lambda(t)} \right]}{\sqrt{(2\pi)^N \det(R(t))}} \]  

(13-38)

where \( N \) is the dimension of \( x \) and the matrix \( \Lambda(t) \) is defined by \( \Lambda(t) \triangleq R^{-1}(t) \).

The FP equation, of which \( p_0(x, t|\xi) \) is a solution, is the original FP equation where \( K_t(x) = f_t(x) \) has been replaced by \( K_t(x) = [Mx] \), so that the quantity \( LP_0(x, t|\xi) \) needed in the sequence equation is given by

\[ LP_0(x, t|\xi) = -\sum_i \frac{\partial}{\partial x_i} [l_i(x)p_0(x, t|\xi)] \]  

(13-39)
The Gaussian Density Function as an Initial Estimate for the Sequence Method

Differentiating and using (13-38) then leads to

\[ Lp_0(x, t|\xi) = \beta(x, t, \xi)p_0(x, t|\xi) \]  
\[ (13-40) \]

where

\[ \beta(x, t, \xi) = [x - \Phi(t)\xi]\Lambda(t)l(x) - \text{tr } [L_x(x)] \]  
\[ (13-41) \]

Making use of (13-40) in (13-15) for the case where \( p_0(x, t|\xi) \) is Gaussian, we have

\[ p_{n+1}(x, t|\xi) = p_0(x, t|\xi) + \int_0^t \int_\Omega p_n(x, t - \sigma|\eta) \times \beta(\eta, \sigma, \xi)p_0(\eta, \sigma|\xi) \ d\eta \ d\sigma \]  
\[ (13-42) \]

On the other hand, (13-30) for the steady-state p.d.f. becomes

\[ p_{n+1}(x, t|\xi) = p_0(x, t|\xi) + \int_0^\infty \int_\Omega p_n(x, \sigma|\eta) \times \beta(\eta, t, \xi)p_0(\eta, t|\xi) \ d\eta \ d\sigma \]  
\[ (13-43) \]

Equations (13-42) and (13-43) represent the sequence method for the case where the initial approximation is Gaussian. The significant feature is that the derivatives have been eliminated and the operator \( L \) has been replaced by the function \( \beta(x, t, \xi) \). These equations can be used directly to produce approximations to the transition p.d.f. If this task is pursued, it will be seen that evaluation of the integral over the probability space \( \Omega \) involves the determination of Gaussian moments of nonlinear functions. These are readily available in closed form for a large class of functions, including those made up of polynomials, exponential functions, sine functions, and cosine functions. Many of these are presented in tabulated form in Appendix I.

Equation (13-43) can be broken down even further if the quantity that is desired is the steady-state variance of some linear combination of states. Let

\[ z(t) = Cx(t) \]  
\[ (13-44) \]

where \( C \) is a row vector, and suppose that it is desired to find the steady-state value of

\[ v(t, \xi) \triangleq E[z^2(t)|\xi] = CE[x(t)x'(t)|\xi]C' \]  
\[ (13-45) \]

Let \( v_n(t, \xi) \) be the \( n \)th approximation to \( v(t, \xi) \). This approximation is formed by using the \( n \)th approximation to the transition p.d.f. from (13-43) to
evaluate the expectation in (13-45). A resulting sequence in the variance is formed; that is,

\[ v_{n+1}(t, \xi) = v_0(t, \xi) + \int_0^\infty \int_\Omega v_0(\sigma, \eta) \beta(\eta, t, \xi) p_0(\eta, t|\xi) \, d\eta \, d\sigma \]  \hspace{1cm} (13-46)

where it is seen from (13-35) that

\[ v_0(t, \xi) = C R(t) C' + C \Phi(t) \xi \xi' \Phi(t) C' \]
\[ = C R(t) C' + \xi' \Phi'(t) C' C \Phi(t) \xi \]  \hspace{1cm} (13-47)

We now define a matrix \( P \) by

\[ P \triangleq \int_0^\infty \Phi'(\tau) C' C \Phi(\tau) \, d\tau \]  \hspace{1cm} (13-48)

and a sequence \( V_n(t, \xi) \) by

\[ V_1(t, \xi) = \int_\Omega x' P x \beta(x, t, \xi) p_0(x, t|\xi) \, dx \]  \hspace{1cm} (13-49)
\[ V_{n+1}(t, \xi) = \int_0^\infty \int_\Omega V_n(\sigma, x) \beta(x, t, \xi) p_0(x, t|\xi) \, dx \, d\sigma \]  \hspace{1cm} (13-50)

By repeated use of (13-46), it is easily shown starting with (13-47), that a series for the steady-state variance is given by

\[ v_n = C R C' + \sum_{i=1}^n V_i \]  \hspace{1cm} (13-51)

where

\[ v_n \triangleq v_n(\infty, \xi) \quad V_n \triangleq V_n(\infty, \xi) \]

Equations (13-49), (13-50), and (13-51) provide a method of obtaining an estimate of the steady-state variance of a quantity in a nonlinear system. Here it is apparent that the integral over \( \Omega \) can be eliminated as long as the Gaussian moments can be evaluated. We shall explore this approach in the next section. In Chapter 15 it will be applied to the problem of phase-locked frequency demodulation.

### 13-4 The Steady-State Variance for a Sinusoidal Nonlinearity

In the previous section it is shown that if the initial estimate to the transition p.d.f. is Gaussian and if it satisfies certain linear portions of the FP
equation, then the steady-state variance of a linear combination of states can be computed using (13-51). In this section it is assumed that the nonlinearity is sinusoidal so that

\[ f(x) = Ax + D \sin(ax) \]  

(13-52)

where \( A \) is a matrix, \( D \) is a column vector, and \( a \) is a row vector. Using (13-52), the first two nonlinear terms of (13-51) are computed and are presented in a form suitable for digital-computer implementation.

If we define

\[ \Gamma \triangleq M - A \]  

(13-53)

then from (13-32) and (13-52) it follows that the nonlinear part of \( f(x) \) is given by

\[ l(x) = D \sin(ax) - \Gamma x \]  

(13-54)

From (13-41) it then follows that

\[ \beta(x, t, \xi) = [x - \Phi(t)\xi']A(t)[D \sin(ax) - \Gamma x] - aD \cos(ax) + \text{tr} [\Gamma] \]  

(13-55)

Adopting the notation that \( \hat{E} \) is the expected value operator with respect to the Gaussian density \( p_0(x, t|\xi) \), then

\[ \nu_1(t, \xi) = \hat{E}[x'Px\beta(x, t, \xi)] \]  

(13-56)

Straightforward manipulations involving matrix algebra, and the evaluation of Gaussian moments using Appendix I, lead to the result

\[ \nu_1(t, \xi) = 2[aR(t)PD \cos[a\Phi(t)\xi]] + D'P\Phi(t)\xi \sin[a\Phi(t)\xi]] \exp[-\frac{1}{2}aR(t)a'] - 2 \text{tr} [\Gamma R(t)P] - 2\xi'\Phi'(t)P\Gamma(t)\xi \]  

(13-57)

which, in the steady state, becomes the first nonlinear term in the series.

\[ \nu_1 = 2[aRPD \exp(-\frac{1}{2}aRa') - \text{tr} (\Gamma RP)] \]  

(13-58)

For simplicity, the next term \( \nu_2 \) will be computed only in the steady state. From (13-50) and (13-57), \( \nu_2 \) is given by
\[ V_2 = \int_0^\infty \mathcal{E}[V_1(\sigma, x)\beta(x)] \, d\sigma \]
\[ = 2 \int_0^\infty \left\{ [aR(\sigma)PD][\cos [a\Phi(\sigma)x]\beta(x)] + D'P\Phi(\sigma)[x \sin [a\Phi(\sigma)x]\beta(x)] \exp \left[ -\frac{1}{2} aR(\sigma)a' \right] \right\} \, d\sigma \]
\[ - \mathcal{E}[x'\Phi'(\sigma)P\Gamma\Phi(\sigma)x\beta(x)] \, d\sigma \] (13-59)

where \( \beta(x) \triangleq \beta(x, \infty, 0) \). Evaluation of each of these terms can be done with matrix algebra, using the Gaussian moments evaluated in Appendix I. Thus

\[ \mathcal{E}[x'\Phi'(\sigma)P\Gamma\Phi(\sigma)x\beta(x)] = aR\Phi'(\sigma)(P\Gamma + \Gamma'P)\Phi(\sigma) \]
\[ \times \exp \left( -\frac{1}{2} aRa' \right) \]
\[ - \text{tr} \left[ \Gamma R\Phi'(\sigma)(P\Gamma + \Gamma'P)\Phi(\sigma) \right] \] (13-60)

\[ \mathcal{E}[\cos [a\Phi(\sigma)x]\beta(x)] = a\Phi(\sigma)[\Gamma R\Phi'(\sigma)a' \]
\[ - D \sinh [a\Phi(\sigma)Ra'] \exp \left[ -\frac{1}{2} aRa' \right] \]
\[ \times \exp \left[ -\frac{1}{2} a\Phi(\sigma)R\Phi'(\sigma)a' \right] \] (13-61)

\[ \mathcal{E}[x \sin [a\Phi(\sigma)x]\beta(x)] = [R\Phi'(\sigma)a'a\Phi(\sigma)\Gamma R\Phi'(\sigma)a' \]
\[ - \Gamma R\Phi'(\sigma)a' - R\Gamma'\Phi(\sigma)a' + [D - R\Phi'(\sigma)a'a\Phi(\sigma)D] \sinh [a\Phi(\sigma)Ra'] \]
\[ \times \exp \left( -\frac{1}{2} aRa' \right) \]
\[ + Ra'a\Phi(\sigma)D \cosh [a\Phi(\sigma)Ra'] \]
\[ \times \exp \left( -\frac{1}{2} aRa' \right) \]
\[ \times \exp \left[ -\frac{1}{2} a\Phi(\sigma)R\Phi'(\sigma)a' \right] \] (13-62)

Equations (13-59), (13-60), (13-61), and (13-62) can now be combined and scalar functions can be defined to produce the second nonlinear term in the expansion.

\[ V_2 = 2 \int_0^\infty \left\{ \chi(t) + [\Delta(t) + \Omega(t)\eta(t)] \exp [\psi(t)] \right\} \]
\[ + \left[ \zeta(t) + \Omega(t)\gamma(t) \right] \sinh [\delta(t)] \exp [\psi(t)] \]
\[ + \nu(t)\gamma(t) \cosh [\delta(t)] \exp [\psi(t)] \exp (\frac{1}{2} aRa') \] dt (13-63)

The scalar functions in (13-63) are defined by

\[ \chi(t) \triangleq \text{tr} \left[ \Gamma R\Phi'(t)(P\Gamma + \Gamma'P)\Phi(t) \right] \]
\[ - aR\Phi'(t)(P\Gamma + \Gamma'P)\Phi(t)D \exp \left( -\frac{1}{2} aRa' \right) \] (13-64)
The Steady-State Variance for a Sinusoidal Nonlinearity

\[ \psi(t) \triangleq -\frac{1}{2} a[R(t) + \Phi(t)R\Phi'(t) - R]a' \]  
(13-65)

\[ \Omega(t) \triangleq -a[R(t) + \Phi(t)R\Phi'(t) - R]PD \]  
(13-66)

\[ \eta(t) \triangleq -a\Phi(t)\Gamma R\Phi'(t)a' \exp(-\frac{1}{2} aRa') \]  
(13-67)

\[ \Delta(t) \triangleq a\Phi(t)\Gamma R\Phi'(t)a'aR - \Gamma R\Phi'(t) \]  

\[-R\Gamma'\Phi'(t)]PD \exp(-\frac{1}{2} aRa') \]  
(13-68)

\[ \gamma(t) \triangleq a\Phi(t)D \exp(-aRa') \]  
(13-69)

\[ \delta(t) \triangleq a\Phi(t)Ra' \]  
(13-70)

\[ \zeta(t) \triangleq (D'P - aRPD)a\Phi(t)D \exp(-aRa') \]  
(13-71)

\[ \nu(t) \triangleq aR\Phi'(t)PD \]  
(13-72)

Equation (13-63), together with (13-64) through (13-72), is now in a form that can be solved for \( V_2 \) with the use of a digital computer. The integral over the probability space \( \Omega \) has been eliminated and only the integral over time remains. The time integral can be solved by numerical integration, or the exponential, \( \sinh \), and \( \cosh \) functions can be expanded in Taylor series and the integration worked out exactly for each term. The scalar functions defined by (13-64) through (13-72) have been carefully selected so that all of them are sums of exponential functions, and if the linear system is stable, they have zero steady-state values. If the arguments \( \psi(t) \) and \( \delta(t) \) of the nonlinear functions approach zero sufficiently fast as time becomes large, the first few terms in Taylor series expansions should produce accurate results.

This latter approach will be used to obtain results in Chapter 15. We begin by first expanding the state transition matrix \( \Phi(t) \) in a residue expansion (Ref. 5) as

\[ \Phi(t) = \sum_{i} Z_i e^{\lambda_i t} \]  
(13-73)

where \( Z_i \) are the residue matrices related to the eigenvalues \( \lambda_i \) respectively of the linear system from which \( p_0(x, t|\xi) \) was derived. In order to write the expansion in this form, we assume that there are no repeated eigenvalues or, equivalently, no multiple poles. The residue matrices \( Z_i \) are easily computed using the algorithm from page 304 of Zadeh and Desoer (Ref. 5). The matrices \( R \) and \( P \) are also easily computed from these residue matrices. By using (13-73), each of the scalar function can now be expanded, that is,

\[ \chi(t) = \sum_{ij} \chi_{ij} e^{(\lambda_i + \lambda_j)t} \]  
(13-74)

\[ \psi(t) = \sum_{ij} \psi_{ij} e^{(\lambda_i + \lambda_j)t} \]  
(13-75)

\[ \Omega(t) = \sum_{ij} \Omega_{ij} e^{(\lambda_i + \lambda_j)t} \]  
(13-76)
\[ \eta(t) = \sum_{ij} \eta_{ij} e^{i\lambda_i + i\lambda_j} t \]  
(13-77)

\[ \Delta(t) = \sum_{ij} \Delta_{ij} e^{i\lambda_i + i\lambda_j} t \]  
(13-78)

\[ \gamma(t) = \sum_i \gamma_i e^{i\lambda_i} \]  
(13-79)

\[ \delta(t) = \sum_i \delta_i e^{i\lambda_i} \]  
(13-80)

\[ \zeta(t) = \sum_i \zeta_i e^{i\lambda_i} \]  
(13-81)

\[ \nu(t) = \sum_i \nu_i e^{i\lambda_i} \]  
(13-82)

where

\[ X_{ij} = \text{tr} \left[ \Gamma R Z_j (P^* + \Gamma P) Z_j \right] \]  
(13-83)

\[ \psi_{ij} = -\frac{1}{2} aZ_i \left( R + \frac{BB'}{\lambda_i + \lambda_j} \right) Z_j a' \]  
(13-84)

\[ \Omega_{ij} = -aZ_i \left( R + \frac{BB'}{\lambda_i + \lambda_j} \right) Z_j PD \]  
(13-85)

\[ \eta_{ij} = -aZ_i \Gamma R Z_j a' \exp\left(-\frac{1}{2} aRa'\right) \]  
(13-86)

\[ \Delta_{ij} = aZ_i (\Gamma R Z_j a' a' R - \Gamma R Z_j - R \Gamma' Z_j) PD \]  
(13-87)

\[ \times \exp\left(-\frac{1}{2} aRa'\right) \]  
(13-88)

\[ \gamma_i = aZ_i D \exp(aRa') \]  
(13-89)

\[ \delta_i = aZ_i Ra' \]  
(13-90)

\[ \zeta_i = (D'P - aRP Da) Z_i D \exp(-aRa') \]  
(13-91)

\[ \nu_i = aRZ_i PD \]  
(13-92)

The exponential, sinh, and cosh functions in (13-63) are now expanded in Taylor series. If terms leading to greater than fourth-order sums are eliminated, the result is

\[ V_2 = \int_0^\infty \left[ \chi(t) + \Delta(t) + \zeta(t) \delta(t) + \nu(t) \gamma(t) \right. \]  
\[ + \Omega(t)[\eta(t) + \gamma(t) \delta(t)] \]  
\[ + \psi(t)[\Delta(t) + \zeta(t) \delta(t) + \nu(t) \gamma(t)] \]  
\[ + \frac{1}{2} \delta(t)\left[ \frac{1}{2} \zeta(t) \delta(t) + \nu(t) \gamma(t) \right] \]  
\[ \left. dt \right] \]  
(13-92)

This integral can now be solved to produce an equation that can be implemented on a digital computer; that is,
Further Studies

\[ V_2 = -2 \sum_{ij} \chi_{ij} + \Delta_{ij} + \zeta_j \delta_j + v_i \gamma_i \]

\[ - 2 \sum_{ijk} \frac{1}{\lambda_i + \lambda_j + \lambda_k + \lambda_l} [\Omega_{ij}(\eta_{kl} + \gamma_k \delta_i) \]

\[ + \psi_{ij}(\Delta_{kl} + \zeta_k \delta_i + v_k \gamma_i) + \frac{1}{2} \delta_j \delta_i \left( \frac{1}{3} \zeta_k \delta_i + v_k \gamma_i \right) ] \] (13-93)

Later we shall apply this method to the problem of frequency demodulation by means of a sinusoidal PLL.

13-5 Further Studies

The sequence method opens up many new areas of study. It can be used, as we shall see in Chapter 15, to compute the mean-squared value of the frequency tracking error in a phase-locked frequency demodulator. It can also provide both steady-state and time-varying information for a large class of problems. Time-varying solutions for the frequency tracking system have not been studied in any detail; however, they represent an important class of problems relating to signal acquisition. In Chapter 16 we consider some solutions to this problem.

The sequence method provides a technique by which a large class of nonlinear systems can be analyzed. Nonlinear filtering theory (Ref. 6) produces a synthesis method in the form of an exact optimal nonlinear filtering equation that relates the optimum estimate of one quantity to others. For the details of this equation, see Chapters 4 and 8 of Bucy (Ref. 6). This nonlinear filtering equation can be implemented as a feedback loop in a dynamical system. However, the loop has inputs that must also be optimum estimates of nonlinear quantities and that must be derived by another implementation of the same equation. The result is the optimum nonlinear system; still, in general, it has infinitely many feedback loops and is therefore impractical. If the system is linear, the equation terminates to form the Kalman-Bucy filter (Ref. 7). Many approximately optimum estimates forming nonlinear systems can be derived using this equation. Snyder (Ref. 8) and Nahi (Ref. 9) present methods of partial linearization that result in approximately optimal systems. Bucy (Ref. 6) presents a method whereby the nonlinearity is expanded in a Taylor series and two terms are retained, thus resulting in another approximately optimal system. The Fokker-Planck techniques represent a method whereby performance information for these systems can be obtained. These two techniques together, therefore, can be used to provide practical designs for new systems that may yield greatly improved performance over that of the phase-locked loop.
For those readers who wish to study the details of nonlinear filtering theory, it should be noted that the theory was started in the early 1960s and was developed by Stratonovich (Ref. 10), Kushner (Ref. 11), Bucy (Ref. 12), Wonham (Ref. 13), Mortensen (Refs. 14 and 15), and Duncan (Ref. 16). Preliminary study can begin here, together with that of finding additional reference material. The real challenge of the future in nonlinear filtering appears to lie in a detailed investigation of the practical synthesis and realization of nonlinear filters. Their mechanization will undoubtedly require special-purpose digital computers. Bucy (Ref. 17) gives an account and extensive bibliography of the theory of nonlinear filtering for stochastic processes.
APPENDIX I

EVALUATION OF GAUSSIAN MOMENTS

Expected values of nonlinear functions involving Gaussian random variables were needed many times in later studies. For this reason, a list of these expected values was tabulated and is included in this appendix.

The quantity $x$, used in the list, is assumed to be a Gaussian column vector with statistical parameters defined by

$$m \triangleq E(x)$$
$$R \triangleq E(xx') - mm'$$
$$\Lambda \triangleq R^{-1}$$

so that its probability density function is given by

$$p(x) = \frac{1}{\sqrt{(2\pi)^N \det(R)}} \exp \left( -\frac{1}{2} ||x - m||_R^2 \right)$$

The following additional definitions are made:

$$g(x) \triangleq$$ an arbitrary scalar function
$$\xi \triangleq$$ an arbitrary row vector
$$v \triangleq$$ an arbitrary column vector
$$A \triangleq$$ an arbitrary matrix
$$\alpha \triangleq m + R\xi'$$
The list of Gaussian moments is as follows:

1. \( E(x_i x_j x_k) = R_{ij} m_k + R_{ik} m_j + R_{jk} m_i + m_i m_j m_k \)

2. \( E(xv'x'') = (v'm)R + mv'R + Rvm' + mv'mm' \)

3. \( E(x_i x_j x_k x_i) = R_{ij} R_{kl} + R_{ik} R_{jl} + R_{il} R_{jk} + R_{ij} m_k m_l \)
   \( + R_{lk} m_i m_l + R_{jl} m_i m_k + R_{jk} m_i m_l \)
   \( + R_{il} m_k m_l + R_{kl} m_i m_j + m_i m_j m_k m_l \)

4. \( E(xx'Ax''x') = R \text{ tr} (AR) + RAR + RA'R + (m'A'm)R \)
   \( + RAm'm + RA'm'm' + m'm'A'R \)
   \( + m'm'AR + mm' \text{ tr} (AR) + mm'Amm' \)

5. \( E[g(x)e^{i\xi}] = \frac{1}{\sqrt{(2\pi)^n \det(R)}} \int g(x) \exp \left( -\frac{1}{2} ||x - \alpha||_R^2 \right) dx \)
   \( \times \exp \left( \xi m + \frac{1}{2} \xi R\xi' \right) \)

6. \( E(e^{i\xi}) = \exp (\xi m + \frac{1}{2} \xi R\xi') \)

7. \( E(xe^{i\xi}) = (m + R\xi') \exp (\xi m + \frac{1}{2} \xi R\xi') \)

8. \( E(xx'e^{i\xi}) = [R + (m + R\xi')(m + R\xi')'] \exp (\xi m + \frac{1}{2} \xi R\xi') \)

9. \( E(x_i x_j x_k e^{i\xi}) = (R_{ij} \alpha_k + R_{ik} \alpha_j + R_{jk} \alpha_i + \alpha_i \alpha_j \alpha_k) \)
   \( \times \exp (\xi m + \frac{1}{2} \xi R\xi') \)

10. \( E(x'Ax'x'e^{i\xi}) = (m + R\xi')'[\text{tr} (AR) + (A + A')R \]
    \( + A(m + R\xi')(m + R\xi')'] \exp (\xi m + \frac{1}{2} \xi R\xi') \)

11. \( E(x_i x_j x_k x_l e^{i\xi}) = (R_{ij} R_{kl} + R_{ik} R_{jl} + R_{il} R_{jk} + R_{ij} \alpha_k \alpha_l \]
    \( + R_{lk} \alpha_i \alpha_l + R_{jl} \alpha_i \alpha_k + R_{jk} \alpha_i \alpha_l \]
    \( + R_{il} \alpha_i \alpha_k + R_{kl} \alpha_i \alpha_j + \alpha_i \alpha_j \alpha_k \alpha_i) \)
    \( \times \exp (\xi m + \frac{1}{2} \xi R\xi') \)

12. \( E[\sin (\xi x)] = \exp (-\frac{1}{2} \xi R\xi') \sin (\xi m) \)

13. \( E[\cos (\xi x)] = \exp (-\frac{1}{2} \xi R\xi') \cos (\xi m) \)

14. \( E[x \sin (\xi x)] = [m \sin (\xi m) + R\xi' \cos (\xi m)] \)
    \( \times \exp (-\frac{1}{2} \xi R\xi') \)

15. \( E[x \cos (\xi x)] = [m \cos (\xi m) - R\xi' \sin (\xi m)] \)
    \( \times \exp (-\frac{1}{2} \xi R\xi') \)
16. \( E[x'x' \sin(\xi x)] = [(R - R\xi' \xi R + mm') \sin(\xi m) + (m\xi R + R\xi'm') \cos(\xi m)] \times \exp\left(-\frac{1}{2} \xi R\xi'\right) \)

17. \( E[x'x' \cos(\xi x)] = [(R - R\xi' \xi R + mm') \cos(\xi m) - (m\xi R + R\xi'm') \sin(\xi m)] \times \exp\left(-\frac{1}{2} \xi R\xi'\right) \)

18. \( E[x_i x_j x_k \sin(\xi x)] = [[m_i R_{jk} + m_j R_{ik} + m_k R_{ij} + m_i m_j m_k \]
\( - m_i(R\xi')(R\xi')_k - m_j(R\xi')(R\xi')_k \]
\( - m_k(R\xi')(R\xi')_j] \sin(\xi m) \]
\( + [(R\xi')(R_{jk} + m_i m_k) \]
\( + (R\xi')(R_{ik} + m_i m_k) \]
\( + (R\xi')(R_{ij} + m_i m_j) \]
\( - (R\xi')(R\xi')(R\xi')_k] \cos(\xi m) \]
\( \times \exp\left(-\frac{1}{2} \xi R\xi'\right) \)

19. \( E[x_i x_j x_k \cos(\xi x)] = [[m_i R_{jk} + m_j R_{ik} + m_k R_{ij} + m_i m_j m_k \]
\( - m_i(R\xi')(R\xi')_k - m_j(R\xi')(R\xi')_k \]
\( - m_k(R\xi')(R\xi')_j] \cos(\xi m) \]
\( - [(R\xi')(R_{jk} + m_i m_k) \]
\( + (R\xi')(R_{ik} + m_i m_k) \]
\( + (R\xi')(R_{ij} + m_i m_j) \]
\( - (R\xi')(R\xi')(R\xi')_k] \sin(\xi m) \]
\( \times \exp\left(-\frac{1}{2} \xi R\xi'\right) \)

20. \( E[x'Ax'x' \sin(\xi x)] = [[m' \text{ tr } (AR) + m'(A + A')R \]
\( + m'Am'm' - m'AR\xi' \xi R \]
\( - \xi RAm\xi R - \xi RAR\xi'm'] \]
\( \times \sin(\xi m) + [\xi R \text{ tr } (AR) \]
\( + \xi R(A + A')R + m'Am\xi R \]
\( + m'AR\xi'm' + \xi RAm'm' \]
\( - \xi RAR\xi' \xi R] \cos(\xi m) \]
\( \times \exp\left(-\frac{1}{2} \xi R\xi'\right) \)
21. \[ E[x'Ax' \cos(\xi x)] = [m' \text{tr}(AR) + m'(A + A')R + m'Am' - m'AR\xiR - \xi RA_m\xiR - \xi RAR\xi m'] \times \cos(\xi m) - [\xi R \text{tr}(AR) + \xi R(A + A')R + m'Am\xiR + m'AR\xi' m' + \xi RAm' - \xi RAR\xi R] \sin(\xi m) \times \exp\left(-\frac{1}{2} \xi R\xi'\right) \]

22. \[ E[x'v'x \sin(\xi x)] = [mv'R + Rvm' + (v'm)R + mm'vm' - m\xi Rv\xi R - R\xi'v'R\xi' m' - R\xi' m' v\xi R] \times \sin(\xi m) + [R\xi'v'(R + mm')] + (R + mm')v\xi R + (R + mm')(\xi Rv) - R\xi'\xi Rv\xi R] \cos(\xi m) \times \exp\left(-\frac{1}{2} \xi R\xi'\right) \]

23. \[ E[x'v'x \cos(\xi x)] = [mv'R + Rvm' + (v'm)R + mm'vm' - m\xi Rv\xi R - R\xi'v'R\xi' m' - R\xi' m' v\xi R] \times \cos(\xi m) - [R\xi'v'(R + mm')] + (R + mm')v\xi R + (R + mm')(\xi Rv) - R\xi'\xi Rv\xi R] \sin(\xi m) \times \exp\left(-\frac{1}{2} \xi R\xi'\right) \]

References

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References


14

SOLUTIONS TO THE FOKKER-
PLANCK EQUATION BY THE
CONDITIONAL EXPECTATION
METHOD

14-1 Introduction

In Chapter 8 it was shown that two approaches exist for system analysis by
Fokker-Planck (FP) techniques. The first approach, the sequence method, to
approximating solutions to the multidimensional FP equation was discussed in
Chapter 13. The second approach, and the subject of this chapter, is to write
the one-dimensional FP equation for the single, non-Markov process \( z(t) \) and
to approximate the resulting conditional expectations. This method is referred
to as the conditional expectation method and represents a generalization of the
approach developed by the author (Ref. 1) in Chapters 11 and 12.

The conditional expectations arise in the evaluation of the intensity coeffi-
cients for the non-Markov FP equation. Two methods are suitable for approxi-
mating these conditional expectations. The first is for use when a general form
for the conditional expectations is known—for example, when the phase error
in a phase-locked loop is studied. The second method is useful when no a priori
information is available and the form cannot be determined. Even though
steady-state results are emphasized in this chapter, these methods apply to the
time-varying situation and are presented within this framework.

In Section 14-3 steady-state solutions to the one-dimensional FP equa-
tion will be derived based on the sequence method and the conditional expec-
tation method. Equations are then presented that allow one to use the conditional
expectation method to arrive at an initial estimate for the transition p.d.f. for
use in the sequence method. Finally, the special case where the nonlinearity
is sinusoidal is treated in detail. The equations in this form are to be used in
Chapter 15 to obtain numerical results for the nonlinear FM demodulation
problem. The matrix notation to be used in what follows is summarized in Ap-
pendix I of Chapter 8 and Appendix I of Chapter 13.

14-2 The Dynamical System Model

The system model to be considered in this chapter is described by the
stochastic differential equation

\[ dx(t) = f[t, x(t)] \, dt + B(t) \, dB(t) \]  \hspace{1cm} (14-1)

and

\[ z(t) = Cx(t) \]  \hspace{1cm} (14-2)

where \( x(t) \) and \( f[t, x(t)] \) are column vectors, \( C \) is a row vector, \( B(t) \) a matrix,
and \( dB(t) \) a column vector of Brownian motion (see Chapter 8). It is assumed,
without loss of generality, that

\[ E[\beta_i(t + \tau) - \beta_i(t)[\beta_j(t + \tau) - \beta_j(t)]] = \begin{cases} |	au| & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]  \hspace{1cm} (14-3)

so that the matrix \( Q \) defined in Chapter 8 is the identity matrix. As discussed
in Chapter 8, the process \( \{x(t)\} \) is known to be a vector Markov process and to
satisfy the FP equation. Using Pawula's results (Ref. 2), it was shown in Chap-
ter 8 that the non-Markov process \( \{z(t)\} \) also satisfies the FP equation; that is,

\[ \frac{\partial p(z; t)}{\partial t} + \frac{\partial J_0(z; t)}{\partial z} = 0 \]  \hspace{1cm} (14-4)

where \( J_0(z; t) \) is the probability current in the \( z \) direction. This current is
defined by

\[ J_0(z; t) \triangleq K_0(z, t)p(z; t) - \frac{1}{2} K_{\infty}(t) \frac{\partial p(z; t)}{\partial z} \]  \hspace{1cm} (14-5)
In Chapter 8 the intensity coefficients, \( K_0(z, t) \) and \( K_{00}(t) \), were shown to be given by

\[
K_0(z, t) = CE[f[t, x(t)]]z(t)
\]

(14-6)

and

\[
K_{00}(t) = CB(t)B'(t)C'
\]

(14-7)

The one-dimensional equation (14-4) is much easier to handle analytically than the multidimensional equation for \( p(x; t) \). In the steady state (14-4) becomes an ordinary differential equation with a general solution given in Chapter 7.

It should be pointed out that occasionally the non-Markov FP equation (14-4) becomes degenerate and fails to yield a solution. This is the case when no input noise in the direction of \( z(t) \) in the state-space produces the result \( CB(t) \), and therefore \( K_{00}(t) \), is zero. In this case it will be found that \( K_0(z, t) \) is also zero and that (14-4) simply becomes an equation stating zero equals zero. Pawula (Ref. 2) discusses this situation and presents methods to overcome it for some cases. If this problem cannot be overcome, the only resort is to a multidimensional equation and the sequence method presented in the previous chapter.

### 14-3 Approximating the Conditional Expectations

The conditional expectations needed for (14-4) cannot, in general, be evaluated exactly; they must be approximated. In Chapter 11 two methods of approximating the conditional expectations were introduced. In this section we shall generalize the method to include cases where the form of the conditional expectations is both known and unknown a priori. We shall assume that the mean values of all random variables are zero. If such is not the case, the mean values can be subtracted out and added back later so that the equations can still be applied.

#### 14-3.1 The Case When the Form of the Conditional Expectation is Known A Priori

Assume the conditional expectations can be approximated by

\[
E[Cf[t, x(t)]|z(t)] \approx \eta(t)g[z(t)]
\]

(14-8)

when \( g(z) \) is known and \( \eta(t) \) is to be determined. The function \( g(z) \) can be selected in many ways. If the conditional expectation is periodic in \( z \) (which is
the case when \( z \) represents the phase error in an SCS, \( g(z) \) can be selected as a sine function—that is, the first term in the Fourier series expansion of \( g(z) \). More logically, if \( z \) is the input to a periodic nonlinearity, \( g(z) \) can be chosen to have the same form as this nonlinearity. The fact that this latter approach is reasonable is borne out by the computer-simulation solutions given in Chapter 10. In any case, for any particular system, the accuracy of the results must be checked to determine the quality of the approximation. This step can usually be accomplished by computer-simulation methods or hardware mechanisms.

The function \( \eta(t) \) in (14-8) can be easily determined by linear mean square estimation. Assume that \( \eta(t) \) is to be selected to minimize the mean square error \( \epsilon(t) \) defined by

\[
\epsilon(t) \triangleq E[(E[Cf[t, x(t)]|z(t)] - \eta(t)g(z(t))|^2)]
\]

(14-9)

Using the orthogonality principle as in Chapter 11, \( \eta(t) \) is then given by

\[
\eta(t) = \frac{E(E[Cf[t, x(t)]|z(t)]g(z(t))]}{E[g^2(z(t))]}
\]

(14-10)

which is equivalent to

\[
\eta(t) = \frac{E[Cf[t, x(t)]g(z(t))]}{E[g^2(z(t))]}
\]

(14-11)

Thus we have turned the problem of approximating the conditional expectations in (14-8) into the problem of approximating the expectations given in the numerator and denominator of (14-11). For example, they can be evaluated with respect to a multidimensional Gaussian p.d.f. Various cases are tabulated in Appendix I of Chapter 13.

Frequently, the denominator in (14-11) can be evaluated more accurately, thereby improving the approximation. When the p.d.f. of \( z \) is to be evaluated (it can easily be evaluated in the steady state), it can be used in the equation

\[
E[g^2(z(t))] = \int_{z_1}^{z_2} g^2(z)p(z; t)\,dz
\]

(14-12)

where \( z \) ranges between \( z_1 \) and \( z_2 \). Since \( p(z; t) \) will be a function of \( E[g^2(z(t))] \), (14-12) cannot be evaluated directly; however, it can be solved for \( E[g^2(z(t))] \) by various numerical methods. We note that in the case where \( z(t) \) represents the phase error in a SCS, it was found in Chapter 11 that the accuracy of the results can be improved by using (14-12) rather than by simply using linear theory to evaluate \( E[g^2(x(t))] \). This same result will be observed in the analysis of the frequency-modulation tracking loop to be discussed in Chapter 15.
14-3.2 The Case When the Form of the Conditional Expectations Is Not Known A Priori

In many cases the form \( g(z) \) of the conditional expectation in (14-8) cannot be determined. This is true, as we shall see, when \( z(t) \) represents the frequency tracking error in a phase-locked frequency demodulator. If \( z(t) \) is not at the input to a nonlinearity and the conditional expectations cannot be determined to be periodic, \( g(z) \) may be difficult to estimate. However, a general method, suggested by Mayfield (Ref. 5), is still available to approximate the conditional expectations. For these situations, the conditional expectations are defined by

\[
E[Cf(t, x(t))|z(t)] \triangleq \int_\Omega Cf(t, x)p(x, t|z) \, dx \quad (14-13)
\]

where \( p(x, t|z) \) is the conditional p.d.f. of \( x \) given \( z \) at time \( t \). Let \( \tilde{p}(x, t|z) \) be the multidimensional Gaussian conditional p.d.f. (see Chapter 1) for the linearized model—that is, for model where \( f(x) \) is replaced by \( f_x(0) \) when \( f(0) = 0 \). In this case \( \tilde{p}(x, t|z) \) is given by

\[
\tilde{p}(x, t|z) \triangleq \frac{1}{\sqrt{(2\pi)^N \det(G(t))}} \exp \left[ -\frac{1}{2} \| x - p(t)z \|^2_{G^{-1}(t)} \right] \quad (14-14)
\]

and \( \| x - p(t)z \|^2_{G^{-1}(t)} \) is the quadratic form associated with the matrix \( G^{-1}(t) \), see Appendix I, Chapter 8. From the theory of Gaussian p.d.f.'s it is known that

\[
p(t) = \frac{\tilde{E}[x(t)z(t)]}{\tilde{E}[z^2(t)]} = \frac{\tilde{E}[x(t)x'(t)]C'}{\tilde{E}[x(t)x'(t)]C'} \quad (14-15)
\]

where \( \tilde{E} \) is used to denote the expectation based on the linearized model. If \( R(t) \) is defined by

\[
R(t) \triangleq \tilde{E}[x(t)x'(t)] \quad (14-16)
\]

then

\[
p(t) = \frac{R(t)C'}{CR(t)C'} \quad (14-17)
\]

The matrix \( G(t) \) in (14-14) is known from the theory of Gaussian p.d.f.'s to be given by

\[
G(t) = \tilde{E}[x(t)x'(t)] - \frac{\tilde{E}[x(t)z(t)]\tilde{E}[z(t)x'(t)]}{\tilde{E}[z^2(t)]} \quad (14-18)
\]
If we use (14-16) and (14-17), \( G(t) \) can be written as

\[
G(t) = R(t) - \rho(t)CR(t)
\]  

(14-19)

Equations (14-17) and (14-19) can be evaluated by using linear system theory (Ref. 3), and the conditional expectations (14-13) can be approximated by

\[
\bar{E}[C_f[t, x(t)]|z(t)] = \int_\Omega C_f(t, x)\bar{p}(x, t|z)\,dx
\]  

(14-20)

**14-4 The Steady-State Solution to the Reduced Fokker-Planck Equation**

Once the conditional expectations have been evaluated or approximated, the steady-state solution to the reduced FP equation can easily be determined. The steady-state solution derived in this section represents a generalization of the solution given in Chapters 10, 11, and 12.

In accordance with the earlier approach, the nonlinear restoring force \( h_0(z; t) \) is defined by

\[
h_0(z; t) \triangleq \frac{2K_0(z, t)}{K_{00}(t)}
\]  

(14-21)

and the potential function by

\[
U_0(z; t) \triangleq -\int_{z_0}^z h_0(\xi; t)\,d\xi
\]  

(14-22)

where \( z_0 \) is arbitrary. In terms of the potential function, (14-4) and (14-5) are seen to be equivalent to

\[
\mathcal{J}_0(z; t) = -\frac{1}{2}K_{00}(t)\exp[-U_0(z; t)]\left(\frac{\partial}{\partial z}\{p(z; t)\exp[U_0(z, t)]\}\right)
\]  

(14-23)

Letting \( t \) go to infinity in (14-23) results in

\[
\frac{d}{dz}\{p(z)\exp[U_0(z)]\} = -\frac{2\mathcal{R}_0}{K_{00}}\exp[U_0(z)]
\]  

(14-24)

Equation (14-24) can be used to determine the steady-state p.d.f. \( p(z) \) in general. The final form for \( p(z) \), however, is more convenient if special cases are considered. The first special case applies in almost all situations where the range of \( z \) is infinite. If the p.d.f. \( p(z) \) and its first derivative are deter-
minded to be zero at the boundaries of the range of $z$ (whether these boundaries are finite or infinite), then from (14-24) it can be seen that the steady-state probability current $J_0$ will be zero. In this case the left-hand side of (14-24) is also zero, and the p.d.f. $p(z)$ is easily seen to be given by

$$p(z) = C_0 \exp \left[ -U_0(z) \right]$$  \hspace{1cm} (14-25)

In general, $J_0$ will not be zero and (14-24) must be integrated to produce

$$p(z) = C_0 \exp \left[ -U_0(z) \right] \left[ 1 + D_0 \int_{z_1}^{z_2} \exp \left[ U_0(\xi) \right] d\xi \right]$$  \hspace{1cm} (14-26)

where $C_0$ and $D_0$ are constants that must be determined and $z_1$ will be taken as the lowest value in the range of $z$. Assume that the range of $z$ is from $z_1$ to $z_2$. For certain problems, the condition will hold that

$$p(z_1) = p(z_2)$$  \hspace{1cm} (14-27)

Equation (14-27) can be used to evaluate $D_0$ with the result that

$$D_0 = \frac{\exp \left[ -U_0(z_1) \right] - \exp \left[ -U_0(z_2) \right]}{\exp \left[ -U_0(z_2) \right] \int_{z_1}^{z_2} \exp \left[ U_0(\xi) \right] d\xi}$$  \hspace{1cm} (14-28)

The constant $C_0$ in both (14-25) and (14-26) can be evaluated by the normalization condition

$$\int_{z_1}^{z_2} p(\xi) d\xi = 1$$  \hspace{1cm} (14-29)

Equations (14-25) and (14-26) can be simplified when $h_0(z)$ is periodic in $z$ with a period $T = z_2 - z_1$. If $h_0(z)$ is periodic, it can be expanded in the Fourier series

$$h_0(z) = a_0 + \sum_{n=1}^{\infty} a_n \exp \left( \frac{i2\pi nz}{T} \right)$$  \hspace{1cm} (14-30)

If the integration in (14-22) is carried out using (14-30), the potential function can be written as

$$U_0(z) = b_0 z + g(z)$$  \hspace{1cm} (14-31)
where \( g(z) \) is periodic with a period \( T \) and \( b_0 \) is a constant. Equations (14-26) and (14-28) can now be combined to produce

\[
p(z) = C_0 \exp \left[ -U_0(z) \right] \left( \int_{z_1}^{z_2} \exp \left[ U_0(\xi) \right] d\xi + \{ \exp [U_0(z_2) - U_0(z_1)] - 1 \} \int_{z_1}^{z} \exp \left[ U_0(\xi) \right] d\xi \right) (14-32)
\]

which is equivalent to

\[
p(z) = C'_0 \exp \left[ -U_0(z) \right] \left( \int_{z}^{z_2} \exp \left[ U_0(\xi) \right] d\xi + \int_{z_1}^{z} \exp \left[ U_0(\xi) + U_0(z_2) - U_0(z_1) \right] d\xi \right) (14-33)
\]

where the new constant \( C'_0 \) is defined in terms of \( C_0 \) by

\[
C'_0 \triangleq \frac{C_0}{\int_{z_1}^{z_2} \exp \left[ U_0(\xi) \right] d\xi} (14-34)
\]

Using (14-31) and the fact that \( g(z) \) is periodic with period \( T \), then

\[
U_0(\xi) + U_0(z_2) - U_0(z_1) = b_0(\xi + T) + g(\xi)
\]

\[
= b_0(\xi + T) + g(\xi + T) = U_0(\xi + T) (14-35)
\]

If (14-35) is inserted into (14-33) and a change of variable is made in the second integral, the result is

\[
p(z) = C'_0 \exp \left[ -U_0(z) \right] \left( \int_{z}^{z_2} \exp \left[ U_0(\xi) \right] d\xi + \int_{z}^{z+T} \exp \left[ U_0(\xi) \right] d\xi \right) (14-36)
\]

or, equivalently,

\[
p(z) = C'_0 \exp \left[ -U_0(z) \right] \int_{z}^{z+T} \exp \left[ U_0(\xi) \right] d\xi (14-37)
\]

Equation (14-37) is the general form for the conditional p.d.f. \( p(z) \) for the case where the system is periodic in \( z \). Notice that if the constant \( b_0 \), de-
fined above, is zero (as with the phase-locked loop with zero frequency offset), then \( U_0(z) \) is periodic in \( z \) and (14-37) is equivalent to the simpler equation (14-25) for \( p(z) \).

14-5 Solution to the Fokker-Planck Equation by a Combination of Methods

Two methods have been presented for the analysis of nonlinear systems by FP techniques. In Chapter 13 the general solution to the multidimensional FP equation was presented in sequence form. This sequence required, as a starting term, an estimate of the transition p.d.f. that can often be determined from a linear model. The linear model is given by the equation

\[
dx(t) = f[x(t)] \, dt + B \, d\beta(t)
\]  

(14-38)

where \( f(x) \) is separated into a linear and a nonlinear part as

\[
f(x) = Mx + I(x)
\]  

(14-39)

The separation in (14-39) is arbitrary. The matrix \( M \) can be chosen by any method and \( I(x) \) is simply equal to \( f(x) - Mx \). In this section the methods of Booton (Ref. 4) are used to compute equivalent gains for the nonlinearities and \( M \) is chosen from them. The matrix \( M \) selected in this way will be a function of the moments of some of the states. If these moments are obtainable by the conditional expectation method, we have a link between the two techniques that can, in some cases, produce very accurate results.

Only the steady-state results are considered in this section although the results are easily extended. Suppose \( f(x) \) can be broken down in terms of functions \( g_i(z_i) \) as

\[
f(x) = Ax + \sum_i d_i g_i(z_i)
\]  

(14-40)

where \( z_i = c_i x \), \( A \) is a matrix, \( d_i \) are column vectors, and \( c_i \) are row vectors. In accordance with Booton (Ref. 4), the nonlinear functions \( g_i(z_i) \) are each approximated by an equivalent gain \( \eta_i \) such that the mean square error \( \epsilon_i \) is minimized, where \( \epsilon_i \) is defined by

\[
\epsilon_i \triangleq E[(g_i(z_i) - \eta_i z_i)^2]
\]  

(14-41)

By the orthogonality principle, \( \epsilon_i \) will be minimized when

\[
\eta_i = \frac{E[z_i g_i(z_i)]}{E(z_i^2)}
\]  

(14-42)
If (14-42) is substituted into (14-40), a linear estimate of \( f(x) \) is formed. The matrix \( M \) can be chosen from this equation to give

\[
M = A + \sum \tilde{d} c_i \frac{E[z_i g_i(z_i)]]}{E(z_i^2)}
\]  

(14-43)

According to Booton's method (Ref. 4), the expectations in (14-43) would now be evaluated by the linear model. If the method of conditional expectations can be used to evaluate the probability density \( p(z_i) \) for each \( z_i \), these expectations can be evaluated directly from it. The resulting matrix \( M \) can then be used in accordance with the methods of Chapter 13 to produce results for any other states. This method will be used to compute the variance of the frequency tracking error in a frequency-modulation tracking loop. The results are given in Chapter 15.

14-6 Steady-State Results for a Sinusoidal Nonlinearity

In this section the results of the previous sections are specialized and applied to the case where the nonlinearity \( g(z) \) is sinusoidal and where steady-state performance is to be evaluated. The equations for the system defined in (14-1) and (14-2) can be written as

\[
f[x(t)] = A x(t) + D \sin [\phi(t)]
\]  

(14-44)

The phase error \( \phi(t) \) is a linear combination of states and is written as

\[
\phi(t) = a x(t)
\]  

(14-45)

where \( a \) is a row vector. Assume that the conditional expectations are periodic in \( \phi(t) \). This property can be established as follows: The coordinates can always be transformed so that \( \phi(t) \) is the first component of the state vector \( x(t) \). If this is done and if the first column of the matrix \( A \) defined in (14-44) above is zero, the other states of the system will be unchanged if \( \phi(t) \) is replaced by \( \phi(t) + 2\pi n \). This suggests that the conditional expectations have this same property and are periodic in \( \phi(t) \). We now derive the steady-state p.d.f.'s for both the periodic state \( \phi(t) \) and for \( z(t) \), which can represent any other combination of states.

The conditional expectations for the \( \phi(t) \) process can now be evaluated by the first method in Section 14-3. Assume that the function \( g(\phi) \), defining the form of the conditional expectations, is given by

\[
g(\phi) = \sin (\phi)
\]  

(14-46)
In accordance with Section 14-3, the conditional expectations can be approximated in the steady state by

\[ E[af(x)|\phi] \approx \frac{E[af(x) \sin(\phi)]}{\sigma_x^2} \sin(\phi) \quad (14-47) \]

where

\[ \sigma_x^2 \triangleq E[\sin^2(\phi)] \quad (14-48) \]

The intensity coefficients given by (14-6) become

\[ K_0(\phi) = \frac{E[af(x) \sin(\phi)]}{\sigma_x^2} \sin(\phi) \quad (14-49) \]

and that of (14-7) reduce to

\[ K_{00} = aBB'a' \quad (14-50) \]

From (14-21) the nonlinear restoring force \( h_0(\phi) \) is given by

\[ h_0(\phi) = \frac{2K_0(\phi)}{K_{00}} = -\alpha \sin(\phi) \quad (14-51) \]

where

\[ \alpha = -\frac{2E[af(x) \sin(\phi)]}{(aBB'a')\sigma_x^2} \quad (14-52) \]

and zero detuning is assumed. If (14-44) is now substituted into (14-52), then

\[ \alpha = -\frac{2aAE[x \sin(\phi)]}{(aBB'a')\sigma_x^2} \quad (14-53) \]

The expectation in the first term of (14-53) must be evaluated via linear theory. Using the results from Appendix I of Chapter 13, we find

\[ \tilde{E}[x \sin(\phi)] = Ra' \exp(-\frac{1}{2}aRa') \]

\[ = \tilde{E}(x\phi) \exp(-\frac{1}{2}\sigma_\phi^2) \quad (14-54) \]

where \( \tilde{E} \) is the expectation operator based on the linearized system \( R = \tilde{E}(xx') \) and \( \sigma_\phi^2 = \tilde{E}(\phi^2) \). It should be pointed out that the correct linear model to use is the model where \( f(x) \) has been replaced by the linear term in its Fourier series expansion—that is, by \( (A + Da)x \). The expression for \( \alpha \) now becomes
\[ \alpha = -\frac{2aA\tilde{E}(x\phi)\exp\left(-\frac{1}{3}\tilde{\sigma}_g^2\right)}{(aBB' a')\sigma_g^2} - \frac{2aD}{aBB' a'} \]  

(14-55)

The variance \( \sigma_g^2 \) could also be computed via linear theory. From Appendix I of Chapter 13 we find that

\[ \tilde{\sigma}_g^2 = \tilde{E}[\sin^2(\phi)] = \frac{1}{2}[1 - \exp(-2\tilde{\sigma}_g^2)] \]  

(14-56)

On the basis of the results given in Chapter 10, it has been found, however, that the results are more accurate if \( \sigma_g^2 \) is computed from the p.d.f. using (14-12).

The potential function \( U_0(\phi) \) can now be determined from (14-22) and (14-51), that is,

\[ U_0(\phi) = -\alpha \cos(\phi) \]  

(14-57)

and the p.d.f. in (14-37) is given by

\[ p(\phi) = \frac{\exp[\alpha \cos(\phi)]}{2\pi I_0(\alpha)}, \quad |\phi| \leq \pi \]  

(14-58)

where \( I_n(\alpha) \), in general, is a modified Bessel function of order \( n \). Equations (14-58) and (14-12) can be combined to produce an equation for \( \sigma_g^2 \).

\[ \sigma_g^2 = \int_{-\pi}^{\pi} [\sin(\phi)]^2 p(\phi) \, d\phi = \frac{1}{\alpha} \frac{I_1(\alpha)}{I_0(\alpha)} \]  

(14-59)

Combining (14-59) with (14-55) yields

\[ \alpha = -\left[ \frac{2aA\tilde{E}(x\phi)\exp\left(-\frac{1}{3}\tilde{\sigma}_g^2\right)}{aBB' a'} \right]\left[ \frac{\alpha I_0(\alpha)}{I_1(\alpha)} \right] - \frac{2aD}{aBB' a'} \]  

(14-60)

which is a nonlinear transcendental equation that can be solved for \( \alpha \). Care should be taken, however, because (14-60) may yield both a positive and a negative solution for \( \alpha \). In this case the positive solution will be the correct one because it is the one that corresponds to a positive value of \( \sigma_g^2 \).

The second method given in Section 14-2.3 can be used to evaluate the steady-state p.d.f. \( p(z) \) of \( [z(t)] \). In this case

\[ K_0(z) = E[\text{CF}(x)|z] \]  

(14-61)

and

\[ K_{00} = \text{CBB'C'} \]  

(14-62)
The first intensity coefficient will be approximated by

$$\tilde{E}[\text{Cf}(x)|z] = CA\tilde{E}(x|z) + CD\tilde{E}[\sin(\phi)|z]$$

(14-63)

where the tilde over the expectation operator denotes that the expectation is taken with respect to a Gaussian p.d.f. The Gaussian conditional expectations in (14-63) can be evaluated with the help of Appendix I of Chapter 13. The results are

$$\tilde{E}(x|z) = \rho$$

(14-64)

and

$$\tilde{E}[\sin(\phi)|z] = \exp\left(-\frac{1}{2}aG\alpha\right)\sin(a\rho z)$$

(14-65)

where, in the steady state,

$$\rho = \frac{RC'}{CRC'}$$

(14-66)

and

$$G = R - \rho CR$$

(14-67)

The nonlinear restoring force is now given by

$$h_0(z) = -\beta z - \alpha \sin(a\rho z)$$

(14-68)

where

$$\beta = -CA\rho$$

(14-69)

and

$$\alpha = -CD \exp\left(-\frac{1}{2}aG\alpha\right)$$

(14-70)

and the potential function by

$$U_0(z) = \frac{\beta}{2}z^2 - \frac{\alpha}{a\rho} \cos(a\rho z)$$

(14-71)

The resulting probability density function is found from (14-71) and (14-25) to be
\[ p(z) = \frac{\exp \left[ -(\beta/2)z^2 + (\alpha/a\beta) \cos (a\beta z) \right]}{\int_{-\infty}^{\infty} \exp \left[ -(\beta/2)z^2 + (\alpha/a\beta) \cos (a\beta z) \right] dz} \]  

(14-72)

for \(|z| \leq \infty\).

Equation (14-72) will be used in the next chapter to evaluate the variance of the frequency tracking error in a phase-locked frequency demodulator. We note that this method is not as general as the sequence method because, as was pointed out earlier, the one-dimensional FP equation fails to yield a solution in some cases.

A more general method, and the most accurate for the special case of the mean-squared frequency tracking error studied in Chapter 4, is the combination method presented in Section 14-5. The matrix \(M\) defined in (14-39) and required to start the sequence method is given by (14-43), which, in the case of the sinusoidal nonlinearity, yields

\[ M = A + Da \frac{E[\phi \sin (\phi)]}{E(\phi^2)} \]  

(14-73)

Since the p.d.f. \(p(\phi)\) of \(\phi\) can be obtained by (14-58), both the expectations in (14-73) can be easily evaluated. The resulting matrix \(M\) can then be used in the equations in Section 13-4 of Chapter 13 to compute the steady-state variance of the process \(\{z(t)\}\). In the next chapter we apply these rather general approximation methods to the special case of frequency demodulation outlined in Chapter 4.

References


PART FIVE
15

FREQUENCY DEMODULATION
BY MEANS OF A PHASE-LOCKED LOOP

15-1 Introduction

The phase-locked loop has been used for years as a demodulator (frequency discriminator) in communication systems employing frequency modulation. Methods for obtaining performance information for such a system, however, have been very inadequate and primarily involve the linear PLL theory discussed in Chapter 4, Section 4-9, and experimental results. In Chapters 10, 11 and 12 emphasis was placed on obtaining the statistical dynamics of the phase error process occurring in phase tracking and demodulation loops designed to give an estimate of the transmitter phase when angle modulation is present and absent.

As noted in Chapters 4 and 12, the phase error process in frequency-modulation tracking systems gives information on loop performance; however, it cannot be used as a direct measure of the performance of a frequency demodulator. The cycle slipping rate observable in the phase error process is probably the best indicator of the demodulator thresholding effect. On the other hand, the mean-squared value of the frequency tracking error is the quantity that directly measures performance of frequency-modulation tracking
loops. It is this quantity that logically should be minimized and is most easily measured.

Chapter 12 extended the Fokker-Planck techniques presented in Chapter 11 so that the variance of the phase error and cycle-slipping rate could be predicted in a phase-locked system designed to track angle modulation. Thus far the only analytical methods presented for determining the mean-squared value of the frequency tracking error were based on the linear PLL theory given in Chapter 4. However, using the analytical tools developed in the previous three chapters, the mean-squared value of the frequency tracking error can now be approximated in the nonlinear region of operation. Consequently, the purpose of this chapter is to demonstrate how this body of theory can be applied to determining the performance of a phase-locked frequency demodulator.

The specific problem we wish to consider was defined in Chapter 4, Sections 4-7 to 4-9. There the linear analysis was given, and, based on that analysis, we shall extend, in what follows, its region of validity to the nonlinear region. In order to attest the validity of the nonlinear theory, we provide a check by using computer-simulation methods. The results in this Chapter are largely due to Mayfield (Ref. 4).

15-2 System Model

The system model was proposed in Chapter 4, Section 4-7. However, for ease of reference and clarity, we illustrate in Fig. 15-1 the equivalent baseband loop model developed in Chapter 4. The signal process \( \{m(t)\} \) is assumed to be a stationary Gaussian process with a rational frequency spectrum. Since frequency-modulation tracking system configurations are so varied, it is difficult to present a set of data that can be applied to every application. Thus data are presented for the special case where the signal is equivalent to white noise

![Fig. 15-1. Equivalent Model for the Frequency-Modulation Tracking System in Terms of the Process \( \{\phi(t)\} \).](image)
shaped by a single-pole filter. The data presented, however, can be applied, at least approximately, to many other problems.

The spectral density of the modulation process is characterized by

\[ S_m(\omega) = \frac{r_s^2}{\omega^2 + \alpha_s^2} \]  \hspace{1cm} (15-1)

while, for most of the results that follow, the spectrum of the phase-noise process \( \{N(t, \phi)\} \) is assumed to be white. However, as shown in Chapter 3, Section 3-5, this assumption becomes suspect when the bandwidth of the signal is of the order of the bandwidth of the noise. For this reason, system performance and comparison data are presented for the case where colored noise has a power spectrum of

\[ S_n(\omega) = \frac{r_n^2}{\omega^2 + \alpha_n^2} \]  \hspace{1cm} (15-2)

The radian cutoff frequency \( \alpha_s \) is selected by the commonly used modification of Carson's rule, which says that

\[ \alpha_n = \alpha_s (2m_f + 1) \]  \hspace{1cm} (15-3)

where \( m_f \) is the modulation index defined by

\[ m_f \triangleq \frac{\Delta \omega}{\alpha_s} = \frac{r_s K_s}{\alpha_s \sqrt{2} \alpha_s} \]  \hspace{1cm} (15-4)

The quantity \( \Delta \omega \) is the root-mean-square frequency deviation in radians per second. Notice that if we set \( r_s = \sqrt{2} \alpha_s \), then \( m_f \) is in agreement with (4-117).

For the case where the phase-noise process \( \{N(t, \phi)\} \) is white, it was shown in Chapter 4, using the Wiener theory and the linear PLL theory, that the closed-loop transfer function minimizing the variance of the phase error was given by

\[ H_p(s) = \frac{\alpha_s [(\gamma - 1)s + \delta \alpha_s]}{s^2 + \alpha_s \gamma s + \alpha_s^2 \delta} \]  \hspace{1cm} (15-5)

where we have set \( \alpha_s = a \) in (4-116) for clarity in what follows. This implied that

\[ F_0(s) = \frac{\alpha_s [(\gamma - 1)^2 K]}{2K_s [(\gamma - 1)s + \alpha_s \delta]} \]  \hspace{1cm} (15-6)

and
\[ F(s) = \frac{\alpha_z[(\gamma - 1)s + \delta \alpha_z]}{AK(s + \alpha_z)} \]  

(15-7)

where

\[ \Delta \omega \triangleq \text{root-mean-square frequency deviation} = \frac{r_s K_r}{\sqrt{2}\alpha_z} \]

\[ m_f \triangleq \text{modulation index} \triangleq \frac{\Delta \omega}{\alpha_z} \]

\[ R \triangleq \text{input signal-to-noise} \triangleq \frac{A^2}{\alpha_z N_0} = \frac{A^2}{aN_0} \]  

(15-8)

\[ \delta \triangleq 2m_f \sqrt{R} \]

\[ \gamma \triangleq \sqrt{1 + 2\delta} \]

and \( z(t) \triangleq m(t) - \gamma(t) \) is defined as the instantaneous frequency tracking error. The loop filter is seen to be the standard lead-lag filter of the second-order phase-locked loop, and the output filter \( F_0(s) \) is a simple single-pole filter. The resulting linearized system model is repeated in Fig. 15-2 for clarity.

![Fig. 15-2. Linear Model for the Frequency-Modulation Tracking System.](image)

It is interesting to relate these parameters to the more conventional parameters; loop gain \( K \), loop natural frequency \( \omega_n \), and \( r \triangleq 4\zeta^2 \), where \( \zeta \) is the loop damping constant. In terms of these parameters, the loop transfer function \( H_p(s) \) can be written

\[ H_p(s) = \frac{\omega^2}{z_1} \left[ \frac{s + z_1}{s^3 + \sqrt{r} \omega_n s + \omega^2_n} \right] \]  

(15-9)

where
Application of the Nonlinear Theory

\[ z_1 \triangleq \frac{AK\omega_n}{AK\sqrt{r - \omega_n}} \quad \text{(15-10)} \]

Comparison of (15-8) and (15-9) with (15-4) results in

\[ \omega_n = \alpha_s\sqrt{\delta}, \quad AK = \alpha_s\delta, \quad r = 2 + \frac{1}{\delta} \quad \text{(15-11)} \]

In order to minimize transient effects, a loop is usually designed with \( r = 2 \). This corresponds to a damping of \( \zeta = 1/\sqrt{2} \). If \( \delta \) is reasonably large, it is evident from (15-11) that \( r \) in the optimum linear loop is approximately equal to two.

The loop design point with \( r = 2 \) varies with input signal-to-noise ratio \( R \). Since \( R \) varies over some range in most real systems, this would suggest an adaptive loop, which, however, creates a more complex system. It is more common to design the loop around a particular value of \( R \), usually one corresponding to \( \sigma_\phi^2 \) fixed at some value near "threshold". The idea is that this practice will ensure a low value of threshold. The values of \( \sigma_\phi^2 \) for this purpose are usually obtained experimentally and commonly range from \( \frac{1}{4} \) to \( \frac{1}{8} \). The value \( \sigma_\phi^2 = 0.2 \) will be used for this design. In fact, it was shown in Chapter 4 that the linear phase error variance is given by

\[ \sigma_\phi^2 = \frac{8m^2_\lambda}{(y - 1)(y + 1)^2} \quad \text{(15-12)} \]

Once \( m_\lambda \) is known, (15-12) can be solved for \( y \) with \( \sigma_\phi^2 = 0.2 \). Equation (15-8) produces \( \delta \) and the loop design is easily determined from (15-5), (15-6), (15-7), (15-9), (15-10), and (15-11).

Since the output filter \( F_\theta(s) \) does not affect the phase error, \( F_\theta(s) \) could be designed independently of \( H_\phi(s) \) for a separate value of \( R \). This approach would have the advantage that the phase error could be reduced near threshold, hopefully reducing threshold, while the frequency tracking error could be minimized at an expected operating point. The result would be an output filter with two zeros and three poles. In the analysis that follows, however, the system was designed entirely at the operating point corresponding to \( \sigma_\phi^2 = 0.2 \).

15-3 Application of the Nonlinear Theory

In this section the results presented in the previous three chapters are applied. For this purpose, it is necessary first to write state-space equations so that the system will be represented by
\[
dx(t) = f[x(t)]\, dt + B\, d\phi(t)
\]
\[
z(t) = Cx(t)
\]

where the nonlinearity is sinusoidal and \( f[x(t)] \) can be written

\[
f[x(t)] = Ax(t) + D\, \sin[\phi(t)]
\]
\[
\phi(t) = ax(t)
\]

Any state-space representation can be used and the results will be identical. The state variables used in this particular application were selected as indicated in Fig. 15-3. There the inputs \( u_i(t), i = 1, 2, 3 \), are white-noise sources with

Fig. 15-3. Frequency-Modulation Tracking System Model Showing State Variables.

unit spectral density functions. Two disturbing noise sources are shown, one white and one colored; however, only one of them is used in any particular case. In order to simplify the model, the loop filter is allowed to include the other system constants, such as loop gain and carrier level. The constants (15-10) for this model are related to the more commonly used constants through
\[ r_0 = \sqrt{r} \omega_n - \alpha_1, \quad r_1 = \omega_n^2 - r_0 \alpha_1, \quad \alpha_1 = \frac{\omega_n^2}{AK} \]  \hspace{1cm} (15-15)

The state vector \( \mathbf{x} \) has dimension four for the white-noise case and five in the colored-noise case. Once the system is designed by the methods discussed in detail in Chapter 4, Section 4-9, and summarized in the previous section, the matrices required in (15-13) and (15-14) are readily determined using Fig. 15-3. The design parameters were computed for the several cases to be studied and are given in Table 15-1.

<table>
<thead>
<tr>
<th>( m_f )</th>
<th>( \omega_n )</th>
<th>( AK )</th>
<th>( r )</th>
<th>( r_1 )</th>
<th>( \alpha_1 )</th>
<th>( \sigma^2_p (\text{rad}^2) )</th>
<th>( R ) (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.598</td>
<td>12.94</td>
<td>2</td>
<td>2.093</td>
<td>3.093</td>
<td>0.200</td>
<td>10.20</td>
</tr>
<tr>
<td>5</td>
<td>6.831</td>
<td>46.67</td>
<td>2</td>
<td>4.356</td>
<td>5.356</td>
<td>0.200</td>
<td>13.38</td>
</tr>
<tr>
<td>10</td>
<td>10.978</td>
<td>120.51</td>
<td>2</td>
<td>7.279</td>
<td>8.279</td>
<td>0.200</td>
<td>15.60</td>
</tr>
<tr>
<td>20</td>
<td>17.580</td>
<td>309.07</td>
<td>2</td>
<td>11.941</td>
<td>12.941</td>
<td>0.200</td>
<td>17.76</td>
</tr>
</tbody>
</table>

| \( \alpha_1 = 1 \) for all cases |

In order to determine the accuracy of results obtained by FP methods, computer-simulation results were carried out (Ref. 4) for certain cases and are included on some of the plots. These results were obtained by implementing Fig. 15-3 as a difference equation on a digital computer. Gaussian random sequences were generated using a standard pseudo-random number subroutine. The filters were implemented via bilinear \( z \)-transform techniques (Refs. 1, 2), which amount to simply replacing the Laplace transform operator \( s \) by

\[
\frac{2}{DT} \left[ \frac{1 - z^{-1}}{1 + z^{-1}} \right]
\]

and treating \( z^{-1} \) as the delay operator by the time shift \( DT \). The result is a difference equation. The difference equation was started at zero initial conditions, then allowed to run for a period of time equal to ten times the longest time constant in the system in order to minimize transient effects. The sample variances were then computed over 50,000 samples with a sample time \( DT \) of 0.005.

In many of the plots presented in this section, curves based on the linear model are included for comparison. These results were computed using linear system theory and the linear model obtained by replacing \( f[x(t)] \)
Fig. 15-4. Variance of the Phase Error versus Input Signal-to-Noise Ratio, $R$.

Fig. 15-5. Variance of the Phase Error versus Input Signal-to-Noise Ratio, $R$. 

624
in (15-13) by the linear term in its Taylor series expansion—that is, by $(\mathbf{A} + \mathbf{Da})\mathbf{x}(i)$.

Once the constants in the state-space equations (15-13) and (15-14) were determined, the equations presented in Sections 13-5 and 14-5 of Chapters 13 and 14, respectively, were applied directly. In order to determine the variance of the phase error, the constant $\alpha$ was computed from (14-60). The modulo-$2\pi$ variance $\sigma_\phi^2$ was then computed directly by numerical integration, using the probability density function $p(\phi)$ as presented in (14-58). In order to demonstrate this theory (i.e., the conditional expectation method where the form of the conditional expectation is known) numerical results are plotted in Figs. 15-4 and 15-5. Computer-simulation results are also included and the comparison is seen to be reasonably good. Figure 15-6, then, presents system performance data based on this method, where the modulation index has been allowed to vary over several different values.

![Graph showing variance of phase error versus input signal-to-noise ratio for various values of $m_f$.](image)

**Fig. 15-6.** Variance of the Phase Error versus Input Signal-to-Noise Ratio for Various Values of $m_f$.

### 15-3.1 Demodulator Performance in the Presence of White Noise

The frequency tracking error is now examined. Since most studies of this nature include plots of an output signal-to-noise ratio versus an input signal-to-noise ratio, this practice will be continued here. In accordance with the theory in Chapter 4, the mean-squared value of the frequency tracking
error \( z(t) = m(t) - y(t) \) is treated as the output noise. The desired output is the signal \( m(t) \) so that the output signal-to-noise ratio is defined by

\[
\rho_0 \triangleq \frac{E[m^2(t)]}{E[z^2(t)]}
\]  

(15-16)

The frequency tracking error \( z(t) \) contains the effects of signal distortion as well as disturbing noise; therefore, as the noise is reduced to zero, \( \rho_0 \) will approach a constant value.

Several methods were developed in the previous chapters for studying the statistical properties of the frequency tracking error. The simplest is the conditional expectation method where, in this case, the form of the conditional expectations is unknown. By means of this method, the variance of the frequency tracking error was computed by numerical integration using the p.d.f. \( p(z) \) presented in (14-72). The numerical results presented in Figs. 15-7 and 15-8 agree fairly well with the simulation results. This method is not, however, as general as would be desired. An examination of the state-space equations and Fig. 15-2 reveals that if an additional pole is added to the output filter, the results become equivalent to the linear results. If the noise is colored and if one additional pole is added to the signal-shaping filter, \( K_0 \) and \( K_{00} \) will both be zero and the one-dimensional FP equation fails to yield any solution.

The sequence method does not have the same problems as the conditional expectation method. Since it converges to the correct solution in the

![Figure 15-7. Output Signal-to-Noise Ratio versus R.](image-url)
Fig. 15-8. Output Signal-to-Noise Ratio versus $R$.

Fig. 15-9. Mean-Squared Value of the Frequency Tracking Error versus $R$ Obtained by Sequence Method.
limit, this method also has a stronger mathematical basis. The accuracy of the sequence method can be increased by taking more and more terms so that any desired accuracy can be reached with enough effort. For the case of interest here, three terms were evaluated for the sinusoidal nonlinearity and they are presented in Section 13-4. In order to demonstrate convergence, the frequency tracking error was computed using one, two, and three terms and starting from the purely linear model. The results are plotted in Figs. 15-9 and 15-10. The convergence appears to be monotonic (continually decreasing error), and with enough terms a reasonable solution may be reached.

![Graph showing the mean-squared value of the frequency tracking error versus R obtained by sequence method.](image)

**Fig. 15-10.** Mean-Squared Value of the Frequency Tracking Error versus $R$ Obtained by Sequence Method.

The starting model is then modified to ensure much faster convergence. This is done simply by selecting the matrix $M$ in (13-32) according to (14-73). If no other method is available, the expectations in (14-73) can be evaluated via linear theory to produce

$$M = A + D \text{e} \exp \left( \frac{\tilde{\sigma}_e^2}{2} \right)$$

(15-17)

where we note that the tilda over $\sigma_e^2$ denotes computation on basis of the linear theory. This results in a considerable improvement over starting from the purely linear model. Since the p.d.f $p(\phi)$ is available from (14-58), however, these expectations can be evaluated directly. They were computed by using...
Fig. 15-11. Mean-Squared Value of the Frequency Tracking Error versus $R$ Obtained by Combination Method.

Fig. 15-12. Mean-Squared Value of the Frequency Tracking Error versus $R$ Obtained by Combination Method.
numerical integration, and the frequency tracking error was evaluated by this combination method using one, two, and three terms from the sequence. The results are presented in Figs. 15-11 and 15-12 and, after three terms, agree quite closely with the simulation results also included. A family of such curves obtained in this same manner are plotted in Fig. 15-13 for use in system performance evaluation.

![Graph showing the relationship between output signal-to-noise ratio and input signal-to-noise ratio.](image)

**Fig. 15-13.** Mean-Squared Value of the Frequency Tracking Error versus R for Various Values of Modulation Index.

### 15-3.2 Demodulator Performance in the Presence of Colored Noise

The colored-noise case (Ref. 4) is now examined using the combination method with three terms. In this case the one-dimensional Fokker-Planck equation produces no solution for $p(\phi)$. The matrix $M$, however, was still selected by (14-73) where the expectations were evaluated from the white-noise case. This method ensures that the first term of the sequence is the same for white and colored noise so that these cases can be accurately compared. The results for both white and colored noise are plotted in Figs. 15-14 and 15-15, and a family of curves with different modulation indices for the colored-noise case are plotted in Fig. 15-16.

The colored-noise results are rather interesting. We see from Figs. 15-14 and 15-15 that the “knee” of the curves occurs about 2 dB lower for colored noise than for white noise, a significant difference. The noise band-
Fig. 15-14. Mean-Squared Value of the Frequency Tracking Error versus $R$ in Colored Noise.

Fig. 15-15. Mean-Squared Value of the Frequency Tracking Error versus $R$ in Colored Noise.
Fig. 15-16. Family of Frequency Demodulator Performance Curves in Colored Noise.

widths are selected according to a version of Carson’s rule commonly used for stochastic signals and given by (15-3). The resulting noise bandwidths defined as \( a_n \) (corresponding to one-half the predetection bandwidths) are given with the loop natural frequencies \( \omega_n \) in Table 15-2. Even for the cases with higher-modulation indices, the noise bandwidths cannot be considered sufficiently higher in frequency than the loop dynamic frequencies that the noise can be approximated as white.

<table>
<thead>
<tr>
<th>( m_f )</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n ) (rad/sec)</td>
<td>5</td>
<td>11</td>
<td>21</td>
<td>31</td>
<td>41</td>
</tr>
<tr>
<td>( \omega_n ) (rad/sec)</td>
<td>3.598</td>
<td>6.831</td>
<td>10.978</td>
<td>14.473</td>
<td>17.580</td>
</tr>
</tbody>
</table>

15-4 Further Studies

In Section 15-2 equations were derived whereby an optimum loop filter and output filter could be selected when operation in the linear region is ex-
pected. These linear methods are suggested as a method of initially designing a phase-locked frequency demodulator even though operation may not be clearly linear. This method, however, has a disadvantage in that even though the loop may be operating in the linear region, it will be optimum only for one value of the input signal-to-noise ratio.

In practice, a loop that is not adaptive is usually desired; that is, one in which the pole-zero configuration of the loop does not change with noise level and that can perform satisfactorily over a reasonable range of input noise level. System optimization subject to these constraints involves both the linear and nonlinear regions of operation and can be accomplished only through a trade-off study involving curves computed using specific system parameters. Because specific systems are too varied, such an optimization study can be accomplished via the FP techniques. Once the second-order loop configuration has been established, optimization within the PLL can be accomplished by varying either the loop damping constant $\zeta$, the loop natural frequency $\omega_n$, or the product of the carrier level times the loop gain $AK$. The loop damping constant is usually fixed at $1/\sqrt{2}$ by considerations involving system transient effects. Some results on the effects of varying $\omega_n$ and $AK$ have been given by Mayfield (Ref. 4); in fact, the computer simulations and numerical results given in this chapter are due to Mayfield (Ref. 4).

At the conclusion of Chapter 12, Section 12-7, reference material was given pertaining to the problem of optimum demodulation of analog signals based on the theory of estimation. In addition, it is worth mentioning the early work of Lehan and Parks (Ref. 5); Youla (Ref. 6) formalized the Lehan and Parks work. Recently Beste (Ref. 7) and Beste and Weber (Ref. 8) have considered the problem of specification and performance evaluation of optimal state variable filters for communication problems. References pertaining to the methods of state-space representation and nonlinear filtering theory are collected and given at the conclusions of Chapters 8 and 13; consequently, there is no need to repeat them here.

References


16

TIME-DEPENDENT SOLUTIONS
TO THE FOKKER-PLANCK
EQUATION

16-1 Introduction

One of the more difficult and unsolved problems that remains in statistical communication theory is that of signal acquisition and synchronization stability in the presence of noise. Yet one of the most important practical problems occurring in the analysis and synthesis of a coherent communication and tracking system is that of specifying the the systems acquisition characteristics and whether or not synchronization stability can be achieved. Undoubtedly the reason why this problem remains largely unsolved concerns the fact that solutions to the synchronization stability and signal acquisition problem are tantamount to obtaining and studying solutions to time-dependent Fokker-Planck equations. Aside from obtaining solutions, the questions of existence and uniqueness of a solution immediately arise. Chapter 13 outlined a sequence method for obtaining time-dependent solutions to FP equations. However, the method requires the use of a digital computer and is therefore limited when it comes to setting forth a general theory; in other words, it appears to be most useful for studying the properties of a well-defined FP equation.

Before proceeding it is perhaps best to motivate, in more mathematical
terms, the need for time-dependent solutions. As pointed out in Chapters 9, 10, and 11, the conditional transition p.d.f.'s $p(\phi, t|\phi_0, t_0, n)$ for a first-order loop and $p(\phi, t|y_0, t_0, n)$ for higher-order loops contain information about how the phase error process diffuses with the passage of time. On the other hand, the restricted "p.d.f.'s" $Q(\phi, t|\phi_0, t_0)$ and $Q(\phi, t|y_0, t_0)$ also contain pertinent information regarding the cycle-slipping process. Since a loop slips cycles during the signal acquisition mode of loop operation, the restricted "p.d.f.'s" are of interest in studying loop acquisition properties in noise. The general question of the existence of these p.d.f.'s is related to the question of synchronization stability in a loop.

To a limited extent, the signal acquisition problem—that is predicting the acquisition time and range, plus the synchronization stability of loop in the absence and presence of noise—was covered in Chapters 9 and 10 for first- and second-order loops. However, as noted there, a general analysis pertaining to loop stability and loop acquisition characteristics was impossible because of the characteristically nonlinear nature of the loop equation.

The material presented in this chapter begins by applying the separation of variables method discussed in Chapter 7. For the first-order PLL it is shown how the eigenfunction expansion method can be used to determine the conditional transition p.d.f. $p(\phi, t|\phi_0, t_0, n)$. The problem reduces to studying a two-point boundary-value problem of Hill's equation (see Chapter 5, Section 5-10). Therefore, in what follows, we shall be concerned with solutions to self-adjoint and nonself-adjoint boundary-value problems. It is a assumed that the reader has a working knowledge of this subject. However, appropriate parts of the theory given in Refs. 1 and 2.

16-2 Development of the Expansion for the Conditional Transition Probability Density Function*

As was shown in Chapter 9, the conditional transition p.d.f. $p(\phi, t|\phi_0, t_0, n)$ corresponding to a first-order SCS satisfies the FP equation

$$\frac{\partial p(\phi, t|\phi_0, t_0, n)}{\partial t} = Lp(\phi, t|\phi_0, t_0, n)$$

(16-1)

Here $p(\phi; t) \triangleq p(\phi, t|\phi_0, t_0, n)$ is a real-valued function of the variable $(\phi; t)$ defined on the cartesian product $\Omega \times T$, where

$$\Omega = \text{the closed interval } \phi \in I(n) = [(2n - 1)\pi, (2n + 1)\pi]$$

*The results given in this Chapter were extracted from the Doctoral dissertation of James R. LaFrieda, (Ref. 14).
\[ T = [t_0, t], \text{ a time interval with } t_0 \text{ chosen as the initial time, and } t \text{ a final time} \]

\[ L = -\frac{\partial}{\partial \phi} \left[ A_0 - AKg(\phi) \right] + \frac{K^2 N_0}{4} \frac{\partial^2}{\partial \phi^2} \text{, a spatial operator defined on some domain } D(L) \subset L^2[2n - 1)\pi, (2n + 1)\pi] \]

\[ L_\phi(\cdot, \cdot) = \text{ the space of square integrable functions defined on } I(n). \]

In what follows, we set \( n = 0 \) without loss in generality (see Chapter 9).

As demonstrated in Chapter 9, Section 9-3.2, \( p(\phi; t) \) must satisfy a set of linear homogeneous boundary conditions, which can be expressed in the form

\[ B_1(p) = 0 : p(-\pi; t) - p(\pi; t) = 0 \]
\[ B_2(p) = 0 : \frac{\partial p(-\pi; t)}{\partial \phi} - \frac{\partial p(\pi; t)}{\partial \phi} = 0 \] (16-2)

and the initial condition

\[ \lim_{t \to t_0} p(\phi; t) = \delta(\phi - \phi_0) \] (16-3)

We may now define the domain \( D(L) \) of the operator \( L \). It is the set of all functions \( u(\phi) \) in \( L^2[-\pi, \pi] \) that are twice continuously differentiable, that satisfy (16-2), and are such that \( Lu \) belongs to \( L^2[-\pi, \pi] \). It will be assumed for the moment that the problem described by (16-1), (16-2), and (16-3) is well posed. That is, a solution exists, it is unique, and it depends continuously on the parameters of the system.

In order to obtain the solution \( p(\phi; t) \), we first consider, as is customary in the method of separation of variables, the following auxiliary problem. Find a nontrivial solution of (16-1) that satisfies the boundary conditions (16-2) and that is representable in the form \( p(\phi; t) = s(t)u(\phi) \), where \( u(\phi) \) is a function dependent only on \( \phi \in [-\pi, \pi] \), and \( s(t) \) depends on \( t \) in \( T \). Differentiating \( p(\phi; t) \) and substituting the result in (16-1), we obtain the eigenvalue-eigenfunction problem

\[ Lu_n(\phi) = -\lambda_n u_n(\phi) \]
\[ B_1(u_n) = u_n(-\pi) - u_n(\pi) = 0 \]
\[ B_2(u_n) = u'_n(-\pi) - u'_n(\pi) = 0, \quad n = 0, 1, 2, \ldots \] (16-4)

where the prime denotes the spatial derivative.

The adjoint operator to \( L \) and the adjoint boundary conditions are found next. Let \( v(\phi) \) be a twice continuously differentiable function and consider the scalar product
\[ \langle v, Lu \rangle = \int_{-\pi}^{\pi} v[a_0(\phi)u'' + a_1(\phi)u' + a_2(\phi)u] \, d\phi \]  
(16-5)

when the primes denote differentiation with respect to \( \phi \), and where

\[
a_0(\phi) = \frac{K^2 N_0}{4} \triangleq D
\]

\[
a_1(\phi) = AK_g(\phi) - \Lambda_0
\]

\[
a_2(\phi) = AK_g \frac{d}{d\phi} \frac{\partial g(\phi)}{\partial \phi}
\]

If we integrate by parts once, so as to transfer the differentiations from \( u \) to \( v \), then (16-5) becomes

\[
\langle v, Lu \rangle = \int_{-\pi}^{\pi} u[(a_0(\phi)v)'' - (a_1(\phi)v)' + a_2(\phi)v] \, d\phi
\]

\[
+ [a_0(\phi)[vu' - uv'] + uv[a_1(\phi) - a_0(\phi)]]\pi_{-\pi}
\]

(16-7)

The bracketed term in the integrand of (16-7) is the formal adjoint of \( L \), and we denote it as \( L^* \). Expanding the brackets, we find

\[
L^* = a_0(\phi) \frac{d^2}{d\phi^2} + [2a_0(\phi) - a_1(\phi)] \frac{d}{d\phi} + [a_0(\phi) - a_0(\phi) + a_2(\phi)]
\]

\[
= D \frac{d^2}{d\phi^2} + [\Lambda_0 - AK_g(\phi)] \frac{d}{d\phi}
\]

(16-8)

and we note that the operator \( L \) is not formally self-adjoint, since \( L \neq L^* \). The remaining term of the integration

\[
J(u, v)\pi_{-\pi} = [D(vu' - uv') + uv(AK_g(\phi) - \Lambda_0)]\pi_{-\pi}
\]

(16-9)

is the bilinear conjunct of \( u \) and \( v \), and it satisfies Green's formula (Refs. 1, 2)

\[
\int_{-\pi}^{\pi} (vLu - uL^* v) \, d\phi = [J(u, v)]\pi_{-\pi}
\]

(16-10)

The adjoint boundary conditions are determined by finding those functions \( v(\phi) \) that satisfy (Ref. 1)

\[
J(u, v)\pi_{-\pi} = 0 \quad \text{for every } u \in DL
\]

(16-11)

From (16-4) and (16-9), we obtain the two linear homogeneous adjoint boundary conditions
\[ B_1^*(v) = 0 : v(-\pi) - v(\pi) = 0 \]
\[ B_2^*(v) = 0 : v'(-\pi) - v'(\pi) = 0 \]  
(16-12)

since, by hypothesis, \( g(\phi) = -g(-\phi) \). Therefore the domain \( D^*(L^*) \) of the adjoint operator is the set of all twice continuously differentiable functions \( v(\phi) \) in \( \mathcal{L}_2[-\pi, \pi] \) that satisfy (16-12) and are such that \( L^*v \) belongs to \( \mathcal{L}_2[-\pi, \pi] \).

We now note that \( D(L) = D^*(L^*) \); however, \( L \neq L^* \). The general second-order differential operator

\[ L = a_0(\phi) \frac{d^2}{d\phi^2} + a_1(\phi) \frac{d}{d\phi} + a_2(\phi) \]  
(16-13)

where \( a_0, a_1, \) and \( a_2 \) are given by (16-6), can be made formally self-adjoint (i.e., \( L = L^* \)) if the appropriate scalar product is used (Refs. 1, 2). In particular, \( L \) can be identified with the differential expression

\[ Lu = -[w(\phi)]^{-1}[r(\phi)u'(\phi)]' + q(\phi)u(\phi) \]  
(16-14)

which is formally self-adjoint if the scalar product is defined as

\[ \langle u, v \rangle \triangleq \int_{-\pi}^{\pi} u(\phi)v(\phi)w(\phi) \, d\phi \]  
(16-15)

where

\[ [w(\phi)]^{-1} \triangleq -a_0(\phi) \exp \left[ -\int \frac{a_1(x)}{a_0(x)} \, dx \right] = -Dr(\phi) \]

\[ r(\phi) \triangleq \exp [\alpha f(\phi) - \beta \phi] \]  
(16-16)

\[ q(\phi) \triangleq a_2(\phi) = AK \frac{\partial g(\phi)}{\partial \phi} = g'(\phi) \]

and we have set

\[ f(\phi) = \int_{-\pi}^{\phi} g(x) \, dx \]  
(16-17)

for simplicity. The parameters \( \alpha \) and \( \beta \) are defined in (9-28). With this weighting, \( L = L^* \); however, if we consider the difference

\[ \langle v, Lu \rangle - \langle u, L^*v \rangle = \langle v, Lu \rangle - \langle Lv, u \rangle \]

then from (15-10) we obtain
\[ \langle v, Lu \rangle - \langle Lv, u \rangle = J(u, v)|_{-\pi}^{\pi} \]
\[ = -\exp[\alpha f(\phi) - \beta \phi](uv' - uv')|_{-\pi}^{\pi} \]  
(16-18)

Since \( g(\phi) \) is odd-symmetric, its integral \( f(\phi) \) is even-symmetric; however, if \( u \) and \( v \) are assumed to be in the same domain—i.e., \( D(L) = D(L^*) \)—then we note that this is impossible, for the conjunct \( J \) does not vanish because of the presence of the nonperiodic factor \( \exp(-\beta \phi) \) in (16-18). Thus we see that \( L = L^* \), but \( D(L) \neq D^*(L^*) \). Hence if \( \beta \neq 0 \), then (16-4) is nonself-adjoint, whereas if \( \beta = 0 \), then (16-4) is self-adjoint.

The adjoint set of eigenvalues \( \lambda_n^* \) and eigenfunctions \( v_n \) are found from

\[ L^* v_n = \lambda_n^* v_n \quad n = 0, 1, 2, \ldots \]  
(16-19)

subject to (16-12), where the \( v_n \)'s are functions of \( \phi \) defined in \([-\pi, \pi]\) only. Note that if \( \lambda_n \) is an eigenvalue of \( L \), then it is also an eigenvalue of \( L^* \) (Ref. 1). The orthogonality of the eigenfunctions that holds when (16-4) is self-adjoint need not hold when (16-4) is nonself-adjoint. It is replaced (Ref. 1) by a biorthogonality relationship involving the eigenfunctions of (16-4) and those of (16-19); that is,

\[ \langle v_n, u_m \rangle = \int_{-\pi}^{\pi} v_n(x) u_m(x) \, dx = 0 \quad \text{if} \quad \lambda_n \neq \lambda_m \]  
(16-20)

Since the magnitudes of the eigenfunctions are arbitrary, they may be normalized to satisfy

\[ \langle v_n, u_m \rangle = \delta_{nm} \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \]  
(16-21)

Assuming that the \( u_n \)'s are complete—that is, they span \( \mathcal{L}_2[-\pi, \pi] \)—an arbitrary function belonging to \( \mathcal{L}_2[-\pi, \pi] \times T \). Thus \( p(\phi; t) \) can be expanded as

\[ p(\phi; t) = \sum_{n=0}^{\infty} s_n(t) u_n(\phi) \]  
(16-22)

where

\[ s_n(t) = \int_{-\pi}^{\pi} p(\phi; t) v_n(\phi) \, d\phi \]  
(16-23)

since, for almost all \( t \in T \), \( p(\phi; t) \in \mathcal{L}_2[-\pi, \pi] \). Substituting (16-22) in (16-1), we have
\[
\sum_{n=0}^{\infty} \delta_n(t)u_n(\phi) = \sum_{n=0}^{\infty} s_n(t)Lu_n(\phi)
\]  
(16-24)

and using (16-4) we obtain

\[
\sum_{n=0}^{\infty} \delta_n(t)u_n(\phi) = -\sum_{n=0}^{\infty} \lambda_n s_n(t)u_n(\phi)
\]  
(16-25)

Multiplying both sides of (16-25) by \(v_n(\phi)\), integrating over \([-\pi, \pi]\), and using (16-21), we obtain the following set of ordinary differential equations:

\[
\delta_n(t) = -\lambda_n s_n(t), \quad n = 0, 1, 2, \ldots
\]  
(16-26)

with the initial conditions

\[
s_n(t_0) = \int_{-\pi}^{\pi} \delta(\phi - \phi_0)v_n(\phi) d\phi = v_n(\phi_0)
\]  
(16-27)

Thus the solution to this countably infinite set of ordinary differential equations, together with the expansion of (16-22), provides the solution to the FP equation (16-1). Note that the solution given by (16-22) satisfies the boundary conditions by the choice of the eigenfunctions \(u_n\); that is, because of (16-4), we have

\[
B_1(p) = \sum_{n=0}^{\infty} s_n(t)B_1(u_n) = 0
\]  
(16-28)

\[
B_2(p) = \sum_{n=0}^{\infty} s_n(t)B_2(u_n) = 0
\]  
(16-29)

so that the boundary conditions are met.

The solution to (16-26) is given by

\[
s_n(t) = v_n(\phi_0) \exp\left[-\lambda_n(t - t_0)\right]
\]  
(16-30)

Multiplying (16-29) by \(u_n(\phi)\) and summing over all \(n\), we obtain the expansion

\[
p(\phi; t) = \sum_{n=0}^{\infty} v_n(\phi_0)u_n(\phi) \exp\left[-\lambda_n(t - t_0)\right]
\]  
(16-31)

where the sum converges in the mean of order two to \(p(\phi; t)\). Note that in the limit as \(t\) approaches \(t_0\), (16-30) reduces to

\[
\lim_{t \to t_0} p(\phi; t) = \sum_{n=0}^{\infty} v_n(\phi_0)u_n(\phi)
\]  
(16-31)
and in order that the initial condition (16-3) be satisfied, the Dirac distribution
\[ \delta(\phi - \phi_0) \] must have the representation
\[ \delta(\phi - \phi_0) = \sum_{n=0}^{\infty} v_n(\phi_0)u_n(\phi) \]  
(16-32)

where the coefficients \( v_n(\phi_0) \) are given by (16-27).

Since \( \delta(\phi - \phi_0) \) does not belong to \( L_2[-\pi, \pi] \), in order to prove the validity of (16-32), let \( H(\phi) \) be an arbitrary function in \( L_2[-\pi, \pi] \) that is continuous at \( \phi = \phi_0 \). Then \( H(\phi) \) has the representation
\[ H(\phi) = \sum_{n=0}^{\infty} H_n u_n(\phi) \]  
(16-33)

where
\[ H_n = \int_{-\pi}^{\pi} H(\phi)v_n(\phi)\,d\phi \]  
(16-34)

and we note that (16-33) converges in the mean of order two, for the \( u_n \)'s are complete. In addition, from the sifting property of the Dirac distribution, we obtain
\[ H(\phi_0) = \int_{-\pi}^{\pi} H(\phi)\delta(\phi - \phi_0)\,d\phi \]  
(16-35)

Now, let us consider the partial sums of (16-32).
\[ \delta_n(\phi - \phi_0) = \sum_{m=0}^{n} v_m(\phi_0)u_m(\phi), \quad n = 1, 2, \ldots \]  
(16-36)

Then the values of the distributions \( \delta_n(\phi - \phi_0) \) acting on \( H(\phi) \) are
\[ H_n(\phi_0) = \int_{-\pi}^{\pi} \delta(\phi - \phi_0)H(\phi)\,d\phi \]
\[ = \sum_{m=0}^{n} H_m u_m(\phi) \]  
(16-37)

Therefore the sequence (16-36) converges weakly, in the sense of weak convergence of linear functionals, to (16-32), since \( \lim_{n \to \infty} H_n(\phi_0) = H(\phi_0) \).

If (16-4) is integrated over \([-\pi, \pi]\), then we obtain the relation
\[ \int_{-\pi}^{\pi} Lu_n(x)\,dx = -\lambda_n \int_{-\pi}^{\pi} u_n(x)\,dx = 0 \]  
(16-38)
Expansion for the Conditional Transition Probability Density Function

provided \( g(\pi) = -g(-\pi) = 0 \), for the boundary conditions of (16-4) imply

\[
\int_{-\pi}^{\pi} Lu_n(x) \, dx = [Du_n(\phi) - [\Lambda_0 - AKg(\phi)]u_n(\phi)]_{-\pi}^{\pi} = 0 \tag{16-39}
\]

Therefore, for any \( n \), either \( \lambda_n = 0 \) or

\[
\int_{-\pi}^{\pi} u_n(\phi) \, d\phi = 0 \tag{16-40}
\]

If we set \( \lambda_0 = 0 \) and solve the homogeneous eigenvalue problems

\[
Lu_0 = 0, \quad B_1(u_0) = 0, \quad B_2(u_0) = 0
\]
\[
L^*v_0 = 0, \quad B^*_1(v_0) = 0, \quad B^*_2(v_0) = 0 \tag{16-41}
\]

then we find that \( \lambda_0 = 0 \) is an eigenvalue, for (16-41) has the nontrivial solutions

\[
u_0(\phi) = C_1 \exp \left[ \beta\phi - \alpha f(\phi) \right] \left[ 1 + C_2 \int_{-\pi}^{\phi} \exp \left[ -\beta x + \alpha f(x) \right] \, dx \right] \]
\[
\tag{16-42}
\]

\[
v_0(\phi) = C_3 + C_4 \int_{-\pi}^{\phi} \exp \left[ -\beta x + \alpha f(x) \right] \, dx \tag{16-43}
\]

where

\[
C_1 = \left[ \int_{-\pi}^{\phi} \exp \left[ \beta\phi - \alpha f(\phi) \right] \left[ 1 + C_2 \int_{-\pi}^{\phi} \exp \left[ -\beta x + \alpha f(x) \right] \, dx \right] \, d\phi \right]^{-1} \tag{16-44}
\]

\[
C_2 = \frac{\exp(-2\beta \pi) - 1}{\int_{-\pi}^{\pi} \exp \left[ -\beta x + \alpha f(x) \right] \, dx} \tag{16-45}
\]

\[
C_3 = 1 \quad \text{and} \quad C_4 = 0 \tag{16-46}
\]

Note that the constants \( C_1, \ldots, C_4 \) are arbitrary; however, since \( C_1 \) and \( C_2 \) have been chosen such that

\[
\int_{-\pi}^{\pi} u_0(\phi) \, d\phi = 1 \quad \text{and} \quad u_0(-\pi) = u_0(\pi) \tag{16-47}
\]

\( C_4 \) must vanish and \( C_3 \) must equal unity in order to satisfy (16-21).

Hence in terms of the eigenfunctions of \( L \) and \( L^* \), normalized according to (16-21), we obtain the expansion
\[ p(\phi; t) = p(\phi) + \sum_{n=1}^{\infty} v_{n}(\phi_{0})u_{n}(\phi) \exp \left[ -\lambda_{n}(t - t_{0}) \right] \]  \hspace{1cm} (16-48)

since \( u_{n}(\phi) \) is the steady-state solution \( p(\phi) \) of the FP equation; that is, \( \lambda_{0} = 0 \) gives \( p(\phi) \).

It should be noted that the expansion (16-48) was developed based on the hypotheses that the spectrum of \( L \) is nonempty and discrete and that the eigenfunctions of \( L \) form a complete set in the sense of mean square convergence on the interval \([-\pi, \pi]\). In addition, we note that the eigenfunction expansion (16-48) is only a formal solution of the boundary value problem (16-1) to (16-3); it is an exact solution only if the series (16-48) converges and can be differentiated term by term twice with respect to \( \phi \) and once with respect to \( t \). This last step is valid provided that the terms of the series have continuous derivatives and that the series of derivatives is uniformly convergent (Ref. 3).

The next section will be concerned with a verification of the preceding hypotheses and with the development of various eigenfunction expansions for the conditional transition p.d.f.

16-3 Spectral Analysis of First-Order Loops in the Absence of Detuning, \( \beta = 0 \)

Our main objective is to determine an eigenfunction expansion for the conditional transition p.d.f. of the first-order phase-locked loop when \( \alpha \) and \( \beta \) are both not equal to zero. It is instructive, however, first to study the problem when \( \beta = 0 \) and later to return to the more complex case when \( \beta \neq 0 \). We begin by keeping \( \beta \) in our equations and later setting it equal to zero.

We first observe that if the normalized cross-correlation function \( g(\phi) \) is not of class \( C^{1}[\, -\pi, \pi \,] \), then the corresponding eigenvalue problem (16-4) is singular because the operator \( L \) is no longer well defined at the discontinuities of \( g(\phi) \). Conversely, if \( g(\phi) \) is in class \( C^{1}[\, -\pi, \pi \,] \), then \( L \) is a regular differential operator because the coefficient \( a_{0}(\phi) = D > 0 \) and the coefficients \( a_{0}(\phi), a_{1}(\phi), \) and \( a_{2}(\phi) \) are all real-valued continuous functions in the finite, closed interval \( |\phi| \leq \pi \).

Dividing the differential expression (16-13) by \( a_{0}(\phi) \), we obtain the eigenvalue problem

\[
Lu = u''(\phi) + (\alpha g(\phi) - \beta)u'(\phi) + \alpha g'(\phi)u(\phi) = -\lambda' u(\phi) \\
B_{1}(u) = u(-\pi) - u(\pi) = 0 \\
B_{2}(u) = u'(-\pi) - u'(\pi) = 0, \quad \lambda' = \frac{\lambda}{D} \]  \hspace{1cm} (16-49)

This differential equation can be transformed into the homogeneous differential
equation

\[ Ly = y''(\phi) - Q(\phi)y'(\phi) = -\lambda' y(\phi) \]  \hspace{1cm} (16-50)

lacking a middle term, by the change of the dependent variable

\[ u(\phi) = \gamma(\phi) \exp \left[ -\frac{\alpha}{2} f(\phi) + \frac{\beta \phi}{2} \right] \]  \hspace{1cm} (16-51)

where

\[ Q(\phi) = -\frac{\alpha}{2} g'(\phi) + \frac{\alpha^2}{4} g^2(\phi) - \frac{\alpha \beta}{2} g(\phi) + \frac{\beta^2}{4} \]  \hspace{1cm} (16-52)

is a real periodic function of period \(2\pi\) that is neither even nor odd, since \(g(\phi)\) is real periodic with period \(2\pi\) and odd-symmetric. Using the transformation (16-51), the periodic boundary conditions of (16-49) are transformed into the nonperiodic conditions

\[ B_1(y) = \exp \left( -\frac{\beta \pi}{2} \right) y(-\pi) - \exp \left( \frac{\beta \pi}{2} \right) y(\pi) = 0 \]

\[ B_2(y) = \exp \left( -\frac{\beta \pi}{2} \right) y'(-\pi) + \frac{\beta}{2} \exp \left( -\frac{\beta \pi}{2} \right) y(-\pi) \]

\[- \exp \left( \frac{\beta \pi}{2} \right) y'(\pi) - \frac{\beta}{2} \exp \left( \frac{\beta \pi}{2} \right) y(\pi) = 0 \]  \hspace{1cm} (16-53)

where we observe that the periodic factor \(\exp \left[ -\alpha f(\phi)/2 \right]\) of the transformation cancels, for the function \(f(\phi)\) is periodic with period \(2\pi\) and even-symmetric. The result is that the operator \(L\) of (16-50) is now formally self-adjoint. However, the eigenvalue problem remains nonself-adjoint, for the domain of \(L^*\), defined by the nonperiodic adjoint conditions

\[ B_1(v) = \exp \left( \frac{\beta \pi}{2} \right) v(-\pi) - \exp \left( -\frac{\beta \pi}{2} \right) v(\pi) = 0 \]

\[ B_2(v) = \exp \left( \frac{\beta \pi}{2} \right) v'(-\pi) + \frac{\beta}{2} \exp \left( \frac{\beta \pi}{2} \right) v(-\pi) \]

\[- \exp \left( -\frac{\beta \pi}{2} \right) v'(\pi) - \frac{\beta}{2} \exp \left( -\frac{\beta \pi}{2} \right) v(\pi) = 0 \]  \hspace{1cm} (16-54)

does not coincide with (16-53).

In the sequel it will be convenient to replace the interval \([-\pi, \pi]\) by \([0, \pi]\). Thus with the change of variables \(\phi = 2x - \pi\) and use of the chain rule, we obtain, without loss of generality,
\[ Ly = y''(x) - Q(x)y(x) = -\eta y(x), \quad 0 \leq x \leq \pi \quad (16-55) \]

\[ B_1(y) = \exp \left( -\frac{\beta \pi}{2} \right) y(0) - \exp \left( \frac{\beta \pi}{2} \right) y(\pi) = 0 \]

\[ B_2(y) = \exp \left( -\frac{\beta \pi}{2} \right) y'(0) + \beta \exp \left( -\frac{\beta \pi}{2} \right) y(0) \]

\[ -\exp \left( \frac{\beta \pi}{2} \right) y'(\pi) - \beta \exp \left( \frac{\beta \pi}{2} \right) y(\pi) = 0 \quad (16-56) \]

where

\[ Q(x) = -2\alpha g'(2x - \pi) + \alpha^2 g^2(2x - \pi) - 2\alpha \beta g(2x - \pi) + \beta^2 \]

\[ Q(x) = Q(x + \pi) \quad (16-57) \]

and

\[ \eta = 4\lambda = \frac{4\lambda}{D} \quad (16-58) \]

When \( \beta = 0 \), the nonself-adjoint problem (16-55) to (16-58) reduces to the regular self-adjoint problem

\[ Ly = y''(x) - Q(x)y(x) = -\eta y(x) \quad (16-59) \]

\[ B_1(y) = y(0) - y(\pi) = 0 \quad (16-60) \]

\[ B_2(y) = y'(0) - y'(\pi) = 0 \]

where the periodic function

\[ Q(x) = -2\alpha g'(2x - \pi) + \alpha^2 g^2(2x - \pi) \quad (16-61) \]

is now even-symmetric, since \( g(\phi) \) is odd. Therefore from the spectral theory of regular self-adjoint eigenvalue problems on a finite interval (Ref. 1), we note that

1. Eigenvalues of (16-59) to (16-61) always exist; the eigenvalues are real, and there can be at most a denumerable number of eigenvalues with no finite point of accumulation.

2. The set of all eigenfunctions (including the eigenfunction corresponding to \( \eta = 0 \)) forms a complete set in \( L_2[0, \pi] \); and eigenfunctions corresponding to distinct eigenvalues are orthogonal.

3. Any function in the domain of definition of the operator \( L \) [or in \( L_2[0, \pi] \)] can be expanded in terms of the eigenfunctions of \( L \) in a
generalized Fourier series that converges uniformly (or in the mean square sense).

Thus, with $\beta = 0$, the eigenfunctions of (16-50) to (16-53) and (16-59) to (16-61) are orthogonal and can always be normalized to satisfy $\langle y_n, y_m \rangle = \delta_{nm}$; however, in this case the eigenfunctions of (16-50) are now orthonormal with respect to the weighting factor given by (16-51). That is,

$$\langle u_n, u_m \rangle = \int_{-\pi}^{\pi} u_n(x)u_m(x) \exp[\alpha f(x)] \, dx = \delta_{nm} \quad (16-62)$$

Hence in the self-adjoint case where $v_n(\phi) = u_n(\phi)$, the formal expansion (16-48) reduces to

$$p(\phi; t) = p(\phi) + \sum_{n=1}^{\infty} u_n(\phi_0)u_n(\phi) \exp[\alpha f(\phi_0)] \exp[-D\lambda_n(t - t_0)] \quad (16-63)$$

where the $u_n$'s are the orthonormal eigenfunctions of (16-49) with corresponding eigenvalues $\lambda_n$. From (16-42) we see that

$$u_n(\phi) = p(\phi) = \frac{\exp[-\alpha f(\phi)]}{\int_{-\pi}^{\pi} \exp[-\alpha f(x)] \, dx} \quad (16-64)$$

when $\beta = 0$. Thus for each $g(\phi) \in C^1[-\pi, \pi]$ there remains the nontrivial problem of determining the eigenfunctions and eigenvalues of (16-49) and of verifying that the expansion of the transition p.d.f. can be differentiated term by term twice with respect to $\phi$ and once with respect to $t$. In general, the eigenfunctions and eigenvalues cannot be found analytically, and one must proceed with the problem of determining the spectrum by means of approximation methods. However, since (16-59) to (16-61) is a linear system with periodic coefficients, some information regarding the nature of the spectrum and the corresponding eigenfunctions can be obtained.

We first note that (16-59) to (16-61) is an eigenvalue problem of Hill's differential equation (Ref. 3), for (16-59) belongs to the class of homogeneous, linear, second-order differential equations with real periodic coefficients and which lacks a middle term. Hill (Ref. 4) studied an equation of this type in his investigation of the motion of the lunar perigee. It should be pointed out, however, that Hill's work was not concerned with the eigenvalue problem to be dealt with here, nor with any other eigenvalue problem. Although the eigenvalue problem (16-59) to (16-61) is defined only on the interval $0 \leq x \leq \pi$, a specific question that arises is the existence of periodic solutions. In particular, it should be pointed out that a linear differential equation need not have a periodic solution, even though the coefficients of the differential equation are
periodic. The solution can be stable, unstable, or periodic. In addition, if a second-order differential equation has a nontrivial periodic solution, then it need not have two linearly independent periodic solutions—that is, all solutions of the equation need not be periodic. An example of the latter case is Mathieu’s equation (Ref. 6)

$$\frac{d^2y}{dx^2} + (a - 2q \cos 2x)y = 0$$

which has a nontrivial periodic solution for certain values of \(a\) and \(q\). As shown by McLachlan (Ref. 6), Mathieu’s equation does not have two linearly independent periodic solutions except when \(q = 0\). The theory of periodic differential equations, including Hill’s equation, is a highly developed field of mathematics, and it is not the purpose of this chapter to contribute to this field. However, in order to construct approximate and numerical solutions of (16-49), we shall now present some specific results that will give a more detailed understanding and classification of the solutions.

We begin with a few general remarks on the differential equation (16-59). If we let \(y = z_1, y' = z_2, z = (z_1, z_2)\), and consider for the moment that the eigenvalue \(\eta\) is a constant parameter, then (16-59) can be rewritten as the system

$$z' = A(x)z$$  \hspace{1cm} (16-65)

where

$$A(x) = \begin{bmatrix} 0 & 1 \\ -\eta - Q(x) & 0 \end{bmatrix}$$  \hspace{1cm} (16-66)

is a continuous matrix of period \(\pi\). Let \(y_1(x), y_2(x)\) be a basis of (16-59)—that is, a system of two linearly independent continuously differentiable solutions so that any solution of (16-59) can be written in the form \(c_1y_1(x) + c_2y_2(x)\), where \(c_1\) and \(c_2\) are suitable real constants. In particular, we choose the so-called normalized solutions \(y_1(x), y_2(x)\) of (16-59), which are uniquely determined by the initial conditions

\[
\begin{align*}
y_1(0) &= 1 & y'_1(0) &= 0 \\
y_2(0) &= 0 & y'_2(0) &= 1 \\
\end{align*}
\]  \hspace{1cm} (16-67)

Setting

$$\Phi(x) = \begin{bmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{bmatrix}$$  \hspace{1cm} (16-68)
we see that $\Phi(x)$ is a fundamental matrix solution of (16-65)—that is, a nonsingular matrix, each of whose columns is a solution of (16-65) and such that $\Phi(0) = I$, where $I$ is the identity matrix. Thus

$$\Phi'(x) = A(x)\Phi(x) \tag{16-69}$$

and since $A(x)$ is periodic with period $\pi$, it follows that

$$\Phi'(x + \pi) = A(x + \pi)\Phi(x) = A(x)\Phi(x + \pi) \tag{16-70}$$

Hence $\Phi(x + \pi)$ is also a fundamental matrix solution of (16-69) with initial value $\Phi(\pi)$. Thus

$$\Phi(x + \pi) = \Phi(x)\Phi(\pi) \tag{16-71}$$

since the expression on the right satisfies (16-69) and has the initial value $\Phi(\pi)$.

Associated with (16-59) is the so-called characteristic equation (Refs. 1 and 2), which is determined by finding the eigenvalues $\mu$ of $\phi(\pi)$.

$$|\Phi(\pi) - \mu I| = \mu^2 - [y_1(\pi) + y_2(\pi)]\mu + [y_1(\pi)y_2'(\pi) - y_2(\pi)y_1'(\pi)] = 0$$

$$= \mu^2 - [y_1(\pi) + y_2(\pi)]\mu + 1 = 0 \tag{16-72}$$

The latter equation follows from the fact that the coefficient of the first derivative is zero in (16-59). This implies that the Wronskian determinant of the basis is equal to unity for all $x$; that is,

$$y_1(\pi)y_2'(\pi) - y_2(\pi)y_1'(\pi) = |\Phi(\pi)| = |\Phi(0)| = 1 \tag{16-73}$$

Until now the eigenvalue $\eta$ has been held fixed and, in this case, the normalized solutions will depend on $\eta$. Defining

$$\Delta(\eta) = y_1(\pi, \eta) + y_2'(\pi, \eta) \tag{16-74}$$

as the discriminant of (16-59), the characteristic equation (16-66) is rewritten as

$$\mu^2 - \Delta(\eta)\mu + 1 = 0 \tag{16-75}$$

From Floquet's theorem (Ref. 1), it is known that the roots $\mu_1, \mu_2$ of (16-75) satisfy $\mu_1\mu_2 = 1$, and if $\mu_1 = \mu_2 = 1$—that is, if (16-75) has a double root at $\mu = 1$—then (16-65), and hence (16-59), has a nontrivial periodic solution $\tilde{y}_1(x)$ of period $\pi$. The second solution $\tilde{y}_2(x)$ is either periodic with period $\pi$ or
\[
\ddot{y}_2(x) = x\dot{y}_1(x) \tag{16-76}
\]
Thus with \(\mu_1 = \mu_2 = 1\), it follows from (16-75) that the solutions \(\eta\) of
\[
\Delta(\eta) = y_1(\pi, \eta) + y_2(\pi, \eta) = 2 \tag{16-77}
\]
can be characterized as yielding the eigenvalues \(\eta\) of (16-59) subject to the periodic boundary conditions (16-60).

To see that these boundary conditions yield periodic eigenfunctions and, conversely, that the solutions of (16-77) yield eigenfunctions satisfying (16-59), we note that the solution of (16-65) can be written as
\[
z(x) = \Phi(x)z(0) \tag{16-78}
\]
where \(z(0)\) is the initial vector. Evidently at \(x = \pi\) we have
\[
z(\pi) = \Phi(\pi)z(0) \tag{16-79}
\]
and if \(\Delta(\eta) = 2\), then (16-75) must have the double root \(\mu = 1\); that is, \(\Phi(\pi)\) has \(\mu = 1\) as an eigenvalue. In this case, the equation
\[
\Phi(\pi)z(0) = \mu z(0) = z(0) \tag{16-80}
\]
has a nonzero solution, and it follows from (16-79) that
\[
z(0) = z(\pi) \tag{16-81}
\]
Thus \(\Delta(\eta) = 2\) implies that the solutions of (16-59) satisfy the boundary conditions (16-60). Conversely, since (16-81) holds from (16-60), we see that (16-80) has a nonzero solution and that the matrix \(\Phi(\pi)\) must have the eigenvalue \(\mu = 1\) [i.e., \(\Delta(\eta) = 2\)], and hence (16-59) must have at least one periodic eigenfunction of period \(\pi\).

In summary, from Floquet's theorem, (16-60) implies that for any continuously differentiable periodic nonlinearity \(g(\phi)\), (16-59) must have at least one periodic eigenfunction of period \(\pi\); and (16-72) is a necessary and sufficient condition in order for (16-59) to have a periodic solution of period \(\pi\). However, in order to find those eigenvalues \(\eta\) satisfying (16-72), it is necessary to find a fundamental matrix for the system (16-62) that has variable, nonlinear coefficients. This problem is a formidable task, for the periodic, nonlinear matrix \(A(x)\) is neither diagonal nor commutative. Thus unless the series for the matricant of \(A(x)\) converges rapidly, the computation of a fundamental matrix \(\Phi(x)\) is quite lengthy, and use of a digital computer is generally the best method to obtain \(\Phi(x)\). This approach will not be used in the sequel, for it does not yield approximate analytic solutions of \(\eta\); and there is more to be
gained by further study of the properties of (16-59) before attempting to construct actual solutions.

There is one important respect in which the eigenfunctions for periodic boundary conditions may differ from the eigenfunctions arising from the separable boundary conditions \( a_0 y(0) - a_1 y'(0) = 0, b_0 y(\pi) - b_1 y'(\pi) = 0 \). In the latter case the eigenvalues of a regular, second-order, self-adjoint operator must be of multiplicity one; that is, to each eigenvalue there is one and only one eigenfunction. For if there were two linearly independent eigenfunctions associated with each eigenvalue, then the complete solution of the differential equation would be a linear combination of these eigenfunctions. Thus any solution would satisfy \( a_0 y - a_1 y' = 0 \) at \( x = 0 \), and this contradicts the fundamental result that at a regular point (such as \( x = 0 \)) we can prescribe \( y \) and \( y' \) freely. However, for periodic boundary conditions, there is no such barrier to multiple eigenfunctions; that is, the eigenvalues may be either simple or degenerate with multiplicity of order two. The proof of this statement and a description of the eigenfunctions and eigenvalues of (16-59) to (16-61) are given by the following oscillation theorem of Haupt (Ref. 1).

**Theorem.** The eigenvalues \( \eta_i, i \geq 0 \), for any regular self-adjoint, second-order differential operator subject to periodic boundary conditions form a sequence such that
\[
-\infty < \eta_0 < \eta_1 < \eta_2 < \eta_3 < \eta_4 < \cdots
\]

For \( \eta = \eta_0 \) there exists a unique eigenfunction \( y_0 \). If \( \eta_{2i+1} < \eta_{2i+2} \) for some \( i \geq 0 \), then there is a unique eigenfunction \( y_{2i+1} \) at \( \eta = \eta_{2i+1} \) and a unique eigenfunction \( y_{2i+2} \) at \( \eta = \eta_{2i+2} \). If, however, \( \eta_{2i+1} = \eta_{2i+2} \), then there are two linearly independent eigenfunctions \( y_{2i+1}, y_{2i+2} \) at \( \eta = \eta_{2i+1} = \eta_{2i+2} \). Furthermore, \( y_0 \) has no zeros in \( 0 \leq x \leq \pi \); \( y_{2i+1} \) and \( y_{2i+2} \), \( i \geq 0 \), each have exactly \( 2i + 2 \) zeros in \( 0 \leq x \leq \pi \).

Thus, from Floquet's and Haupt's theorem, we see that the coexistence of two linearly independent eigenfunctions of period \( \pi \) is equivalent to the occurrence of a double eigenvalue or root of (16-77).

Since the periodic function \( Q(x) \) of (16-59) is even when the loop cross-correlation function \( g \) is odd, it is possible to establish some simple but useful relations between the values of the eigenfunctions of (16-59) to (16-61) when two linearly independent eigenfunctions coexist. In particular, the eigenvalue problem (16-59) to (16-61) can be characterized as an eigenvalue problem with the separated endpoint conditions
\[
y_0(0) = y_0\left(\frac{\pi}{2}\right) = y_0(\pi) = 0
\]
\[
y'(0) = y'_e\left(\frac{\pi}{2}\right) = y'_e(\pi) = 0
\]  

(16-82)
where \(y_e\) and \(y_o\) are odd and even eigenfunctions of period \(\pi\) respectively. The proof of (16-82) is easily established by first observing that if \(y(x)\) is a solution of (16-59), then \(y(-x)\) is also a solution, for the equation is unchanged when \(-x\) is written for \(x\). Of course, \(y(-x)\) will normally not be equal to \(y(x)\). Secondly, since the normalized solutions \(y_1(x), y_2(x)\) form a basis for (16-59), it follows that \(y_1(-x), y_2(-x)\) will also be solutions that are expressible as a linear combination of \(y_1(x), y_2(x)\); that is,

\[
y_1(-x) = c_1y_1(x) + c_2y_2(x) \\
y_2(-x) = c_3y_1(x) + c_4y_2(x)
\]  

(16-83)

where \(c_1, c_2, c_3,\) and \(c_4\) are real constants. Setting \(x = 0\) in (16-83) yields \(c_1 = 1, c_3 = 0\). Differentiating (16-83) with respect to \(x\) and then setting \(x = 0\) yields \(c_2 = 0, c_4 = -1\), so that \(y_1(-x) = y_1(x)\) and \(y_2(-x) = -y_2(x)\); that is, the normalized solutions \(y_1(x), y_2(x)\) are even and add functions respectively. Hence if (16-59) to (16-61) has two linearly independent eigenfunctions of period \(\pi\), corresponding to the same eigenvalue, then one eigenfunction must be even

\[
y_e(x) = \sum_{n=0}^{\infty} A_n \cos 2nx
\]  

(16-84)

and the other eigenfunction must be odd

\[
y_o(x) = \sum_{n=1}^{\infty} B_n \sin 2nx
\]  

(16-85)

for \(y_e(x)\) and \(y_o(x)\) are necessarily multiplies of the normalized solutions \(y_1(x)\) and \(y_2(x)\) respectively; and (16-82) follows.

### 16-4 Eigenfunction Expansions for First-Order Loops in the Absence of Detuning, \(\beta = 0\)

Whenever two linearly independent eigenfunctions of (16-59) to (16-61) coexist, the conditional transition p.d.f. of a first-order SCS may formally be written as

\[
p(\phi; t) = p(\phi) + \sum_{n=1}^{\infty} \left[ u_{nc}(\phi_0)u_{ne}(\phi) + u_{nc}(\phi_0)u_{no}(\phi) \right] \\
\times \exp \left[ \alpha f(\phi_0) \right] \exp \left[ -D\lambda_n(t - t_0) \right]
\]  

(16-86)
where \( u_{ne}(\phi), u_{no}(\phi) \) are, respectively, even and odd, orthonormal, periodic eigenfunctions of period \( 2\pi \) satisfying (16-49), (16-62), and the separated endpoint conditions

\[
\begin{align*}
\dot{u}_{ne}(-\pi) &= \dot{u}_{ne}(0) = u_{ne}(\pi) = 0 \\
\dot{u}_{no}(-\pi) &= \dot{u}_{no}(0) = u_{no}(\pi) = 0
\end{align*}
\]  

(16-87)

The expansion (16-86) follows directly from (16-22) to (16-27), with \( v_n(\phi) \) replaced by \( u_n(\phi) \), and from the fact that whenever two linearly independent eigenfunctions coexist or correspond to the same eigenvalue, their linear combination

\[
u_n(\phi) = a_n u_{ne}(\phi) + b_n u_{no}(\phi),
\]

(16-88)

where \( a_n \) and \( b_n \) are arbitrary constants, is also an eigenfunction.

We now obtain formal expansions of the steady-state joint p.d.f. corresponding to (16-63) and (16-86). Since the \( \phi \) modulo-\( 2\pi \) process is stationary in the steady state, the joint p.d.f.

\[
p_\phi(\phi, \phi_0, \tau) = p(\phi; t)p(\phi_0), \quad \tau = t - t_0 > 0
\]

(16-89)

corresponding to (16-63) and (16-86), respectively, is

\[
p_\phi(\phi, \phi_0, \tau) = p(\phi)p(\phi_0) + h^{-1}(\alpha) \sum_{n=1}^{\infty} u_n(\phi_0)u_n(\phi) \exp \left[-D\lambda_n^2 \tau\right]
\]

(16-90)

\[
p_\phi(\phi, \phi_0, \tau) = p(\phi)p(\phi_0) + h^{-1}(\alpha) \sum_{n=1}^{\infty} [u_n(\phi_0)u_{ne}(\phi)

+ u_{no}(\phi_0)u_{no}(\phi)] \exp \left[-D\lambda_n^2 \tau\right]
\]

(16-91)

where

\[
h(\alpha) = \int_{-\pi}^{\pi} \exp \left[-\alpha f(x)\right] dx
\]

(16-92)

From the steady-state joint p.d.f., the steady-state autocorrelation function

\[
R_\phi(\tau) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(\phi)F(\phi_0)p_\phi(\phi, \phi_0, \tau) d\phi d\phi_0
\]

(16-93)

of the stochastic process \( \psi(t) = F[\phi(t)] \), where \( F \) is any memoryless, time-invariant transformation, is easily found to be
\[ R_p(\tau) = \sigma_0^2 + h^{-1}(\alpha) \sum_{n=1}^{\infty} \sigma_n^2 \exp \left[ -D\lambda_n^\prime |\tau| \right] \] (16-94)

\[ R_p(\tau) = \sigma_0^2 + h^{-1}(\alpha) \sum_{n=1}^{\infty} (\sigma_{ne}^2 + \sigma_{no}^2) \exp \left[ -D\lambda_n^\prime |\tau| \right] \] (16-95)

where

\[ \sigma_n = \int_{-\pi}^{\pi} F(x)u_n(x) \, dx \quad n \geq 0 \] (16-96)

\[ \sigma_{ne} = \int_{-\pi}^{\pi} F(x)u_{ne}(x) \, dx \quad n > 1 \] (16-97)

and

\[ \sigma_{no} = \int_{-\pi}^{\pi} F(x)u_{no}(x) \, dx \quad n > 1 \] (16-98)

The spectral density can be evaluated using (16-94) and (16-95) to give

\[ S_p(\omega) = \sum_{n=1}^{\infty} \frac{2D\lambda_n^\prime\sigma_n^2 h^{-1}(\alpha)}{\omega^2 + (\lambda_n^\prime D)^2} \] (16-99)

\[ S_p(\omega) = \sum_{n=1}^{\infty} \frac{2D\lambda_n^\prime(\sigma_{ne}^2 + \sigma_{no}^2) h^{-1}(\alpha)}{\omega^2 + (\lambda_n^\prime D)^2} \] (16-100)

16-4.1 The Special Case Where \( \alpha = \beta = 0 \)

As an example of the application of the preceding method, we shall first consider the simplest case when the \( \phi \) modulo-2\( \pi \) process is simply a diffusion; that is, the input to the input to the loop consists of noise only. In this case, which corresponds to \( \alpha = \beta = 0 \), the eigenvalue problem (16-49) has the eigenvalues \( \lambda_n^\prime = n^2 \), where \( n \) is a non-negative integer; and corresponding to the eigenvalues \( \lambda_n^\prime = 0 \), and \( \lambda_n^\prime = n^2, n > 0 \), we obtain the complete orthonormal set of eigenfunctions

\[ u_n(\phi) = (2\pi)^{-\frac{1}{4}} (p(\phi))^i \]

\[ u_{ne}(\phi) = (\pi)^{-\frac{1}{2}} \cos n\phi \] (16-101)

\[ u_{no}(\phi) = (\pi)^{-\frac{1}{2}} \sin n\phi \]

Substituting (16-101) into (16-86) yields the conditional transition p.d.f.

\[ p(\phi; t) = \frac{1}{2\pi} + \frac{1}{\pi} \cos n(\phi - \phi_0) \exp \left[ -Dt^2(t - t_0) \right] \] (16-102)
and from (15-91) the steady-state joint p.d.f. is

\[ p_2(\phi, \phi_0, \tau) = \frac{1}{4\pi^2} + \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \cos n(\phi - \phi_0) \exp \left(-Dn^2\tau\right) \]  

(16-103)

In the limit as \( t \to t_0 \), we note that (16-102) converges to the Dirac distribution (Ref. 7); that is,

\[ \lim_{t \to t_0} p(\phi; t) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \cos n(\phi - \phi_0) = \delta(\phi - \phi_0) \]  

(16-104)

and (16-103) becomes

\[ \lim_{\tau \to 0} p_2(\phi, \phi_0, \tau) = \frac{1}{2\pi} \delta(\phi - \phi_0) \]  

(16-105)

which agrees with the theory of Wang and Uhlenbeck (Ref. 8). Furthermore, as \( t \to \infty \), (16-102) converges to the steady-state p.d.f. \( p(\phi) = (2\pi)^{-1} \), which is simply the mean of \( \delta(\phi - \phi_0) \) over the interval \( I(n) \). In other words, since probability is conserved over the interval for all \( t \), the conditional transition p.d.f. diffuses about the initial point \( \phi = \phi_0 \), at \( t = t_0 \), to its initial mean value.

From (16-95) and (16-103), with \( \psi = \phi, \psi = \sin \phi, \) and \( \psi = \cos \phi \), respectively, we easily obtain the steady-state autocorrelation functions

\[ R_\phi(\tau) = \sum_{n=1}^{\infty} \frac{2}{n^2} \exp \left[-Dn^2|\tau|\right] \]  

(16-106)

\[ R_{\sin \phi}(\tau) = R_{\cos \phi}(\tau) = \frac{\exp \left[-D|\tau|\right]}{2} \]  

(16-107)

From (16-106) and (16-107), the corresponding spectral densities are given by

\[ S_{\phi}(\omega) = \sum_{n=1}^{\infty} \frac{4D}{\omega^2 + n^4D^2} \]  

(16-108)

\[ S_{\sin \phi}(\omega) = S_{\cos \phi}(\omega) = \frac{D}{\omega^2 + D^2} \]  

(16-109)

Computing the steady-state variance of \( \phi \) from (16-106), we obtain

\[ \sigma_\phi^2 = R_\phi(0) - R_\phi(\infty) = \sum_{n=1}^{\infty} \frac{2}{n^2} = \frac{\pi^2}{3} \]  

(16-110)
which agrees with the fact that, in the steady state, the $\phi$ modulo-$2\pi$ process is uniformly distributed, over $[-\pi, \pi]$, with zero mean. This method of comparing the steady-state variance, obtained from the time-dependent solution of the FP equation, with its exact value, determined from the known steady-state p.d.f. (16-42), serves as an excellent check on the accuracy of the solutions of the eigenvalue problem (16-49) when $\alpha$ and $\beta$ are not equal to zero.

It is interesting to note that this simple example yields a complete description of the changes that occur in the steady-state spectrum of the open-loop nonlinear system shown in Fig. 16-1, where $F(\phi)$ is any time-invariant, memoryless nonlinearity. When $F(\phi) = \sin \phi$ or $F(\phi) = \cos \phi$, we observe from (16-107) that both nonlinearities have identical output autocorrelation functions that are exponential functions; whereas the input autocorrelation function (16-106), of the uniform Markov $\phi$ modulo-$2\pi$ process, is an infinite sum of exponentials. This surprising result (16-107) agrees with the results of Barrett and Lampard (Ref. 9). Namely, they have shown that a sine wave process with a constant amplitude and a uniformly distributed phase has a joint p.d.f. belonging to their class $\Lambda$ of stationary random processes, for which the joint p.d.f. is expressible as

$$p_{\phi}(\phi, \phi_0, \tau) = p(\phi)p(\phi_0) \sum_{n=0}^{\infty} a_n(\tau)F_n(\phi)F_n(\phi_0)$$  \hspace{1cm} (16-111)$$

where $[F_n(\phi)]$ is a set of polynomials that are orthonormal with respect to the first-order p.d.f. $p(\phi)$ and where the coefficients $a_n$ are given by

$$a_n(\tau) = \int \int p(\phi, \phi_0, \tau)F_n(\phi)F_n(\phi_0) \, d\phi \, d\phi_0$$  \hspace{1cm} (16-112)$$

Furthermore, it is shown that the autocorrelation function of any first-order
Markov process in $\Lambda$ must be an exponential function, which agrees with (16-107).

Comparing the foregoing properties of $\Lambda$ to that of the $\varphi$ modulo-$2\pi$ process—that is, (16-62), (16-64), (16-90), (16-91), (16-99), and (16-100)—disproves the conjecture of Tausworthe (Ref. 10) that $\varphi$ modulo-$2\pi$ process belongs to $\Lambda$. With the assumption that $\{\varphi(t)\}$ is in $\Lambda$, Tausworthe obtained, for the first-order phase-locked loop with $\beta = 0$, the linear spectral approximation

$$S_p(\omega) = \frac{K^2 N_0}{2[\omega^2 + (AK\gamma_{i,p})^2]}$$  \hspace{1cm} (16-113)

where

$$\gamma_{i,p}^2 = \frac{R_{\sin q}(0)}{R_p(0)}$$  \hspace{1cm} (16-114)

In the limit as $\alpha$ approaches zero, (16-113) reduces to

$$S_p(\omega) = \frac{K^2 N_0}{2\omega^2} = \frac{2D}{\omega^2}$$  \hspace{1cm} (16-115)

which does not agree with (16-108). Thus the spectral method of Tausworthe fails to take into account the effect of the modulo-$2\pi$ nonlinearity shown in Fig. 16-1. (See Chapter 9 for a brief discussion of the spectral method).

16-5 Spectral Analysis of First-Order Sinusoidal PLLs in the Absence of Detuning, $\beta = 0$.

We now consider the self-adjoint eigenvalue problem corresponding to the first-order PLL in the absence of detuning. If we substitute $g(\phi) = \sin \phi$ into (16-49) and (16-50), we obtain

$$u''(\phi) + \alpha \sin \phi u'(\phi) + (\lambda' + \alpha \cos \phi)u(\phi) = 0$$  \hspace{1cm} (16-116)

$$y''(\phi) + \left(\lambda' - \frac{\alpha^2}{8} + \frac{\alpha}{2} \cos \phi + \frac{\alpha^2}{8} \cos 2\phi\right)y(\phi) = 0$$  \hspace{1cm} (16-117)

where $u$ and $y$ are related by the transformation (16-51) with $\beta = 0$. When we introduce the change of variables $\phi = 2x - \pi$, these equations become

$$u''(x) - 2\alpha \sin 2x u'(x) + (\eta - 4\alpha \cos 2x)u(x) = 0$$  \hspace{1cm} (16-118)

$$y''(x) + \left(\eta - \frac{\alpha^2}{2} - 2\alpha \cos 2x + \frac{\alpha^2}{2} \cos 4x\right)y(x) = 0$$  \hspace{1cm} (16-119)

where $\eta = 4\lambda'$. 
Equation (16-118) is recognized as a special case of the three-parameter equation

$$\frac{d^2\psi}{dz^2} + \left[ \sin 2z \frac{d\psi}{dz} + (\eta - p\xi \cos 2z)\psi \right] = 0 \quad (16-120)$$

that is called Ince’s equation (Ref. 11), where $\xi$, $\eta$, and $p$ are real constants. Its formally self-adjoint form

$$\frac{d^2w}{dz^2} + \left[ \frac{\xi^2}{8} - (p + 1)\xi \cos 2z + \frac{\xi^2}{8} \cos 4z \right]w = 0 \quad (16-121)$$

for which (16-119) is a special case, is called the Whittaker-Hill equation, for its algebraic form is Whittaker’s confluent hypergeometric equation (Ref. 12). Ince has shown that when $p$ is a nonnegative integer, (16-120) possesses, for certain discrete values of $\eta$ that are dependent on $\xi$, even or odd periodic solutions of period $\pi$ of the form

$$\psi(z) = \sum_{n=0}^{\infty} A_n \cos 2nz$$

$$\quad \psi(z) = \sum_{n=1}^{\infty} B_n \sin 2nz$$

(16-122)

which are called Ince polynomials, for they are finite trigonometric series. For other values of $p$ (integral or not) the equation possesses, for special values of $\eta$, even or odd periodic solutions of period $\pi$; however, excluding the case where $p$ is a nonnegative integer, these solutions are nonterminating Ince polynomials. From (16-118) we see that $p = -2$, so that the periodic eigenfunctions of (16-116) or (16-118) are nonterminating even or odd trigonometric series of period $2\pi$ and $\pi$ respectively.

Thus far very little is known about the nonterminating periodic solutions of Ince’s equation when $p$ is a negative integer, because if the parameter $\xi$ is given, the equation represents a two-parameter eigenvalue problem for the parameters $\eta$ and $p$. The theory of such problems is still in a very undeveloped state. However, (16-119) represents a one-parameter eigenvalue problem, similar to Mathieu’s equation, since $p$ is now fixed and the loop signal-to-noise ratio $\alpha$ shall be considered as a known parameter.

There is one important respect in which Ince’s equation differs from Mathieu’s equation. In the latter case, we recall that two linearly independent eigenfunctions never coexist for the same eigenvalue. However, for Ince’s equation, it is possible for coexistence to occur. We now show that coexistence occurs for the first-order phase-locked loop in the absence of detuning. The coexistence problem for Ince’s equation, written in the form
(1 + a \cos 2x)y'' + b(\sin 2x)y' + (c + d \cos 2x)y = 0 \quad (16-123)

where \( a, b, c, \) and \( d \) are real parameters and \( |a| < 1 \), is discussed in Magnus and Winkler (Ref. 4). Equation (16-123) corresponds to (16-118) when \( a = 0, b = -2\alpha, c = \eta, \) and \( d = -4\alpha \). Their first important result, which is a necessary condition in order for coexistence to occur, is given by the following theorem.

**Theorem.** If Ince's equation (16-123) has two linearly independent solutions of period \( \pi \), then the polynomial

\[
Q(\mu) = 2a\mu^2 - b\mu - \frac{d}{2}
\]

has a zero at one of the points

\[
\mu = 0, \pm 1, \pm 2, \ldots
\]

For the Ince type (16-118), it is obvious that \( Q(\mu) \) is a linear function with a zero at the point \( \mu = -1 \); furthermore, the necessary condition for coexistence is satisfied independent of the value of \( \alpha \).

Corresponding to the case when \( \mu \) is a negative integer, a necessary and sufficient condition for coexistence to occur is given by the following theorem.

**Theorem.** Assume that \( Q(\mu) \) has a negative integral zero. Let \(-k - 1 = \mu, \) where \( k = 0, 1, 2, \ldots \). If Ince's equation (16-123) has one infinite solution of period \( \pi \), it will have two linearly independent solutions of period \( \pi \) if and only if the coefficient of \( \cos 2kx \) (for an even solution) or of \( \sin 2kx \) (for an odd solution) in the expansions

\[
y(x) = \sum_{k=0}^{\infty} A_k \cos 2kx \quad y(x) = \sum_{k=1}^{\infty} B_k \sin 2kx
\]

vanishes.

Note that the preceding theorem allows us to decide whether two linearly independent solutions of period \( \pi \) exist for (16-123) provided that we know that one such solution exists. Now since \( p = -2 \) for (16-118), the existence of an even or odd, nonterminating periodic solution of period \( \pi \) is obvious. Furthermore, since \( k = 0 \) for (16-118), if we assume that the eigenfunction is odd, then the coefficient \( B_0 \) vanishes trivially; and if the eigenfunction is assumed to be even, then from (16-40) it is clear that the coefficient \( A_0 \) must also vanish.

In summary, the eigenvalues of the first-order PLL in the absence of detuning are degenerate for all values of \( \alpha \), and to each eigenvalue there corresponds an even and odd eigenfunction. Thus it follows that the conditional
transition p.d.f. of the phase error process $\phi$ is given by (16-86), and the auto-
correlation and spectral density of the process $\psi = F(\phi)$ are given by equations
(16-95) and (16-100), where $f(\phi) = -\cos \phi$ and $h(\alpha) = 2\pi I_0(\alpha)$.

16-6 Perturbation Solutions

Since we are primarily concerned with the nonlinear behavior of the
loop—that is, the region of small signal-to-noise ratios $0 \leq \alpha \leq 10$, where
non-Gaussian statistics exist—we now obtain approximate analytical expressions
for the eigenvalues of (16-117), which are valid for small $\alpha$. To obtain these
approximate solutions, solutions of (16-117) are constructed by a perturbation
method, similar to that originally used by Mathieu (Ref. 12) for Mathieu’s
equation. No attempt is made to prove the convergence of the asymptotic
series obtained, for we have previously established that eigenvalues and eigen-
functions of (16-117) exist for all values of $\alpha$; and from general arguments it
can be shown that these quantities are continuously differentiable with respect
to $\alpha$—that is, they are analytic functions of $\alpha$ and hence possess convergent
power series expansions.

When $\alpha = 0$, (16-117) reduces to the unperturbed equation

$$y''(\phi) + \lambda' y(\phi) = 0, \quad -\pi \leq \phi \leq \pi$$  \hspace{1cm} (16-124)

with periodic eigenfunctions $\cos n\phi$, $\sin n\phi$ of period $2\pi$, where the eigenvalues
$\lambda'_n = n^2$, $n$ being a non-negative integer. We now obtain perturbation solutions
of (16-117) that reduce to $\cos n\phi$, $\sin n\phi$ and $n^2$ as $\alpha \to 0$. For example, for an
odd solution we assume

$$y_n(\phi) = \sin n\phi + \alpha g_1(\phi) + \alpha^2 g_2(\phi) + \alpha^3 g_3(\phi) + \cdots$$ \hspace{1cm} (16-125)

$$\lambda'_n(\alpha) = n^2 + c_1 \alpha + c_2 \alpha^2 + c_3 \alpha^3 + \cdots$$ \hspace{1cm} (16-126)

Substituting the foregoing series into (16-117), collecting terms with
like powers of $\alpha$, and equating each group to zero, we obtain a sequence of
second-order linear differential equations with constant coefficients for the functions $g_i(\phi), i = 1, 2, 3$, etc. From the general solutions of these equations, subject to the restrictions that $y_n(\phi)$ be odd, periodic with period $2\pi$, and normalized such that the coefficient of $\sin n\phi$ in the trigonometric series for $y_n(\phi)$
shall be unity for all values of $\alpha$, the coefficients $c_i$ and the functions $g_i$ are
found successively. The effect of the normalization condition is such that the
functions $g_i, i = 1, 2, 3, \ldots$, will contain no terms in $\sin n\phi$, for any coefficient
of $\sin n\phi$ is set equal to zero.
To illustrate the method, we show the first few steps in the solution for \( y_1(\phi) \) and \( L_1(\alpha) \). Letting \( n = 1 \) in (16-125) and (16-126) and then substituting these equations into (16-117), we obtain

\[
-\sin \phi + \alpha g_1'' + \alpha^2 g_2'' + \cdots \\
+ \left[ 1 + c_1 \alpha + c_2 \alpha^2 + \cdots \right] \left[ \sin \phi + \alpha g_1 + \alpha^2 g_2 + \cdots \right] \\
+ \left[ \frac{\alpha^2}{8} \left( \cos 2\phi - 1 \right) + \frac{\alpha}{2} \cos \phi \right] \left[ \sin \phi + \alpha g_1 + \alpha^2 g_2 + \cdots \right] = 0
\]

(16-127)

The term independent of \( \alpha \) in this equation vanishes identically; equating successive powers of \( \alpha \) to zero yields a sequence of differential equations, of which we display the first three:

\[
Lg_1 = -c_1 \sin \phi - \frac{1}{2} \sin \phi \cos \phi \\
Lg_2 = -\left( c_1 + \frac{\cos \phi}{2} \right) g_1 - c_2 \sin \phi - \frac{1}{8} \left( \cos 2\phi - 1 \right) \sin \phi \\
Lg_3 = -\left[ c_2 + \frac{1}{8} \left( \cos 2\phi - 1 \right) \right] g_1 - \left( c_1 + \frac{1}{2} \right) g_2 - c_3 \sin \phi
\]

(16-128)

where the differential operator \( L = d^2/d\phi^2 + 1 \) in each. Using the method of undetermined coefficients, the general solution of the first equation becomes

\[
g_1(\phi) = a_1 \sin \phi + a_2 \cos \phi + c_1 \frac{1}{2} \phi \cos \phi + \frac{1}{12} \sin 2\phi
\]

(16-129)

where \( a_1 \) and \( a_2 \) are arbitrary constants. Since \( g_1 \) must be odd, it follows that \( a_2 = 0 \); since \( g_1 \) is to be periodic, the term in \( \phi \cos \phi \) must also vanish, hence \( c_1 = 0 \); and since \( g_1 \) must contain no term in \( \sin \phi \), \( a_1 = 0 \) also. So we obtain

\[
c_1 = 0 \quad g_1(\phi) = \frac{\sin 2\phi}{12}
\]

(16-130)

We now substitute \( c_1 \) and \( g_1 \) into the second differential equation and simplify, using the identity

\[
\sin a \pm \sin b = 2 \sin \frac{1}{2}(a \pm b) \cos \frac{1}{2}(a \mp b)
\]

(16-131)

to obtain
\[ L g_2 = \left( \frac{1}{6} - c_2 \right) \sin \phi - \frac{\sin 3\phi}{12} \]  

(16-132)

The general solution for \( g_2 \) is given by

\[ g_2(\phi) = a_3 \sin \phi + a_4 \cos \phi + \left( \frac{c_2}{2} - \frac{1}{12} \right) \phi \cos \phi + \frac{\sin 3\phi}{96} \]  

(16-133)

Applying the same restrictions as on \( g_1(\phi) \) to \( g_2(\phi) \) gives \( a_3 = a_4 = (c_2/2 - 1/12) = 0 \), so

\[ c_2 = \frac{1}{6} \quad g_2(\phi) = \frac{\sin 3\phi}{96} \]  

(16-134)

Consequently, under the normalization used here, we have

\[ \lambda_1'(\alpha) = 1 + \frac{\alpha^2}{6} + O(\alpha^3) \]  

(16-135)

\[ y_1(\phi) = \sin \phi + \frac{\alpha}{12} \sin 2\phi + \frac{\alpha^2}{96} \sin 3\phi + O(\alpha^3) \]  

(16-136)

Note that \( \lambda_1' \) is known to \( O(\alpha^3) \) before even actually solving the second differential equation, for the nonhomogeneous term \( \sin \phi \), which also appears in the homogeneous solution, always yields nonperiodic terms that must vanish. Additional terms are successively obtained by continuing the same process, although the equations become progressively more complicated. Asymptotic expansions have been obtained for the first three eigenvalues to \( O(\alpha^3) \) and for the corresponding even and odd eigenfunctions to \( O(\alpha^4) \).

The results obtained are summarized below.

\( n = 1 \)

\[ \lambda_1'(\alpha) = 1 + \frac{\alpha^2}{6} - \frac{\alpha^4}{2^3 \cdot 3^3} + O(\alpha^5) \]

\[ y_{1,1}(\phi) = \sin \phi + \frac{\alpha}{12} \sin 2\phi + \frac{\alpha^2}{96} \sin 3\phi \]

\[ + \alpha^3 \left[ \frac{7}{2^8 \cdot 3} \sin 2\phi + \frac{\sin 4\phi}{2^7 \cdot 15} \right] + O(\alpha^4) \]  

(16-137)

\[ y_{1,2}(\phi) = \cos \phi + \alpha \left[ \frac{\cos 2\phi}{12} - \frac{1}{4} \right] + \frac{\alpha^2}{96} \cos 3\phi \]

\[ + \alpha^3 \left[ \frac{29}{2^7 \cdot 3^3} \cos 2\phi + \frac{\cos 4\phi}{2^7 \cdot 15} + \frac{1}{192} \right] + O(\alpha^4) \]
\( n = 2 \)

\[
\lambda'_2(\alpha) = 4 + \frac{2\alpha^2}{15} + \frac{11\alpha^4}{8 \cdot 15^3} + O(\alpha^5)
\]

\[
y_{2,0}(\phi) = \sin 2\phi - \alpha \left[ \frac{\sin \phi}{20} + \frac{\sin 3\phi}{10} \right] + \frac{\alpha^2}{160} \sin 4\phi
\]

\[
- \alpha^3 \left[ \frac{11}{2^{7} \cdot 3^3 \cdot 5} \sin \phi + \frac{31 \sin 3\phi}{2^7 \cdot 3^3 \cdot 5^3} + \frac{\sin 5\phi}{2^7 \cdot 3^3 \cdot 5^3} \right] + O(\alpha^5)
\]

\[
y_{2,1}(\phi) = \cos 2\phi + \alpha \left[ \frac{\cos 3\phi}{20} - \frac{\cos \phi}{12} \right] + \frac{\alpha^2}{160} \left[ \frac{\cos 4\phi}{2^7 \cdot 3^3 \cdot 5} + \frac{1}{96} \right]
\]

\[
+ \alpha^3 \left[ \frac{23}{2^7 \cdot 3^3 \cdot 5} \cos \phi + \frac{31 \cos 3\phi}{2^7 \cdot 3^3 \cdot 5^3} + \frac{\cos 5\phi}{2^7 \cdot 3^3 \cdot 5^3} + O(\alpha^5) \right]
\]

\( n = 3 \)

\[
\lambda'_3 = 9 + \frac{9\alpha^2}{70} + \frac{4689\alpha^4}{512 \cdot 35^3} + O(\alpha^5)
\]

\[
y_{3,0}(\phi) = \sin 3\phi + \alpha \left[ \frac{\sin 4\phi}{28} - \frac{\sin 2\phi}{20} \right]
\]

\[
+ \alpha^2 \left[ \frac{\sin 5\phi}{224} - \frac{\sin \phi}{160} \right]
\]

\[
+ \alpha^3 \left[ \frac{\sin 6\phi}{2^7 \cdot 63} + \frac{59 \sin 4\phi}{2^7 \cdot 7^5} - \frac{11}{2^7 \cdot 5^3 \cdot 7} \sin 2\phi \right] + O(\alpha^5)
\]

\[
y_{3,1}(\phi) = \cos 3\phi + \alpha \left[ \frac{\cos 4\phi}{28} - \frac{\cos 2\phi}{20} \right]
\]

\[
+ \alpha^2 \left[ \frac{\cos \phi}{160} + \frac{\cos 5\phi}{224} \right]
\]

\[
+ \alpha^3 \left[ \frac{\cos 6\phi}{2^7 \cdot 63} - \frac{59 \cos 4\phi}{2^7 \cdot 7^5} - \frac{11 \cos 2\phi}{2^7 \cdot 5^3 \cdot 7} + \frac{1}{2^7 \cdot 15} \right] + O(\alpha^5)
\]

Examining the first two terms of \( \lambda'_n(\alpha) \) for \( n = 1, 2, 3 \), we observe that the eigenvalues can be expressed analytically as

\[
\lambda'_n(\alpha) = n^2 + \frac{n^2\alpha^2}{2(4n^2 - 1)} + O(\alpha^4)
\]

(16-140)

The same result is also found to be valid for \( n > 3 \). We also observe that the eigenvalues contain only even powers of \( \alpha \), similar to the property of the even characteristic values \( a_r(q), b_r(q), r = 2n \), of Mathieu’s equation. In fact, since it is recognized that (16-119) reduces to Mathieu’s equation for small \( \alpha \) (i.e., if we neglect terms in \( \alpha^2 \)), it is interesting to compare our results with those of
Mathieu’s equation. Since the eigenfunctions of (16-117) and (16-119) are of period $2\pi$ and $\pi$, respectively, the eigenvalues $\eta$ of (16-119), yielding periodic solutions of period $\pi$, can be determined by substituting $r = 2n$ into (16-140) and recalling that $\eta = 4\lambda'$. So we obtain

$$
\eta_r(x) = 4\lambda'_r(x) = r^2 + \frac{r^2\alpha^2}{2(r^2 - 1)} + O(x^4), \quad r = 2n = 2, 4, 6, \ldots
$$

(16-141)

Now, in Whittaker and Watson (Ref. 13), it is shown that for $r < 7$, $a_r(q) \neq b_r(q)$, but for $r \geq 7$ and $q$ not too large, $a_r(q)$ is approximately equal to $b_r(q)$ and is given by

$$
a_r(q) = b_r(q) = r^2 + \frac{q^2}{2(r^2 - 1)} + O(q^4)
$$

(16-142)

thus showing the effect of the terms in $\alpha^2$ of (16-119); that is, these terms cannot be neglected, even for small $\alpha \ll 1$, for they have a significant effect on the eigenvalue.

We now verify that the formal eigenfunction expansion for $p(\phi; t)$ satisfies the FP equation (16-1); that is, the series converges and can be differentiated term by term twice with respect to $\phi$ for all $\phi$ in $[-\pi, \pi]$ and once with respect to $t$ for all $t > t_0$. In fact, it turns out that the series can be differentiated term by term with respect to $\phi$ or $t$ any number of times. Because of the presence of the factors $\exp[-\lambda'_r(t - t_0)]$, because the eigenvalues $\lambda'_r$ are all positive and monotonically increasing in $n$, and because the eigenfunctions are bounded on the interval $-\pi \leq \phi \leq \pi$, the series (16-86) converges uniformly for all $\phi \in [-\pi, \pi]$ and for all $t > t_0$. This is a consequence of the Weierstrass $M$-test, for

$$
\|u_{nn}(\phi)u_{nn}(\phi_0) + u_{no}(\phi)u_{no}(\phi_0)\exp(\alpha \cos \phi_0)\| < M, \quad n = 1, 2, 3, \ldots
$$

$$
\|u_{nn}(\phi)u_{nn}(\phi_0) + u_{no}(\phi)u_{no}(\phi_0)\exp(\alpha \cos \phi_0)\exp[-D\lambda_n(t - t_0)]\|
$$

(16-143)

$$
< M \exp[-D\lambda_n(t - t_0)] \cong M_n
$$

and the series $M_n$ is a series of constants whose convergence follows from the ratio test. In the same way we can verify that the series (16-86) remains uniformly convergent in the range stated when differentiated as often as desired with respect to $\phi$ and $t$.

Finally, because the eigenvalues increase with respect to the square of $n$, it follows that the infinite series can be approximated using only a sufficient finite number of terms, thus making the expansion of $p(\phi; t)$ useful in engineering applications.
16-7 Iterative Techniques and the Calculation of Eigenvalues

Since it is impossible to construct a fundamental matrix for (16-116), any practical attempt to solve the eigenvalue problem has to be a numerical approach. In order to solve the eigenvalue problem, we have taken the approach of developing the simplest numerical techniques that are consistent in giving reasonably accurate results and that are computationally feasible on present-day computers. In particular, two iterative digital computer programs were developed, each of which makes use of the approximate analytical expressions for the eigenvalues. Because the computer programs are quite large, they will not be presented; however, we now outline the iterative methods and present the numerical results, (Ref. 14).

The first iterative method consists of initially setting the unknown iteration eigenvalue \( \lambda_n(\epsilon) \), where \( \epsilon \) is the iteration parameter, equal to its approximate value \( \lambda_n(\alpha) \). That is, initially we set

\[
\lambda_n(\epsilon) = n^2 + \epsilon
\]

(16-144)

where

\[
\epsilon \triangleq \frac{n^2 \alpha^2}{2(4n^2 - 1)}
\]

(16-145)

Substituting this value into (16-116), the differential equation is solved from \( \phi = -\pi \) to \( \phi = 0 \) and then from \( \phi = \pi \) to \( \phi = 0 \) for the even and odd solutions, subject to the initial conditions

\[
\begin{align*}
  u_{no}(\pm \pi) &= 0 \\
  u'_{no}(\pm \pi) &= (-1)^n(n + \epsilon) \\
  u_{ne}(\pm \pi) &= (-1)^n \left(1 + \frac{\epsilon}{n}\right) \\
  u'_{ne}(\pm \pi) &= 0
\end{align*}
\]

(16-146)

The initial conditions follow from the separated endpoint conditions (16-87) and from the fact that as \( \alpha \to 0, \epsilon \to 0, u_{no}(\phi) \to \sin n\phi, u_{ne}(\phi) \to \cos n\phi \) and \( \lambda_n \to n^2 \). By defining the iteration error as the difference \( u_n(0^+) - u_n(0^-) \), the iteration parameter \( \epsilon \) is successively increased and decreased until the error changes sign, thus locating the eigenvalue. To ensure reasonable accuracy, iterations were performed until the error was less than \( 10^{-5} \), and the eigenfunctions were obtained in increments of one degree in \( \phi \), using a modified Runge-Kutta algorithm with an integration upper-error bound of \( 10^{-6} \). Computation time for obtaining 30 eigenvalues and 60 normalized eigenfunctions was approximately 45 seconds on an IBM-360 computer. The relatively small amount of time required was principally due to the excellent initial estimate provided by the approximate eigenvalue \( \lambda_n(\alpha) \).
The second iterative computer program not only determines the eigenvalue but also the approximate Fourier coefficients of the even and odd eigenfunctions. This method makes use of the fact that a three-term recurrence relation exists among the Fourier coefficients of Ince’s equation, (Ref. 11); however, for (16-116) or (16-117) it is found that Fourier coefficients satisfy a five-term recurrence relation. That is, if we let

\[ y_e(\phi) = \sum_{n=0}^{\infty} A_n \cos n\phi \quad y_o(\phi) = \sum_{n=0}^{\infty} B_n \sin n\phi \]  

(16-147)

substitute these into (16-117), collect together like terms of \(\cos n\phi\) and \(\sin n\phi\), and equate each group to zero, we obtain, for the even periodic eigenfunctions

\[
\left( \lambda' - \frac{\alpha^2}{8} \right) A_0 + \frac{\alpha}{4} A_1 + \frac{\alpha^2}{16} A_2 = 0, \quad n = 0
\]

\[
\frac{\alpha}{2} A_0 + \left( \lambda' - 1 - \frac{\alpha^2}{16} \right) A_1 + \frac{\alpha}{4} A_2 + \frac{\alpha^2}{16} A_3 = 0, \quad n = 1
\]

\[
\frac{\alpha^2}{8} A_0 + \frac{\alpha}{4} A_1 + \left( \lambda' - \frac{\alpha^2}{8} - 4 \right) A_2 + \frac{\alpha}{4} A_3 + \frac{\alpha^2}{16} A_4 = 0, \quad n = 2
\]

\[
\left( \lambda' - \frac{\alpha^2}{8} - n^2 \right) A_n + \frac{\alpha}{4} (A_{n-1} + A_{n+1}) + \frac{\alpha^2}{16} (A_{n-2} + A_{n+2}) = 0, \quad n \geq 3
\]  

(16-148)

and for the odd periodic eigenfunctions

\[
\left( \lambda' - 1 - \frac{3}{16} \alpha^2 \right) B_1 + \frac{\alpha}{4} B_2 + \frac{\alpha^2}{16} B_3 = 0, \quad n = 1
\]

\[
\frac{\alpha}{4} B_1 + \left( \lambda' - \frac{\alpha^2}{8} - 4 \right) B_2 + \frac{\alpha}{4} B_3 + \frac{\alpha^2}{16} B_4 = 0, \quad n = 2
\]

\[
\frac{\alpha^2}{16} B_1 + \frac{\alpha}{4} B_2 + \left( \lambda' - \frac{\alpha^2}{8} - 9 \right) B_3 + \frac{\alpha}{4} B_4 + \frac{\alpha^2}{16} B_5 = 0, \quad n = 3
\]

\[
\left( \lambda' - \frac{\alpha^2}{8} - n^2 \right) B_n + \frac{\alpha}{4} (B_{n-1} + B_{n+1}) + \frac{\alpha^2}{16} (B_{n-2} + B_{n+2}) = 0, \quad n \geq 4
\]  

(16-149)

Note that (16-148) and (16-149) constitute an infinite set of algebraic equations; hence for an explicit computation of the Fourier coefficients \(A_n, B_n\) and the eigenvalue \(\lambda'\), we actually have to solve an infinite set of equations. How-
ever, to circumvent this problem of dimensionality, an approximation procedure was employed. Namely, the systems (16-148) and (16-149) were truncated at the fifth equation, thus yielding two systems of five equations in six unknowns—$A_0, \ldots, A_5$ and $B_1, \ldots, B_6$. The coefficients $A_5, B_6$ are set equal to zero and the coefficients $A_4$ and $B_5$ are set equal to unity, thereby giving a nonhomogeneous system of five equations in four unknowns. One equation in each system of equations is therefore redundant, and it is used as an iteration error equation to find the eigenvalue and the approximate Fourier coefficients. For example, with $\alpha = 4$, $B_6 = 0$, and $B_3 = 1$, we obtain for the odd eigenfunction the nonhomogeneous system

\[
(\lambda' - 4)B_1 + B_2 + B_3 = 0, \quad n = 1
\]
\[
B_1 + (\lambda' - 6)B_2 + B_3 + B_4 = 0, \quad n = 2
\]
\[
B_1 + B_2 + (\lambda' - 11)B_3 + B_4 = -1, \quad n = 3 \quad (16-150)
\]
\[
B_2 + B_3 + (\lambda' - 18)B_4 = -1, \quad n = 4
\]
\[
B_3 + B_4 = 27 - \lambda', \quad n = 5
\]

To find the eigenvalue $\lambda'$, we use the first equation as our iteration control—that is, we let

\[
[\lambda'(\epsilon) - 4]B_1 + B_2 + B_3 = E[\lambda'(\epsilon)] \quad (16-151)
\]

be the iteration error, set $\lambda'(\epsilon)$ equal to the approximate value $\lambda'(\alpha)$, solve the last four equations, compute the error $E[\lambda'(\epsilon)]$, check its sign, and repeat the same iteration process used in the previous method. The solution for the algebraic equations was obtained by means of Gauss-elimination with complete pivoting at all elimination steps with an error tolerance of $10^{-5}$ to ensure reasonable accuracy. Computation time for obtaining 20 eigenvalues was less than 2 minutes on an SDS-930 computer.

16-8 Numerical Results for the Case $\beta = 0$

The eigenvalues $\lambda'_n, n = 1, \ldots, 4$ are tabulated in Table 16-1 and the even and cdd periodic eigenfunctions $u_n(\phi), n = 1, \ldots, 4$ are given in Figs. 16-2 and 16-3 for $\alpha = 2$. To aid in recognizing the quality of the perturbation solutions $\lambda'_n(\alpha)$, these values are given in Table 16-2. In this table the first three eigenvalues are given by (16-137) through (16-139), respectively, and for $n \geq 4$ the values given are determined from (16-123). Comparing the values of Table 16-1 and Table 16-2, it is seen that the perturbation solutions yield excellent estimates of the computed values not only for small $\alpha$ but throughout
the nonlinear region of operation $0 \leq \alpha \leq 6$. In fact, the perturbed value $\lambda_i(\alpha = 6)$—that is, the value with the least fidelity—differs from its computed value with a percentage error of 2 per cent.

In computing the eigenvalues via the system of (16-148) and (16-149), these equations were first truncated at $n = 5$ and then at $n = 6$, where in each case the first equation was used as the iteration control. Increasing the dimensionality by one equation had the effect of changing only the fifth significant digit; so for all practical purposes five equations are sufficient in computing the eigenvalue. In addition, the values obtained, using this iterative
Table 16-2 Eigenvalues as Obtained from Perturbation Solutions

<table>
<thead>
<tr>
<th>α</th>
<th>n = 1</th>
<th>n = 2</th>
<th>n = 3</th>
<th>n = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.16551</td>
<td>4.13374</td>
<td>9.12878</td>
<td>16.12698</td>
</tr>
<tr>
<td>2</td>
<td>1.64816</td>
<td>4.53985</td>
<td>9.51770</td>
<td>16.50794</td>
</tr>
<tr>
<td>3</td>
<td>2.40610</td>
<td>5.23300</td>
<td>10.17444</td>
<td>17.14286</td>
</tr>
<tr>
<td>4</td>
<td>3.37060</td>
<td>6.23763</td>
<td>11.11182</td>
<td>18.03174</td>
</tr>
<tr>
<td>5</td>
<td>4.44557</td>
<td>7.58796</td>
<td>12.34779</td>
<td>19.17460</td>
</tr>
<tr>
<td>6</td>
<td>5.50000</td>
<td>9.32800</td>
<td>13.90540</td>
<td>20.57143</td>
</tr>
</tbody>
</table>

Fig. 16-3. Odd Eigenfunctions for $\beta = 0$.

method, differed only in the fifth significant digit from the values obtained using the first iterative method.

In the limit as $n \to \infty$, (16-140) reduces to

$$\lambda_n(\alpha) = n^2 + \frac{\alpha^2}{8}$$  \hspace{1cm} (16-152)

which describes the asymptotic behavior of the eigenvalues—that is, for large
λ'. Corresponding to (16-152), the asymptotic normalized eigenfunctions of (16-116) are easily seen to be

\[ u_{n\alpha}(\phi) = (\pi)^{-1/2} \cos n\phi \exp \left( \frac{\alpha}{2} \cos \phi \right) \]
\[ u_{n\alpha}(\phi) = (\pi)^{-1/2} \sin n\phi \exp \left( \frac{\alpha}{2} \cos \phi \right) \]  

(16-153)

Note that as \( \alpha \to 0 \), these equations are consistent with (16-101). It turns out that (16-153) yields an excellent approximation to the eigenfunctions, not only for very large \( n \) but also commencing at \( n \geq 5 \). This result is evidenced by the values given in Table 16-3, where it is seen that for \( n = 5, \alpha = 2 \) and 4, the computed values differ from their corresponding asymptotic values (16-152) by percentage errors of 0.02 and 0.15 percent respectively.

The conditional (\( n = 0 \)) transition p.d.f., given by (16-117), is shown in Figs. 16-4, 5, and 6 at four different instants of time. In computing these functions, the first ten terms of (16-86) were used. As may be seen from these figures, as \( \alpha \) increases (i.e., as the drift force increases), the transition time \( Tr \) (the time required for the steady-state distribution to be established) decreases.

---

**Fig. 16-4.** Conditional Transition p.d.f. for a First-Order Sinusoidal PLL, \( n = 0 \).
Table 16-3 Asymptotic, Perturbation, and Computed Eigenvalues

<table>
<thead>
<tr>
<th>α = 2</th>
<th>Asymptotic Values</th>
<th>Perturbation Values</th>
<th>Computed Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = 5</td>
<td>25.5</td>
<td>25.50504</td>
<td>25.50514</td>
</tr>
<tr>
<td>n = 6</td>
<td>36.5</td>
<td>36.50349</td>
<td>36.50358</td>
</tr>
<tr>
<td>n = 7</td>
<td>49.5</td>
<td>49.50256</td>
<td>49.50264</td>
</tr>
<tr>
<td>n = 8</td>
<td>64.5</td>
<td>64.50195</td>
<td>64.50202</td>
</tr>
<tr>
<td>n = 9</td>
<td>81.5</td>
<td>81.50154</td>
<td>81.50160</td>
</tr>
<tr>
<td>n = 10</td>
<td>100.5</td>
<td>100.50125</td>
<td>100.50130</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>α = 4</th>
<th>Asymptotic Values</th>
<th>Perturbation Values</th>
<th>Computed Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = 5</td>
<td>27.0</td>
<td>27.02019</td>
<td>27.04007</td>
</tr>
<tr>
<td>n = 6</td>
<td>38.0</td>
<td>38.01398</td>
<td>38.02794</td>
</tr>
<tr>
<td>n = 7</td>
<td>51.0</td>
<td>51.01025</td>
<td>51.02051</td>
</tr>
<tr>
<td>n = 8</td>
<td>66.0</td>
<td>66.00784</td>
<td>66.01613</td>
</tr>
<tr>
<td>n = 9</td>
<td>83.0</td>
<td>83.00618</td>
<td>83.01268</td>
</tr>
<tr>
<td>n = 10</td>
<td>102.0</td>
<td>102.00500</td>
<td>102.01065</td>
</tr>
</tbody>
</table>

Fig. 16-5. Conditional Transition p.d.f. for a First-Order Sinusoidal PLL, n = 0.
Table 16-4: Autocorrelation Factors for the Cases $F(\phi) = \phi$, $F(\phi) = \sin \phi$

<table>
<thead>
<tr>
<th>$\alpha = 2$</th>
<th>$\alpha = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$\sigma_{n\phi}$</td>
</tr>
<tr>
<td>1</td>
<td>3.19580</td>
</tr>
<tr>
<td>2</td>
<td>-0.29830</td>
</tr>
<tr>
<td>3</td>
<td>0.42412</td>
</tr>
<tr>
<td>4</td>
<td>-0.31248</td>
</tr>
<tr>
<td>5</td>
<td>0.25436</td>
</tr>
<tr>
<td>6</td>
<td>-0.21361</td>
</tr>
<tr>
<td>7</td>
<td>0.18392</td>
</tr>
<tr>
<td>8</td>
<td>-0.16140</td>
</tr>
<tr>
<td>9</td>
<td>0.14375</td>
</tr>
<tr>
<td>10</td>
<td>-0.12953</td>
</tr>
</tbody>
</table>

Fig. 16-6. Conditional Transition p.d.f. for a First-Order Sinusoidal PLL, $n = 0$.

The correlation factors $\sigma_n$, given by (16-98), for $F(\phi) = \phi$ and $F(\phi) = \sin \phi$ are tabulated in Table 16-4. Recalling that the autocorrelation and spectral density functions depend on $\sigma_n^2$, it is evident from these results that only
the first few eigenvalues and eigenfunctions are significant in evaluating these functions. Moreover, as $\alpha$ increases, we note that only the first eigenvalue and eigenfunction are significant; therefore, for large $\alpha$, the expansions (16-95) and (16-100) can be adequately approximated by using only the first term.

As a check on the accuracy of the eigenvalues given in Table 16-1, and hence on the corresponding eigenfunctions, from (16-95) and the values of Table 16-4 it is found that the steady-state variance is given by

$$
\sigma_{\phi}^2 = R_{\phi}(0) = \frac{1}{2\pi I_0(\alpha)} \sum_{n=1}^{10} \sigma_{n,\phi}^2 = \begin{cases} 
0.75314, & \alpha = 2 \\
0.29791, & \alpha = 4 
\end{cases} \quad (16-154)
$$

![Graph](image)

**Fig. 16-7.** Autocorrelation Function for a First-Order Sinusoidal PLL.
Computing the variance of $\phi$ from (9-46), we find that $\sigma_\phi^2 = 0.76445$ for $\alpha = 2$ and $\sigma_\phi^2 = 0.29882$ for $\alpha = 4$.

The autocorrelation function

$$R_\phi(\tau) = \sum_{n=1}^{10} \sigma_{n,\phi}^2 \frac{\exp \left[ -\lambda'_n D |\tau| \right]}{2\pi I_0(\alpha)}$$

(16-155)

and the normalized spectral density function (as a function of frequency)

$$S_\phi(f) = \frac{1}{2\pi I_0(\alpha)} \sum_{n=1}^{10} \frac{\lambda'_n \sigma_{n,\phi}^2}{(\lambda'_n)^3 + (2\pi f/D)^2}$$

(16-156)

are illustrated in Figs. 16-7 and 16-8 respectively.

**Fig. 16-8.** Normalized Spectral Density for a First-Order Sinusoidal PLL.
16-9 Spectral Analysis of First-Order Loops in the Presence of Detuning, $\beta \neq 0$

In the present section our main objectives are to establish that the transition p.d.f. can be obtained as an eigenfunction expansion, to obtain the form of the expansions of practical interest, and to develop a first-order approximation for the eigenvalues.

Consider the nonself-adjoint eigenvalue problem

$$Lu = u''(\phi) + (\alpha \sin \phi - \beta)u'(\phi) + \alpha \cos \phi u(\phi) = -\lambda' u(\phi)$$
$$B_1(u) = u(-\pi) - u(\pi) = 0 \quad B_2(u) = u'(-\pi) - u'(\pi) = 0$$

(16-157)

which corresponds to the case $\omega \neq \omega_0$ and $g(\phi) = \sin \phi$. For this case the operation of the loop consists of frequency acquisition, acquiring and tracking the phase of the received signal. Making the change of variables $\phi = 2x - \pi$ and reducing $Lu = -\lambda' u$ to its formally self-adjoint form, using (16-55) to (16-58), we obtain

$$Ly = -y''(x) + Q(x)y(x) = \eta y(x), \quad 0 \leq x \leq \pi$$
$$B_i(y) = a_{1i}y(0) + a_{2i}y'(0) + a_{3i}y(\pi) + a_{4i}y'(\pi) = 0 \quad i = 1, 2, \ldots$$

(16-158)

where

$$Q(x) = \frac{\alpha^2}{2} + \beta^2 + 2\alpha \cos 2x - 2\alpha^2 \cos 4x + 2\alpha \beta \sin 2x$$

(16-159)

$$\{a_{ij}\} = \begin{bmatrix} \beta & e^{-\beta x/2} & -\beta & e^{\beta x/2} \\ e^{-\beta x/2} & 0 & -e^{\beta x/2} & 0 \end{bmatrix}, \quad i = 1, 2 \quad j = 1, 4$$

(16-160)

and $\eta = 4\lambda'$. Since $Q(x)$ is sufficiently many times differentiable on the finite closed interval $0 \leq x \leq \pi$, it follows that $L$ is a regular differential operator. It can also be shown (Ref. 14) that the nonself-adjoint eigenvalue problem (16-158) is well posed; that is, there exists a denumerable number of eigenvalues and, moreover, that the corresponding set of eigenfunctions form a basis for the entire space $L_2[0, \pi]$. Furthermore, because the boundary conditions of (16-158) are "regular" in the sense of Birkhoff (Ref. 15) and "strongly regular" in the sense of Tamarkin (Ref. 16) and Mikhailov (Ref. 17), the conditional transition p.d.f. $p(\phi; t)$ can be expanded in a unique series in terms of the eigenfunctions of (16-158). We also note that this result is independent of the loop nonlinearity provided that it is periodic, odd-symmetric, and of class $C^1$. 
16-10 Eigenfunction Expansions for First-Order Loops in the Presence of Detuning, $\beta \neq 0$

In order to gain some physical insight into the effect of the nonself-adjoint term—$\beta \mu'(\phi)$—in (16-157) on the nature of the spectrum, and hence its effect on the form of the eigenfunction expansions of interest, let us consider the nonself-adjoint problem

$$Lu = u''(\phi) - \beta u'(\phi) = -\lambda' u(\phi)$$
$$u(-\pi) - u(\pi) = 0 \quad u'(-\pi) - u'(\pi) = 0$$

(16-161)

that results from setting $\alpha = 0$ in (16-157). This corresponds physically to the transient behavior of the loop during a temporary loss of signal; it corresponds mathematically to imposing a constant drift force on the $\phi$ modulo-$2\pi$ diffusion process that has the real degenerate spectrum given by (16-101).

From the definition of $L^*$ and $D^*$, the adjoint system of (16-161) is

$$L^* v = v''(\phi) + \beta v'(\phi) = -\lambda' v(\phi)$$
$$v(-\pi) - v(\pi) = 0 \quad v'(-\pi) - v'(\pi) = 0$$

(16-162)

Two linearly independent solutions of $Lu = -\lambda' u$ and $L^* v = -\lambda' v$ are

$$u_{1,2}(\phi, \lambda') = \exp \left[ (\beta \pm \sqrt{\beta^2 - 4\lambda'}) \phi \right]$$

(16-163)

and in order to satisfy the periodic boundary conditions, we find that

$$\lambda'_n = n^2 + i\beta n \quad \lambda''_n = n^2 - i\beta n$$

(16-164)

where $n$ is a non-negative integer.

Substituting $\lambda'_n, \lambda''_n$ into (16-163) yields the complete, biorthogonal set of complex eigenfunctions

$$u_0(\phi) = (2\pi)^{-1/2}, \quad v_0(\phi) = 1, \quad \lambda'_0 = 0$$
$$u_n(\phi) = \tilde{v}_n(\phi) = (2\pi)^{-1/2} e^{in\phi}, \quad \lambda'_n = n^2 + i\beta n$$
$$\tilde{u}_n(\phi) = v_n(\phi) = (2\pi)^{-1/2} e^{-in\phi}, \quad \lambda''_n = n^2 - i\beta n, \quad n > 0$$

(16-165)

which have been normalized such that

$$\langle u_n, v_m \rangle = \delta_{nm}$$

(16-166)
Thus when the nonself-adjoint differential expression \(-\beta u'(\phi)\) is added to the self-adjoint expression \(u''(\phi)\), we see that the degeneracy of the self-adjoint spectrum is completely lifted, for to each eigenvalue there now corresponds only one eigenfunction. Moreover, the spectrum is now complex, with the eigenvalues and eigenfunctions appearing in complex conjugate pairs.

The question now arises as to whether the same phenomenon occurs in the nonself-adjoint case when \(\alpha \neq 0\)—that is, when \(-\beta u'(\phi)\) is added to (16-116) to give (16-157).

Rewriting (16-161) in its formally self-adjoint form and defining \(\eta = \eta^* - \beta^2\), we obtain

\[ \begin{align*}
L_y = -y''(x) = \eta' y & \quad B_i(y) = 0, \quad i = 1, 2, \ldots \\
\end{align*} \tag{16-167} \]

where the boundary conditions are exactly those of (16-158), and where

\[ \eta_n^* = 4n^2 \pm i4\beta n \tag{16-168} \]

Rewriting (16-158) as

\[ L_y = -y''(x) + Q'(x)y = \eta'' y \tag{16-169} \]

where

\[ \begin{align*}
Q'(x) = 2\alpha \cos 2x - 2\alpha^2 \cos 4x + 2\alpha \beta \sin 2x & \\
\eta'' = \eta - \beta^2 - \frac{\alpha^2}{2} & \tag{16-170}
\end{align*} \]

and noting that

\[ |Q'(x)| \leq 2\alpha(1 + \beta) + 2\alpha \Delta M \quad \text{for } 0 \leq x \leq \pi \tag{16-171} \]

it is clear that for \(\eta'' \gg M\) the spectrum of (16-169) must approach that of (16-167). In terms of \(\lambda\) we find

\[ \begin{align*}
\lambda'_n = n^2 + \frac{\alpha^2}{8} + i\beta n & \quad \bar{\lambda}'_n = n^2 + \frac{\alpha^2}{8} - i\beta n & \tag{16-172}
\end{align*} \]

for

\[ |\lambda'| \gg \frac{\beta^2}{2} + \frac{\alpha}{2} (1 + \beta) + \frac{5}{8} \alpha^2 \tag{16-173} \]

Thus for large \(|\lambda'|\), but not necessarily values approaching infinity, the eigenvalues and eigenfunctions of (16-158) occurring in complex conjugate pairs. In the sequel, we obtain a similar result by using perturbation theory.
We note that the biorthogonality of \([y_n(x)], [w_n(x)]\) implies the biorthogonality of \([u_n(x)], [v_n(x)]\), since with \(x = (\phi - \pi)/2\)

\[
y_n(\phi) = u_n(\phi) \exp\left(-\frac{\alpha}{2} \cos \phi - \frac{\beta \phi}{2}\right)
\]
\[
w_n(\phi) = v_n(\phi) \exp\left(\frac{\alpha}{2} \cos \phi + \frac{\beta \phi}{2}\right)
\] (16-174)

Therefore in terms of the complex forms

\[
\lambda'_n = \lambda_{nr} + i\lambda_{ni}
\]
\[
u_n(\phi) = u_{nr} + iu_{ni}
\] (16-175)
\[
v_n(\phi) = v_{nr} + iv_{ni}
\]

the conditional transition p.d.f. assumes the following form:

\[
p(\phi, t) = p(\phi) + 2 \text{Re} \left\{ \sum_{n=1}^{\infty} v_n(\phi_0)u_n(\phi) \exp[-D\lambda'_n(t - t_0)] \right\}
\]
\[
= p(\phi) + 2 \sum_{n=1}^{\infty} \left[ v_{nr}(\phi_0)u_{nr}(\phi) - v_{ni}(\phi_0)u_{ni}(\phi) \right]
\times \cos D\lambda_{nr}(t - t_0) \exp[-D\lambda_{nr}(t - t_0)]
- 2 \sum_{n=1}^{\infty} \left[ v_{nr}(\phi_0)u_{nr}(\phi) + v_{ni}(\phi_0)u_{ni}(\phi) \right]
\times \sin [D\lambda_{ni}(t - t_0)] \exp[-D\lambda_{nr}(t - t_0)]
\] (16-176)

where \(p(\phi)\) is the steady-state p.d.f. given by (16-42), and where \(u_{nr}, u_{ni}, v_{nr}, v_{ni}\) satisfy the following equations:

\[
u_{nr}' + (\alpha \sin \phi - \beta)u_{nr}' + (\lambda_{nr} + \alpha \cos \phi)u_{nr} = \lambda_{ni}u_{ni}
\]
\[
u_{ni}' + (\alpha \sin \phi - \beta)u_{ni}' + (\lambda_{nr} + \alpha \cos \phi)u_{ni} = -\lambda_{ni}u_{nr}
\]
\[
u_{nr}' + (\beta - \alpha \sin \phi)v_{nr}' + \lambda_{nr}v_{nr} = \lambda_{ni}v_{ni}
\]
\[
u_{ni}' + (\beta - \alpha \sin \phi)v_{ni}' + \lambda_{nr}v_{ni} = -\lambda_{ni}v_{nr}
\] (16-177)

subject to periodic boundary conditions.

In a manner similar to the development of (16-90) to (16-100), the steady-state autocorrelation and spectral density functions may be obtained for the process \(\psi(t) = F(\phi)\), where \(F\) is any memoryless, time-invariant transformation.

Considering the case \(\alpha = 0, \beta = 0\), which corresponds to a physical loss of signal, or, equivalently, the case \(g(\phi) = 0, \beta = 0\), which corresponds to
the received signal being orthogonal to the reference signal, the statistics of \( \phi \) during this period are given by following expansions:

\[
p(\phi, \tau|\phi_0) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos n(\phi - \phi_0 - \Lambda_0 \tau) \exp[-Dn^2\tau]
\]

(16-178)

\[
p_2(\phi, \phi_0, \tau) = \frac{1}{4\pi^2} + \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \cos n(\phi - \phi_0 - \Lambda_0 \tau) \exp[-Dn^2\tau]
\]

(16-179)

\[
R_\phi(\tau) = \sum_{n=1}^{\infty} \frac{2}{n^2} \cos n\Lambda_0 \tau \exp[-Dn^2|\tau|]
\]

(16-180)

\[
S_\phi(\omega) = \sum_{n=1}^{\infty} \left\{ \frac{2D}{(n\Lambda_0 + \omega)^2 + n^4D^2} + \frac{2D}{(n\Lambda_0 - \omega)^2 + n^4D^2} \right\}
\]

(16-181)

where \( \tau = |t - t_0| \) and \( D\beta = \Lambda_0 \).

Letting \( F(\phi) = \sin \phi \) and \( F(\phi) = \cos \phi \), we find

\[
R_{\sin \phi}(\tau) = R_{\cos \phi}(\tau) = \frac{\cos \Lambda_0 \tau \exp[-D|\tau|]}{2}
\]

(16-182)

\[
S_{\sin \phi}(\omega) = S_{\cos \phi}(\omega) = \frac{D/2}{(\Lambda_0 + \omega)^2 + D^2} + \frac{D/2}{(\Lambda_0 - \omega)^2 + D^2}
\]

(16-183)

The effect of the detuning term \( \Lambda_0 \) is such that it phase-modulates the terms \( \cos n(\phi - \phi_0) \) that determine the steady-state autocorrelation and spectral density functions. The effect of this modulation is seen in \( S_\phi(\omega) \). Namely, the components of \( S_\phi(\omega) \), given by (16-108), are halved in amplitude and displaced about the frequencies \( \omega_n = n\Lambda_0 \) so that the spectral density of \( \phi \) for \( \beta \neq 0 \) is similar to that of a bandpass process. The same phenomenon occurs for the sine wave processes \( F(\phi) = \sin \phi \) and \( F(\phi) = \cos \phi \).

16-11 Perturbation Estimates of the Nonself-Adjoint Spectrum, \( \beta \neq 0 \)

By perturbing the self-adjoint spectrum, we now develop a first-order estimate for the spectrum of the nonself-adjoint problem (16-157).

To begin with, the perturbed or nonself-adjoint operator will be denoted as

\[
L = L_0 - \beta \frac{d}{d\phi}
\]

(16-184)

where the unperturbed operator

\[
L_0 = \frac{d^2}{d\phi^2} + \frac{d}{d\phi} (\alpha \sin \phi)
\]

(16-185)
Since each unperturbed eigenvalue $\lambda'_n (n \geq 1)$ is twofold degenerate, for the zeroth-order perturbed eigenfunction we take the linear combination

$$u_{\beta n}(\phi) = a_n(\beta)u_{n0}(\phi) + b_n(\beta)u_{n0}(\phi)$$  \hspace{1cm} (16-186)

and for the first-order approximation to the perturbed eigenvalue we let

$$\lambda'_{\beta n} = \lambda'_n + \beta \frac{\partial \lambda'_{\beta n}}{\partial \beta} \bigg|_{\beta = 0}$$  \hspace{1cm} (16-187)

so that as the perturbation parameter $\beta \to 0$, $a_n(\beta) \to a_n$, $b_n(\beta) \to b_n$, and hence $u_{\beta n}(\phi) \to u_n(\phi)$, $\lambda'_{\beta n} \to \lambda'_n$.

Adding and subtracting $\lambda'_n u_{n0}(\phi)$ to both sides of the perturbed equation

$$L u_{\beta n}(\phi) = -\lambda'_{\beta n} u_{\beta n}(\phi)$$  \hspace{1cm} (16-188)

and then differentiating with respect to $\beta$ yields

$$L_0 \frac{du_{\beta n}}{d\beta} + \lambda'_n \frac{du_{\beta n}}{d\beta} = (\lambda'_n - \lambda'_{\beta n}) \frac{du_{\beta n}}{d\beta} + \frac{du_{\beta n}}{d\beta}$$

$$= \beta \frac{du_{\beta n}}{d\phi} - u_{\beta n} \frac{\partial \lambda'_{\beta n}}{\partial \beta}$$  \hspace{1cm} (16-189)

Taking the limit as $\beta \to 0$, the foregoing equation reduces to

$$L_0 \left( \frac{du_{\beta n}}{d\beta} \right) \bigg|_{\beta = 0} + \lambda'_n \left( \frac{du_{\beta n}}{d\beta} \right) \bigg|_{\beta = 0} = \left( \frac{du_{\beta n}}{d\beta} \right) \bigg|_{\beta = 0} - u_n(\phi) \left( \frac{\partial \lambda'_{\beta n}}{\partial \beta} \right) \bigg|_{\beta = 0}$$  \hspace{1cm} (16-190)

Since the set of eigenfunctions $\{u_n(\phi), u_{n0}(\phi), u_{n0}(\phi)\}$ of the unperturbed operator $L_0$ form a complete orthonormal set with respect to the scalar product

$$\langle u, v \rangle = \int_{-\pi}^{\pi} u(x)v(x)w(x) \, dx$$  \hspace{1cm} (16-191)

where the weighting function

$$w(\phi) = \exp (-\alpha \cos \phi)$$  \hspace{1cm} (16-192)

any function in $L_2[-\pi, \pi]$ can be represented as

$$f(\phi) = a_0 u_0(\phi) + \sum_{k=1}^{\infty} a_k u_{k0}(\phi) + b_k u_{k0}(\phi)$$  \hspace{1cm} (16-193)
Thus letting

$$
\left( \frac{du_n}{d\beta} \right)_{\beta=0} = a_0(\beta)u_0(\phi) + \sum_{k=1}^{\infty} [a_k(\beta)u_k(\phi) + b_k(\beta)u_{k-1}(\phi)]
$$

(16-194)

and defining the operator

$$
L_0^\prime = L_0 + \lambda_n'
$$

(16-195)

the left-hand side of (16-190), which is in the range of $L_0$, has the representation

$$
L_0^\prime \left( \frac{du_n}{d\beta} \right)_{\beta=0} = a_0(\beta)[\lambda_n' - \lambda_n]u_0(\phi) + \sum_{k=1}^{\infty} [\lambda_n' - \lambda_k'][a_k(\beta)u_k(\phi) + b_k(\beta)u_{k-1}(\phi)]
$$

(16-196)

However, since the set $(u_0, u_{ne}, u_{no})$ is orthonormal with respect to the scalar product (16-191), from (16-196) it follows that the function

$$
L_0 \frac{du_n}{d\beta} \bigg|_{\beta=0}
$$

and hence the right side of (16-190) is orthogonal to the functions

$$
u_{ne}(\phi) \exp (-\alpha \cos \phi) \quad u_{no}(\phi) \exp (-\alpha \cos \phi)
$$

(16-197)

Expressing the right-hand side of (16-190) in terms of $(u_0, u_{ne}, u_{no})$, and multiplying by the preceding functions, we obtain the following relations:

$$
\begin{align*}
a_n[\langle u_{ne}', u_{ne} w(\phi) \rangle - \bar{\tau} \langle u_{ne}, u_{ne} w(\phi) \rangle] \\
+ b_n[\langle u_{no}', u_{ne} w(\phi) \rangle - \bar{\tau} \langle u_{no}, u_{ne} w(\phi) \rangle] &= 0 \\
\end{align*}
$$

(16-198)

$$
\begin{align*}
a_n[\langle u_{ne}', u_{no} w(\phi) \rangle - \bar{\tau} \langle u_{ne}, u_{no} w(\phi) \rangle] \\
+ b_n[\langle u_{no}', u_{no} w(\phi) \rangle - \bar{\tau} \langle u_{no}, u_{no} w(\phi) \rangle] &= 0 \\
\end{align*}
$$

(16-199)

where

$$
\bar{\tau} \triangleq \left( \frac{\partial \lambda_n'}{\partial \beta} \right)_{\beta=0}
$$

(16-200)
Making use of the orthonormal property
\[
\langle u_{ne}, u_{ne} \phi \rangle = \langle u_{no}, u_{no} \phi \rangle = 1 \tag{16-201}
\]
the properties that the functions \( u_{ne}, u'_{ne} \phi \) are even, that the functions \( n_{no}, u'_{ne} \phi \) are odd, and solving the resulting algebraic equations
\[
\tilde{t} a_n = b_n \langle u'_{no}, u_{ne} \phi \rangle \\
a_n \langle u'_{ne}, u_{no} \phi \rangle = \tilde{t} b_n \tag{16-202}
\]
for \( \tilde{t} \) yields
\[
\tilde{t} = \pm \sqrt{\langle u'_{no}, u_{ne} \phi \rangle \langle u'_{ne}, u_{no} \phi \rangle} \tag{16-203}
\]
so that the first-order approximation to the nonself-adjoint eigenvalue problem is
\[
\lambda'_{\beta n} = 0 \\
\lambda'_n = n^2 + \frac{n^2 \alpha^2}{2(4n^2 - 1)} + \beta \tilde{t}, \quad n \geq 1 \tag{16-204}
\]
Since, for \( n \geq 5 \), the normalized periodic sine eigenfunctions of \( L_0 \) are approximately given by (16-153), substituting these solutions into the inner-products of (16-203) and evaluating the integrals yields
\[
\langle u'_{ne} \phi, \ u_{no} \phi \rangle w(\phi) = -n \\
\langle u'_{ne} \phi, \ u_{ne} \phi \rangle w(\phi) = +n \tag{16-205}
\]
so that
\[
\lambda'_{\beta n} = n^2 + \frac{n^2 \alpha^2}{2(4n^2 - 1)} \pm i \beta n \tag{16-206}
\]
a result that is consistent with the exact values obtained for the case \( \alpha = 0 \), \( \beta \neq 0 \), and the asymptotic values for the case \( \alpha \neq 0 \), \( \beta \neq 0 \).

16-12 Development of the Expansion for the Restricted Probability Density Function \( Q(\phi; t) \)

In this section we present the solution to the two-point boundary-valued problem defined in (9-74) and (9-75). We look for a solution of the form \( Q(\phi; t) = r(t)R(\phi) \). Substitution of this form into (9-74) leads to (Ref. 22)
\[ \dot{r}(t) = r(t_0) \exp \left[ -D \lambda(t - t_0) \right] \quad (16-207) \]

and the Sturm-Liouville problem

\[ R''(\varphi) + [\alpha g(\varphi) - \beta] R'(\varphi) + \alpha g'(\varphi) R(\varphi) = -\lambda R(\varphi) \quad (16-208) \]

with \( R(\varphi_{11}) = 0 \) and \( R(\varphi_{12}) = 0 \). The self-adjoint form is an eigenvalue problem of Hills' equation

\[ z''(\varphi) - G(\varphi) z(\varphi) = -\lambda z(\varphi) \quad (16-209) \]
\[ z(\varphi_{11}) = 0, \quad z(\varphi_{12}) = 0 \]

where

\[ z(\varphi) = R(\varphi) \exp \left[ \frac{(-\beta \varphi + \alpha f(\varphi))}{2} \right] \quad (16-210) \]

and \( G(\varphi) \) is defined by the right-hand side of (16-52) which \( \phi \) replace by \( \varphi \). Using the initial condition (16-3), (16-207) and the orthonormal property

\[ \int_{\varphi_{11}}^{\varphi_{12}} R_n(\varphi) R_m(\varphi) \exp \left[ \alpha f(\varphi_0) - \beta \varphi \right] d\varphi = \delta_{nm} \quad (16-211) \]

we obtain the spectral representation

\[ Q(\varphi; t) = \sum_{n=1}^{\infty} R_n(\varphi_0) R_n(\varphi) \cdot \exp \left[ \alpha f(\varphi_0) - \beta \varphi_0 - 4 \lambda_n B_L(t - t_0) / \alpha \right] \quad (16-212) \]

where the eigenvalues \( \{ \lambda_n \} \) satisfy \( 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \), and where the eigenfunctions \( \{ R_n(\varphi) \} \) have the property that each \( R_n(\varphi) \) has exactly \( n \) zeros in \( [\varphi_{11}, \varphi_{12}] \) with \( R_n(\varphi) \) being periodic in \( \varphi \) when \( \Lambda_0 = 0 \). Integrating (16-212) with respect to \( \varphi \) yields the probability that \( \varphi(t) \) never reaches the boundaries during \([t_0, t]\). Thus one minus this probability is the probability \( P(t|\varphi_0) \) reaching the boundaries during \([t_0, t]\), that is,

\[ P(t|\varphi_0) = 1 - \sum_{n=1}^{\infty} A_n R_n(\varphi_0) \exp \left[ \alpha f(\varphi_0) - \beta \varphi_0 - 4 \lambda_n B_L(t - t_0) / \alpha \right] \quad (16-213) \]

which is the expansion for (9-87). Here

\[ A_n \triangleq \int_{\varphi_{11}}^{\varphi_{12}} R_n(\varphi) d\varphi \quad (16-214) \]

Differentiating (16-213) we obtain the p.d.f of the first-passage time, namely,
\[ p(t|\varphi_0) = \frac{4B_L}{\alpha} \sum_{n=1}^{\infty} \lambda_n A_n R_n(\varphi_0) \exp \left[ \alpha f(\varphi_0) - \beta \varphi_0 - 4\lambda_n B_L(t - t_0)/\alpha \right] \]

(16-215)

Therefore, the \( n \)th moment of the first-passage time to reach \( \varphi_{11} \) or \( \varphi_{12} \) is given by

\[ \tau^n(\varphi_0|\varphi_0) = \sum_{k=1}^{\infty} \frac{n!}{\lambda_k^n} \left( \frac{\alpha}{4B_L} \right)^n R_k(\varphi_0) A_k \exp \left[ \alpha f(\varphi_0) - \beta \varphi_0 \right] \]

(16-216)

This result should be compared with the recursive formula (9-76).

When \( g(\varphi) = \sin \varphi \) the boundaries are \( \varphi_{11} = -\varphi_i \) and \( \varphi_{12} = \varphi_i \). Considering (16-208) with \( \alpha = 0 \) produces

\[ \lambda_n = \frac{(n\pi)^2}{4\varphi_i^2} + \frac{\beta^2}{4} \]

(16-217)

\[ R_k(\varphi) = \frac{1}{\sqrt{\varphi_i}} \sin \left[ k\pi \left( \frac{\varphi - \varphi_0}{2\varphi_i} \right) \right] \exp \left( \beta \varphi / 2 \right) \]

Fig. 16-9. Probability of Slipping a Cycle at Time \( B_L t \).
LaFrieda (Ref. 22) has perturbed this spectrum for \( \varphi = \pi \) to produce approximate expressions for the eigenvalues; however, we omit these tabulations here. He also solves (16-208) numerically to produce the results of Fig. 16-9. Superposed is the plot of (9-73) to confirm the fact that for \( \alpha \geq 4 \) the p.d.f. of the mean time to first slip is approximately exponential.

16-13 Conclusions

It is clear from the material presented in this chapter that a great deal of effort must be expended before the signal acquisition and synchronization stability problem is understood in the presence of noise. The study will be made extremely difficult and tedious because most solutions of interest must be obtained via a digital computer. The work presented in this chapter is due to one of my students, Dr. James R. LaFrieda, and was first published in his doctoral dissertation (Ref. 14). Another student, Dr. Albert Chang, helped write the computer program that generated most of the figures pertaining to \( p(\varphi; t) \) and its properties. Other material relative to the theory of two-point boundary-valued problems can be found in Refs. 18, 19, 20, and 21.

References


INDEX

Absorbing boundaries, random walk with, 292–296
Acquisition:
  aids, 475–481
  behavior, 374–379
  frequency, 497–500
  probability, 410–412
  range, 452–465
  signal time, 410–412, 452, 558
  techniques, 475–481
Analog demodulators, 131
Angle demodulation theory, 158–161
Angle modulation, 566
  carrier tracking in the presence of, 85
  Doppler tracking in the presence of, 85
  Doppler tracking of suppressed carrier signals, 95
  extraction in the presence of carrier tracking, 85–91
  PLL theory, 158–161
  suppressed carrier signals with, 95
  Average value, 5

Backward equation, 331–332
Bandpass limiter (BPL), 153–158, 186–192
Bandpass PLL design, 175–177
Bandwidth:
  equivalent noise, 33
  loop, 136–140
  spectral, 15
Baseband modulation reconstruction, 97–99
Bessel functions:
  of imaginary argument, 425–428
  of imaginary order, 425–428
Boundary conditions, 386–387, 530–532, 541–542
Brownian motion, 268–273, 348–350
Carrier tracking, 82–83, 143–158
  angle modulation extraction in the presence of, 85–91
  phase-locked loop mechanizations, 177–180
  in the presence of angle modulation, 85
  sinusoidal phase-locked loops, 143–158, 177–180
  of suppressed carrier signals with angle modulation present, 95
Carriers, hybrid, 103–106
Changing flux theorem, 261–262
Changing flux theorem (Cont'd)
proof of, 281–283
Characteristic function, 6
Characterization, 444–447
performance, 446
Circular moments, derivation of, 429–433
Coherent communications, 82–91, 548–551
Coherent demodulation, 192–195
Coherent receivers, 99–107
Coherent transponders, 107–112, 131, 179, 180
Conditional transition probability density functions, 567–568
expansion for, 636–644
Correlation time, 15
Critical damping, 139
Cycle slipping:
average rate of, 400–404
characterization of the probabilities, 404
mean time, 409
moments, 407
probabilities, 494–497
rate, 494–497

Damped systems, 226–227
Damping:
critical, 139
over-, 139
under-, 139
Damping factor, 139
Data-aided loops, 100–102
Decision-directed feedback loops, 96–97
Delay-locked loops, 111–112
Delta-correlated processes, 19–20
Demodulation:
angle theory, 158–161
coherent, 192–195
frequency, 166–175, 617–634
nonlinear theory application, 621–632
system model, 618–621
phase, 161–166
Demodulators, analog, 131
Design point, 155
Deterministic external forces, 218–220
Deterministic signals, 3, 4
Detuning, 245
normalized, 375

Detuning (Cont'd)
residual, 398
Deviation ratio, 167
Differential operators, 278–279
Differential equation, integro-, 415–416
Differential rule, Itô, 352–355
Diffusion approximation, 337–339
Diffusion coefficient, 404–406
Diffusive flow law, 291–292
Digital phase-locked loops, 112–115
Doppler tracking, 84, 143–158
in the presence of angle modulation, 85
rates, 145–146
sinusoidal phase-locked loops, 143–158
of suppressed carrier signals with angle modulation present, 95
Dynamical system model, 601–602
Eigenfunction expansion for first-order loops, 652–657, 676–679
Eigenvalues, calculation of, 665–667
Entrainment principle (see Phase-lock principle)
Events, 4
Extrasystole, 132
False-lock, 468–473
Field, the, 4
Figures of merit, 447–449
Filtering:
causal, 168–173
of stationary processes, 44–47
to maximize the output signal-to-noise ratio, 53–56
noncausal, of stationary random processes, 41–43
optimum, of polynomial-type signals, 47–48
optimum linear theory, 37–57
Filters:
optimum linear, 40–41
time-invariant linear, 32–34
First-order loop, 136
performance, 373–420
First-passage time:
model, 541–542
moments of, 542–546
problems, 311–320
First-slip time formula, 434–436
Flow equation, probability, 262–265
Flow law, diffusive, 291–292
Flux, probability, 258–259
density, 260–262
divergence of, 259–260
Flux theorem, 261–262
proof of, 281–283
Fokker-Planck equation, 336–337, 345–359
basic rules of Itô calculus, 347–355
dimensional, 525–527
for first-order SCS, 385–386
intensity coefficients, 355–358
multidimensional, 326–332
one-dimensional, 298–311
reduction of, 566
solutions by conditional expectation method, 600–613
dynamical system model, 601–602
steady-state results for sinusoidal nonlinearity, 609–613
steady-state solution, 605–608
solutions by the sequence method, 577–599
Gaussian density function as an initial estimate, 585–588
steady-state variance for sinusoidal nonlinearity, 588–593
state-space representation, 346–347
time-dependent solutions to, 304–311, 635–636
calculation of eigenvalues, 665–667
conditional transition probability density function, 636–644
eigenfunction expansions for first-order loops, 652–657, 676–679
iterative techniques, 665–667
perturbation estimates of nonself-adjoint spectrum, 679–682
perturbation solutions, 660–664
restricted probability density function, 682–685
spectral analysis of first-order loops, 644–652, 675
spectral analysis of first-order sinusoidal PLLs, 657–660
Fourier transform (Cont’d)
tracking a constant offset, 144
uncertainty region, 149
Frequency demodulation, 166–175, 617–634
nonlinear theory application, 621–632
system model, 618–621
Frequency modulation, 131
Frequency tracking error, 160, 166
Gaussian density function, 585–588
Gaussian moments, 595–598
Gaussian noise, 412–416
Gaussian process, 17–18
narrowband, 20–23
Gauss’s law, 258–259
Gradient field, 277–278
Hill’s equation, 250
Hybrid tracking loops, 103–106
Hybrid loops, elimination of the sub-carrier reference in, 106–107
Imaginary argument, Bessel functions, 425–428
Imaginary order, Bessel functions, 425–428
Impulsive noise, 412–416
characterization of, 30–32
Initial conditions, 386–387, 527–530
Integrals:
Itô, 350–352
line, 279
surface, 279–280
volume, 280
Integrators:
imperfect, 140
perfect, 139–140
Integro-differential equation, 415–416
Intensity coefficients, 330–331
Fokker-Planck, 355–358
Iterative techniques, 665–667
Itô calculus:
basic rules of, 347–355
intensity coefficients, 355–358
Itô differential rule, 352–355
Itô integral, 350–352
Laplace’s equation, 273–274
Least-squares estimation, 537–538
Limit cycle, 452
Limiter performance factor, 154
Limiter suppression factor, 156
Line integrals, 279
Linear approximation:
quasi-, 417
spectral, 417–418
Linear modulation extraction, 95–96
Linear PLL model, 131–142
Linear-spectral theory, 184
Loop equation, 73–81
unified, 115–117
Loop mechanization, 66–72
Loop threshold, 155, 404–406
Loop (see also Phase-locked loop):
bandwidth, 136–140
baseband modulation reconstruction, 97–99
closed-transfer function, 136
Costas or in phase-quadrature (I-Q), 93–95
data-aided, 100–102
decision-directed feedback, 96–97
delay-locked, 111–112
digital phase-locked, 112–115
double tracking system, 109–111
extension to an arbitrary filter, 554–556
figures of merit, 447–449
first-order:
eigenfunction expansion for, 652–657, 676–679
spectral analysis of, 644–652, 675
gain, 141
hybrid, elimination of the subcarrier reference in, 106–107
modulation tracking, 91–99, 103–106
phase-locked mechanizations, 177–180
second-order:
comparison to third-order, 152–153
imperfect, 473
perfect, 474–475
signal acquisition, 449–481
sinusoidal phase-locked, 617–634
with applications, 130–185
baseband PLL design, 175–177
carrier tracking, 143–158, 177–180
Doppler tracking, 143–158
frequency demodulation, 166–175
linear PLL angle demodulation theory, 158–161
Loop (Cont'd)
linear PLL model, 131–142
phase demodulation, 161–166
phase-locked loop mechanizations, 177–180
spectral analysis of, 657–660
superheterodyne PLL receiver, 180–184
squaring, 92–93
stability as determined by root locus plots, 140–141
suppressed carrier (subcarrier), 91–99
third-order:
comparison to second-order, 152–153
performance of, 556–558
range in, 558
tracking Doppler rates, 145–146
unified equation for, 115–117
Loss of lock, 406–407
Markov processes, 296–298, 336–337
first-order, 18–19, 298–304
vector, 320–326
Matrix notation, 360
Maxwell's curl equations, 265–268
Mean-square phase error, 134
Mean-square tracking error, 135
Modulation:
age, 566
carrier tracking in the presence of, 85
Doppler tracking in the presence of, 85
Doppler tracking of suppressed carrier signals, 95
extraction in the presence of carrier tracking, 85–91
PLL theory, 158–161
suppressed carrier signals with, 95
baseband reconstruction, 97–99
frequency, 131
linear extraction, 95–96
phase, 131
Modulation index, 167
Modulation tracking loops, 91–99, 103–106
Moments of the mean time to first loss, 491–494
Moments of the mean time to first slip, 407–410
Narrowband process, 186–192
  Gaussian, 20–23
Natural frequency, 139
Noise:
  Gaussian, 412–416
  impulse, 412–416
    characterization of, 30–32
  -phase process, 81–82
pseudo-noise receivers, 107–112
  shot, characterization of, 30–32
  thermal, characterization, 29–30
  -to-signal ratios, 192–199
    filtering to maximize, 53–56
white, filter performance formulas for,
  48–53
Nondeterministic signals, 3, 4
Nonlinear oscillations, 213–256
  coupled, 240–250
  of damped systems with nonlinear
    restoring forces, 226–227
Hill's equation, 250
  linear-free, 215–217
  linear oscillations in presence of de-
    terministic external force, 218–220
  pendulum with damping force propor-
    tional to absolute velocity, 230–
    231
  phase-locked regenerative receiver,
    240–250
  relaxation, 231–240
  self-sustained, 231–240
  synchronization principle, 240–250
  types of singularities, 227–230
  of undamped systems with nonlinear
    restoring forces, 221–226
Nonlinear restoring forces, 221–227
Nonlinear theory:
  frequency demodulation application,
    621–632
  of second-order SCSs, 481–497
Nonself-adjoint spectrum, 679–682
Nonstationary processes:
  periodic, 25–26
    with stationary increments, 23–25
Normalization condition, 5
Nyquist criterion, 141–142
Nyquist diagram, 141–142
Operation equations, 242–244
Oscillations, nonlinear, 213–256
  coupled, 240–250
Oscillations (Cont'd)
  of damped systems with nonlinear re-
    storing forces, 226–227
Hill's equation, 250
  linear-free, 215–217
  linear oscillations in presence of de-
    terministic external force, 218–
    220
  pendulum with damping force propor-
    tional to absolute velocity, 230–
    231
  phase-locked regenerative receiver,
    240–250
  relaxation, 231–240
  self-sustained, 231–240
  synchronization principle, 240–250
  types of singularities, 227–230
  of undamped systems with nonlinear
    restoring forces, 221–226
Overdamping, 139
Paroxysmal tachycardia, 132
Particles escape over a potential wall,
  339–341
Periodic extension, 527–530, 564–566
Periodic nonstationary processes, 25–26
Perturbation estimates of the nonself-
  adjoint spectrum, 679–682
Perturbation solutions, 660–664
Phase demodulation, 161–166
Phase detector characteristic, 458–463,
  465–468
  to maximize acquisition, 463–465
Phase error:
  density for the sinusoidal PLL, 388–
    397
  mean-square, 134
  statistical dynamics of, 398–400
  steady-state, 143–146
  steady-state probability density of, 387,
    485–491, 569–570
  steady-state variance of, 500–501
  techniques for approximating the
    variance of, 417–420
Phase-jumps, 381
Phase-lock principle, 65–126
  applications to coherent communica-
    tions, 82–91
  coherent receivers, 99–107
  coherent transponders, 107–112
digital phase-locked loop, 112–115
Phase-lock principle (Cont’d)
  loop equation, 73–81
  loop mechanization, 66–72
  modulation tracking loops, 91–99
  pseudo-noise tracking receivers, 107–112
  statistical properties of the phase-noise process, 81–82
  suppressed carrier (subcarrier) loops, 91–99
  unified loop equation, 115–117
Phase-locked loop mechanization, 177–180
Phase-locked loop receivers, super-
  heterodyne, 177, 178, 180–184
Phase-locked loops (see also Sinusoidal
  phase-locked loops):
  angle demodulation theory, 158–161
  bandpass design, 175–177
  frequency demodulation using, 166–175
  linear model, 131–142
  operation, 412–416
  phase demodulation using, 161–166
  phase error density for, 388–397
  spectral analysis of first-order sinus-
  oidal, 657–660
  stability of imperfect second-order, 141–142
  tracking with second-order, 153–158
Phase-locked regenerative receivers, 240–250
Phase modulation, 131
Phase-noise process, 81–82
Poisson’s equation, 273–274
Polynomial-type signals, 47–48
Potential case, the, 333–336
Potential wall, particles escape over, 339–341
Power engineering, 67–70
Probability acquisition, 410–412
Probability current, 314–315
  density, 332–333, 400–404
Probability cycle slipping, 494–497
Probability density function, 5
  conditional transition, 567–568
  expansion for, 636–644
  restricted, 682–685
Probability distribution function, 5
Probability flow equation, 262–265
Probability flux, 258–259
Probability flux (Cont’d)
  density, 260–262
  divergence of, 259–260
Probability of losing lock, 407
Projection method, 324–326
Quasi-linear approximation, 417
Quasi-linear theory, 184
Random processes, 3–35
  delta-correlated, 19–20
  first-order Markov, 18–19, 293–304
  Gaussian, 17–18
  narrowband, 20–23
  impulse noise, characterization of, 30–32
  nonstationary transformations on, 26–29
  periodic, 25–26
  with stationary increments, 23–25
  passage through a time-invariant
    linear filter, 32–34
  shot noise, characterization of, 30–32
  spectral densities, 13–17
  stationary, 13–17
  noncausal filtering of, 41–43
  thermal noise, characterization of,
    29–30
  variables, 4–10
Random walk, 286–290
  with absorbing boundaries, 292–296
Range:
  hold in, 455
  pull in, 449
Realization, 5
Receivers:
  coherent, 99–107
  phase-locked regenerative, 240–250
  pseudo-noise tracking, 107–112
  signal acquisition, 499–500
  superheterodyne phase-locked, 177,
    178, 180–184
Recursive formula, 315–316
Relaxation oscillations, 231–240
Restoring forces, nonlinear, 221–227
Rhythms, 70–72
Root locus plots, 140–141
Second-order loops:
  design, 137–140
  performance, 442–503
Self-sustained oscillations, 231–240
Shot noise, characterization of, 30–32
Signal acquisition:
  aids, 475–481
  with imperfect second-order loops, 449–481
  probability, 410–412, 497–500
  properties of, 474–475
  range, 452–463
  receivers, 499–500
  techniques, 475–481
  time, 452–458
Signal amplitude suppression factor, 154, 190
Signal distortion, 161
Signal-to-noise ratios, 192–199
  filtering to maximize, 53–56
Signals:
  acquisition range, 452–465
  acquisition time, 410–412, 558
  deterministic, 3, 4
  nondeterministic, 3, 4
  optimum reference, 553–554
  polynomial-type, 47–48
Singularities, types of, 227–230
Sinusoidal nonlinearity
  steady-state results for, 609–613
  steady-state variance for, 588–593
Sinusoidal phase-locked loops, 617–634
  with applications, 130–185
  bandpass PLL design, 175–177
  carrier tracking, 143–158, 177–180
  Doppler tracking, 143–158
  frequency demodulation, 166–175
  linear PLL angle demodulation theory, 158–161
  linear PLL model, 131–142
  phase demodulation, 161–166
  phase-locked loop mechanisms, 177–180
  spectral analysis of, 657–660
  superheterodyne PLL receiver, 180–184
Smoluchowski equation, 321–324
Spectral densities, 13–17
Spectral linear approximation, 417–418
Spectral linear theory, 184
Squaring loops, 92–93
Square-wave input, 553–554
State-space representation, 346–347
Stationary increments, 23–25
Stationary processes, 13–17
  noncausal filtering of, 41–43
Statistical behavior, 379–406
Steady-state:
  conditional expectations, 536–541
  density, 421–424, 536–541
  density function, 414–415
  of the phase error, 387, 485–491, 500–501, 569–570
  results for a sinusoidal nonlinearity, 609–613
  solution to the Fokker-Planck equation, 605–608
  variance for a sinusoidal nonlinearity, 588–593
Steady-state solutions, 307–309
Steady-state tracking error, 143
Stochastic differential equations, 345–359
  basic rules of Itô calculus, 347–355
  FP intensity coefficients, 355–358
  state-space representation, 346–347
Stochastic methods, 285–344
  diffusion approximation, 337–339
  diffusive flow law, 291–292
  first-passage time problem, 311–320
  Fokker-Planck equation, 336–337
  multidimensional, 326–332
  one-dimensional, 298–311
Markov processes, 296–298, 336–337
  first-order, 18–19, 298–304
  vector, 320–326
  particles escape over a potential wall, 339–341
  the potential case, 333–336
  probability current density, 332–333, 400–404
random walk, 286–290
  with absorbing boundaries, 292–296
Stochastic field theory, 257–284, 336, 337
  changing flux theorem, 261–262
  proof of, 281–283
Gauss's law, 258–259
Laplace's equation, 273–274
Maxwell's curl equations, 265–268
Poisson's equation, 273–274
potential equations, 265–268
probability flow equation, 262–265
probability flux, 258–259
  density, 260–262
  divergence of density, 259–260
Stochastic field theory (Cont'd) produced by Brownian motion, 268–273
Stoke's theorem, 260–261
Stoke's theorem, 260–261
Superheterodyne phase-locked loop receivers, 177, 178, 180–184
Surface integrals, 279–280
Synchronization principle, 240–250
Synchronous control systems, 115–117
acquisition behavior, 374–379
moments of the mean time to first slip, 407–410
moments of signal acquisition time, 410–412
PLL operation, 412–416
signal acquisition probability, 410–412
statistical behavior, 379–406
synchronization stability, 374–379
techniques for approximating the variance, 417–420
first-order, 373–420
higher-order, 521–561
average number of cycles slipped, 546–548
boundary conditions, 530–532, 541–542
dimensional Fokker-Planck equation, 525–527
equivalent model, 522–525
extensions to an arbitrary loop filter, 554–556
first-passage time model, 541–542
first-passage time moments, 542–546
initial conditions, 527–530
new flow of probability, 546–548
optimum reference signal, 553–554
performance of a third-order loop, 556–558
periodic extension method, 527–530
representation, 522–525
steady-state conditional expectations, 536–541
steady-state density, 536–541
synthesis of optimum, 548–553
transition probability density functions, 532–535
with random modulation inputs, 562–573

Synchronous control systems (Cont'd)
conditional transition probability density functions, 567–568
periodic extension, 564–566
reduction of the Fokker-Planck equation, 566
steady-state probability density, 569–570
transmitter-receiver characterization, 563–564
second-order, 442–503
frequency acquisition time, 497–500
nonlinear theory, 481–497
signal acquisition, 449–481
signal acquisition probability, 497–500
steady-state variance of the phase error, 500–501
synthesis of optimum, 548–553

Telecommunication, 67–70
Thermal noise, characterization of, 29–30
Third-order loops:
design, 145–146
performance, 145, 556–558
Time-invariant linear filters, 32–34
Tracking, 551–553
accelerating targets, 473
angle demodulation modulation, 84
carrier, 82–83, 143–158
angle modulation extraction in the presence of, 85–91
phase-locked mechanizations, 177–180
in the presence of angle modulation, 85
sinusoidal phase-locked loops, 143–158, 177–180
of suppressed carrier signals with angle modulation present, 95
a constant frequency offset, 144
a constant phase offset, 144
Doppler, 84, 143–158
in the presence of angle modulation, 85
rates, 145–146
sinusoidal phase-locked loops, 143–158
Tracking (Cont'd)
of suppressed carrier signals with angle modulation present, 95
double loop PN system, 109–111
frequency error, 160, 166
mean-square error, 135
modulation loops, 91–99, 103–106
pseudo-noise receivers, 107–112
steady-state error, 143
Transient performance, 377–379
Transition probability density functions, 532–535
Transponders, coherent, 107–112, 131, 179, 180
Transort processes, 257–284
carriers and, 274–275
changing flux theorem, 261–262
proof of, 281–283
Gauss's law, 258–259
probability flow equation, 262–265
probability flux, 258–259
density, 260–262
divergence of density, 259–260
Stoke's theorem, 260–261
Transreceiver instabilities, 134
Undamped systems, 221–226

Underdamping, 139
Universe, 4

Van der Pol equation, 236–240
Variables, random, 4–10
Variance:
steady-state of the phase error, 500–501
techniques for approximation, 417–420
Vector field concepts, 277–280
derivative operators, 278–279
gradient, 277–278
line integral, 279
surface integral, 279–280
volume integral, 280
Vector Markov processes, 320–326
Volterra functional expansions, 418–419
Volume integrals, 280

White noise, filter performance formulas for, 48–53
Wiener process, 348–350
Wiener-Hopf equation, 44–47

Yovits-Jackson formulas, 58–60